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ORIGINAL PAPER

Equitable Vertex Arboricity of Graphs with Low Maximum Degree

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Abstract
An equitable tree-k-coloring of a graph is a vertex coloring using k distinct colors such that every color class induces a forest and the sizes of any two color classes differ by at most one. The equitable vertex arboricity conjecture states that every graph with maximum degree Δ has an equitable tree-m-coloring for every $m \geq \lceil \frac{\Delta+1}{2} \rceil$. In this paper, we verify this conjecture for graphs with maximum degree at most 6.

Keywords Equitable partition · Vertex arboricity · Vertex coloring

1 Introduction

We only consider simple and fiite graphs in this paper. For a graph G, $V(G)$, $E(G)$, $\Delta(G)$, and $\delta(G)$ denote the vertex set, the edge set, the minimum degree, and the maximum degree of G, respectively. By $|G|$, we denoted the value of $|V(G)|$. For two disjoint subsets $U, W \subseteq V(G)$, $e(U, W)$ denotes the number of edges that have one end-vertex in U and the other in W. For a vertex $v \in V(G)$ and a set $S \subseteq V(G)$, $N_S(v)$ is the set of vertices in S that are adjacent to v, and $d_S(v)$ is the number of vertices in S that are adjacent to v. For convenience, we write $N_G(v)$ and $d_G(v)$ instead of $N_{V(G)}(v)$ and $d_{V(G)}(v)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S.

An *equitable tree-k-coloring* of a graph is a vertex k-coloring such that each color class induces a forest and the size of any color class is $\lceil |G|/k \rceil$ or $\lfloor |G|/k \rfloor$. The minimum integer k such that G has a equitable tree-k-coloring is the *equitable vertex arboricity* of G, denoted by $va_{eq}(G)$. The *equitable vertex arboricity threshold* of G is the smallest k such that G has an equitable tree-k'-coloring for every $k' \geq k$,

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denoted by $va^*_{eq}(G)$. The difference between $va_{eq}(G)$ and $va^*_{eq}(G)$ can be any large, see $K_{n,n}$ for an easy example.

The equitable tree-k-coloring was initially introduced by Wu, Zhang and Li [14] in 2013. They conjectured that there is a constant c independent of G such that $va^*_{eq}(G) \le c$ for every planar graph G. In 2016, Esperet, Lemoine, and Maffray verified this conjecture by proving $va^*_{eq}(G) \le 4$ for every planar graph G. It is still open whether every planar graph can be equitably partitioned into three induced forests. Concerning this problem, Kim, Oum, and Zhang [7] proved that every planar graph can be equitably partitioned into three induced graphs, two of which are forests; and every planar graph can be equitably partitioned into three induced 2-degenerate graphs. Another interesting conjecture posed by Wu, Zhang and Li [14] is the following so-called equitable vertex arboricity conjecture (EVAC for short).

Conjecture 1.1 *If G is a graph with maximum degree Δ, then G has an equitable tree-m-coloring for every $m \ge \lceil \frac{\Delta+1}{2} \rceil$. In other words, $va^*_{eq}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ for every graph G.*

In 2014, Zhang and Wu [20] proved that every graph G has an equitable tree-$\lceil \frac{\Delta(G)+1}{2} \rceil$-coloring provided $\Delta(G) \ge |G|/2$. Later in 2020, Zhang and Niu [20] verified EVAC for graphs G with $\Delta(G) \ge (|G| - 1)/2$. For graphs with small maximum degree, Zhang [18] showed that every subcubic graph has an equitable tree-m-coloring for every $m \ge 2$. In 2017, Chen, Gao, Shan, Wang, and Wu [1] considered EVAC for degenerate graphs. They proved that EVAC holds for every 5-degenerate graphs. For graphs with larger degeneracy, Zhang, Niu, Li, and Li [19] proved that every d-degenerate graph with maximum degree at most Δ has an equitable tree-m-coloring for every $m \ge \lceil \frac{\Delta+1}{2} \rceil$ provided that $\Delta \ge 9.818d$. Other specific problems related to the equitable tree coloring of graphs have also been investigated by numerous authors [2–6, 8–13, 15–17].

Following this line of thought, in this paper, we prove the following, which implies that EVAC completely holds for graphs with maximum degree 6.

Theorem 1.1 *If G is a graph with maximum degree $\Delta \le 6$, then G has an equitable tree-m-coloring for every $m \ge \max\{\lceil \frac{\Delta+1}{2} \rceil, 4\}$.*

2 The Proof of the Maim Theorem

Lemma 2.1 *Let $m \ge 4$ be an integer and $\mathcal{G}$ be the class of graphs with $\Delta(G) \le 6$. If G has an equitable tree-m-coloring for every graph $G \in \mathcal{G}$ with $|G|$ divisible by m, then G has an equitable tree-m-coloring for every graph $G \in \mathcal{G}$.*

Proof We prove this Lemma by induction on the order n of G. We assume $m \nmid n$ because otherwise we are already done. Let t be an integer such that $mt < n < m(t+1)$. Let $u \in V(G)$. By the induction hypothesis, $G - u$ has an equitable tree-m-coloring φ with color classes $V_1, V_2, \ldots, V_m$ such that $|V_i| = t$ or $t + 1$ for all $i \in [m]$.

Since $d_G(u) \leq 6$, we may assume that each of the colors $4, 5, \ldots, m$ appears at most once among $N_G(u)$. If $|V_i| = t$ for some $i \geq 4$, then by adding u to V_i, we get an equitable tree-m-coloring of G. Hence we assume that $|V_i| = t + 1$ for all $i \geq 4$. It follows

$$m(t+1) - 1 \geq n \geq 1 + (m-3)(t+1) + 3t = m(t+1) - 2,$$

and thus $n = m(t+1) - 1$ or $n = m(t+1) - 2$.

If $n = m(t+1)-1$, then $G' := G \cup K_1$ is a graph of order $m(t+1)$ and $\Delta(G') \leq 6$. By our assumption, G' has an equitable m-tree coloring φ' with the size of each color class being exactly $t+1$. Restricting φ' to G, we obtain an equitable tree-m-coloring of G.

If $n = m(t+1) - 2$, then let $G' := G \cup K_2$, where the K_2 is denoted as vw. By our assumption, G' has a tree-m-coloring φ' with color classes $V_1, \ldots, V_m$ such that $|V_i| = t+1$ for each $i \in [m]$. If v and w are in different color classes of φ', then restricting φ' to G, we obtain an equitable tree-m-coloring of G. If v and w are in a same color class of φ', say V_1, then we look into two cases. First, if there exist a vertex $x \in V_i$ for some $i \in \{2, \ldots, m\}$, say $i = 2$, such that $d_{V_1}(x) \leq 1$, then G has an equitable tree-m-partition $(V_1 \cup \{x\} \setminus \{v, w\}, V_2 \setminus \{x\}, V_3, \ldots, V_m)$. Next, if $d_{V_1}(x) \geq 2$ for each $x \in \cup_{i=2}^m V_i$, then

$$6(t-1) = 6(|V_1| - 2) \geq e(V_1, \cup_{i=2}^m V_i) \geq 2(m-1)(t+1) \geq 6(t+1),$$

a contradiction. □

By Lemma 2.1, it is sufficient to verify Theorem 1.1 for graphs G with $m \mid |G|$. Now we set $|G| = mt$, where t is a positive integers, and prove the theorem by induction on $|E(G)|$.

Choose a vertex $x \in V(G)$ such that $d := d_G(x) = \delta(G) \leq 6$. Let $N_G(x) = \{x_1, \ldots, x_d\}$. By induction, the graph $G \setminus \{xx_1\}$ has an equitable tree-m-coloring φ with color classes $V_1, \ldots, V_m$ such that $|V_i| = t$ for each $1 \leq i \leq m$. Clearly, φ is also an equitable tree-m-coloring of G unless x and x_1 belong to a same color class, say V_1, and meanwhile, there is a cycle passing through xx_1 in $G[V_1]$. Let x_2 be the other neighbor of x on that cycle. Since x has at most 4 neighbors among $\cup_{i=2}^m V_i$, we may assume

$$d_{V_i}(x) \leq 1,\ 4 \leq i \leq m. \tag{2.1}$$

Let I_1 be a set of isolated vertices in $G[V_1]$, and let $V_1' = V_1 \setminus \{I_1 \cup \{x\}\}$. Since $x_1, x_2 \notin I_1$, $V_1' \neq \emptyset$ and thus $2 \leq |V_1'| \leq t - 1$.

A vertex $v \in V_i$ is *movable* to a color class V_j with $j \neq i$ if $G[V_j \cup \{v\}]$ contains no cycles. In other words, if v is not movable to V_j, then $d_{V_j}(v) \geq 2$ and there is a vertex $u \in N_{V_j}(v)$ such that $d_{V_j}(u) \geq 1$. For vertices $u \in V_i$ and $v \in V_j$, *exchanging* u and v refers to moving u into V_j and v into V_i.

Lemma 2.2 *Vertices in* $\cup_{i=4}^m V_i$ *are not movable to* V_1'.

Proof Suppose, to the contrary, that $v \in \cup_{i=4}^{m} V_i$ that is movable to V_1'. Now, exchanging the vertices x and v, we obtain an equitable tree-m-coloring of G. □

Lemma 2.3 *There exists a vertex $w \in V_2 \cup V_3$ that is movable to V_1'.*

Proof Otherwise, $d_{V_1'}(w) \geq 2$ for each $w \in V_2 \cup V_3$. By Lemma 2.2, $d_{V_1'}(v) \geq 2$ for each $v \in \cup_{i=4}^{m} V_i$. For every vertex $v \in V_1'$, $d_{V_1'}(v) \geq 1$ by the definition of V_1'. So $d_{G \setminus V_1'}(v) \leq \Delta - 1$. Hence

$$(\Delta - 1)(t - 1) \geq e(V_1', \cup_{i=2}^{m} V_i) \geq 2(m-1)t \geq (\Delta - 1)t,$$

a contradiction. □

By Lemma 2.3, we assume, without loss of generality, that V_2 has a vertex w_2 that is movable into V_1'. Let $V_2' = V_2 \setminus \{w_2\}$

Lemma 2.4 *Every vertices in $\cup_{i=4}^{m} V_i$ are not movable to V_2'.*

Proof Otherwise, there is a vertex $v \in \cup_{i=4}^{m} V_i$ movable to V_2'. For convenience, we assume $v \in V_4$. Moving w_2, v, x into V_1, V_2, V_4, respectively, we obtain an equitable tree-m-coloring of G. □

For two vertices $x \in V_i$ and $y \in V_j$ with $i \neq j$, if both $G[V_i \cup \{y\} \setminus \{x\}]$ and $G[V_j \cup \{x\} \setminus \{y\}]$ are forests, then we say that (x, y) is an *exchangeable pair* between V_i and V_j.

Lemma 2.5 *Suppose that $d_{V_2'}(v) = 2$ for every vertex $v \in V_i$ with $4 \leq i \leq m$. Let $z \in V_2'$.*

(a) Let $v \in V_i$, where $4 \leq i \leq m$. If $G[V_2' \cup \{v\} \setminus \{z\}]$ has no cycles, then

- *(a1) $d_{V_2'}(z) \geq 1$;*
- *(a2) $d_{V_1'}(z) \geq 2$ provided $vw_2 \notin E(G)$;*
- *(a3) $d_{V_i}(z) \leq 3$ provided $vw_2 \notin E(G)$, and moreover, the equality implies $zx \notin E(G)$.*

(b) If there is a vertex $v \in V_i$ with $4 \leq i \leq m$ such that $zv \in E(G)$ and $vw_2 \notin E(G)$, then $d_{V_i}(z) \leq 2$ and (z, v) forms an exchangeable pair.

(c) Let y_1 and y_2 be two nonadjacent vertices in V_i such that (z, y_1) and (z, y_2) are exchangeable pairs.

- *(c1) If $4 \leq i \leq m$, or $i = 3$ and $d_{V_2'}(v) = 2$ for every vertex $v \in V_3$, then $G[V_2' \cup \{y_1, y_2\} \setminus \{z\}]$ is a forest provided $d_{V_i}(x) \leq 1$.*
- *(c2) Suppose $4 \leq i \leq m$ and suppose $y_1 w_2 \notin E(G)$ or $y_2 w_2 \notin E(G)$. If $d_{V_i}(z) \neq 1$, or $d_{V_i}(z) = 1$ and the neighbor of z in V_i is not adjacent to w_2, then*
  - *(c2.1) $G[V_i \cup \{z, w_3\} \setminus \{y_1, y_2\}]$ is a forest for each vertex $w_3 \in V_3$ such that $d_{V_i}(w_3) \leq 1$;*
  - *(c2.2) $d_{V_2'}(x) + d_{V_i}(x) \leq 3$.*

Proof (a1) Since $G[V_2' \cup \{v\} \setminus \{z\}]$ has no cycles, if there is a cycle in $G[V_2' \cup \{v\}]$ then it would pass z. This is impossible if $d_{V_2'}(z) = 0$. Hence, moving w_2, v, x into V_1, V_2, V_i, respectively, we obtain an equitable tree-m-coloring of G, a contradiction.

(a2) Suppose that $d_{V_1'}(z) \le 1$ and $vw_2 \notin E(G)$. If $G[V_2 \cup \{v\} \setminus \{z\}]$ is a forest, then moving z, x, v into V_1, V_i, V_2, respectively, we obtain an equitable tree-m-coloring of G, a contradiction. Hence $G[V_2 \cup \{v\} \setminus \{z\}]$ contains a cycle C_1, and by the symmetry of z and w_2 (note that both z and w_2 are movable to V_1'), $G[V_2 \cup \{v\} \setminus \{w_2\}]$ contains a cycle C_2. Since $G[V_2' \cup \{v\} \setminus \{z\}]$, $G[V_2 \setminus \{z\}]$, and $G[V_2 \setminus \{w_2\}]$ have no cycles,

- C_1 passes v, w_2, and does not pass z;
- C_2 passes v, z, and does not pass w_2.

Since $vw_2 \notin E(G)$ and $d_{V_2'}(v) = 2$, $d_{V_2}(v) = 2$ and we set $N_{V_2}(v) = \{a, b\}$. Clearly, the path avb are on both C_1 and C_2. This implies that $G[V(C_1) \cup V(C_2) \setminus \{v\}]$ contains a cycle, a contradiction.

(a3) By (a1) and (a2), $d_{V_i}(z) \le \Delta - 1 - 2 \le 3$. Moreover, if $d_{V_i}(z) = 3$, then $d_{V_2'}(z) = 1$ and $d_{V_1'}(z) = 2$, which implies $zx \notin E(G)$.

(b) Since $d_{V_2'}(v) = 2$ and $zv \in E(G)$, $G[V_2' \cup \{v\} \setminus \{z\}]$ has no cycles.

By (a3), $d_{V_i}(z) \le 3$. If $d_{V_i}(z) = 3$, then z has two neighbors y_1 and y_2 in V_i such that they are not adjacent (otherwise there would be a triangle in the graph induced by V_i) and $zx \notin E(G)$ by (a3). Now, moving w_2 into V_1, y_1, y_2 into V_2, and z, x into V_i, we obtain an equitable tree-m-coloring of G, a contradiction. Hence $d_{V_i}(z) \le 2$ and therefore (z, v) forms an exchangeable pair between V_2' and V_i.

(c1) Suppose that $G[V_2' \cup \{y_1, y_2\} \setminus \{z\}]$ contains a cycle C. Since (z, y_1) and (z, y_2) are exchangeable pairs, $G[V_2' \cup \{y_1\} \setminus \{z\}]$ and $G[V_2' \cup \{y_2\} \setminus \{z\}]$ have no cycles. This concludes that

- C passes y_1 and y_2.

If $zy_1 \in E(G)$, then $d_{V_2'}(y_1) = 2$ and $y_1y_2 \notin E(G)$ implies that y_1 has degree one in $G[V_2' \cup \{y_1, y_2\} \setminus \{z\}]$, and thus it cannot be contained in any cycles there, a contradiction. Hence $zy_1 \notin E(G)$, and by symmetry, $zy_2 \notin E(G)$.

If $G[V_2' \cup \{y_1\}]$ has no cycles, then we move w_2, y_1, x to V_1, V_2, V_i, respectively. This gives an equitable tree-m-coloring of G. Hence $G[V_2' \cup \{y_1\}]$ contains a cycle C_1, and by symmetry, $G[V_2' \cup \{y_2\}]$ contains a cycle C_2.

Since $G[V_2' \cup \{y_1\} \setminus \{z\}]$ and $G[V_2' \cup \{y_2\} \setminus \{z\}]$ have no cycles, we conclude the following:

- C_1 passes y_1 and z;
- C_2 passes y_2 and z.

Denote C by $a_1y_1b_1 \cdots a_2y_2b_2 \cdots a_1$ (it is possible that $a_1 = b_2$ or $a_2 = b_1$). Since $d_{V_2'}(y_1) = d_{V_2'}(y_2) = 2$, we have $a_1, b_1 \in V(C_1)$ and $a_2, b_2 \in V(C_2)$. Let $P(z, a_1)$ be the path on C_1 from z to a_1 that does not pass y_1, $P(b_2, z)$ be the path on C_2 from b_2 to z that does not pass y_2, and $P(a_1, b_2)$ be the path on C from a_1 to b_2 that does not pass y_1 and y_2. We walk along a trail that connects by turn $P(z, a_1)$, $P(a_1, b_2)$ and $P(b_2, z)$, and find a cycle in $G[V_2']$, a contradiction.

(c2.1) Suppose that $G[V_i \cup \{z, w_3\} \setminus \{y_1, y_2\}]$ has a cycle C. Since (z, y_1) is an exchangeable pair, $G[V_i \cup \{z\} \setminus \{y_1, y_2\}]$ has no cycles. Therefore, C passes w_3, and thus w_3 has degree at least two in $G[V_i \cup \{z, w_3\} \setminus \{y_1, y_2\}]$. This implies $zw_3 \in E(G)$, as $d_{V_i}(w_3) \leq 1$. On the other hand, $d_{V_i}(w_3) \leq 1$ implies that $G[V_i \cup \{w_3\} \setminus \{y_1, y_2\}]$ has no cycles. Therefore, C passes z, and thus z has degree at least two in $G[V_i \cup \{z, w_3\} \setminus \{y_1, y_2\}]$. Now, since $zw_3 \in E(G)$, we conclude $d_{V_i}(z) \geq 1$, and by (a1) and (a2), we further have $d_{V_i}(z) \leq \Delta - (1 + 2 + 1) \leq 2$.

If $d_{V_i}(z) = 2$, then $zx \notin E(G)$. We obtain an equitable tree-m-coloring of G by moving w_2 into V_1, y_1, y_2 into V_2, and x, z into V_i, respectively. Note that $G[V_2 \cup \{y_1, y_2\} \setminus \{z, w_2\}]$ is a forest by (c1).

If $d_{V_i}(z) = 1$, then let $y \in V_i$ such that $yz \in E(G)$ (it is possible that $y \in \{y_1, y_2\}$). By Lemmas 2.2 and 2.4, $d_{V_1'}(y) \geq 2$ and $d_{V_2'}(y) \geq 2$. If $yy_1, yy_2 \in E(G)$, then $d_{V_i \setminus \{y_1, y_2\}}(y) = 0$ and $yw_3 \notin E(G)$. This implies that $G[V_i \cup \{z, w_3\} \setminus \{y_1, y_2\}]$ has no cycles, a contradiction. Hence we assume by symmetry that $yy_1 \notin E(G)$. By (b), (z, y) is an exchangeable pair because $yw_2 \notin E(G)$, and then by (c1), $G[V_2 \cup \{y, y_1\} \setminus \{z, w_2\}]$ is a forest. Since $d_{V_i}(z) = 1$ and $zy \in E(G)$, $G[V_i \cup \{x, z\} \setminus \{y, y_1\}]$ is a forest. Therefore, we obtain an equitable tree-m-coloring of G by moving w_2 into V_1, y, y_1 into V_2, and x, z into V_i, respectively.

(c2.2) Suppose that $d_{V_2'}(x) + d_{V_i}(x) \geq 4$ for some $4 \leq i \leq m$. Since $d_{V_1'}(x) \geq 2$ and $\Delta \leq 6$, $d_{V_2'}(x) + d_{V_i}(x) = 4$. If $d_{V_i}(x) = 0$, then we move w_2 into V_1, y_1, y_2 into V_2, and z, x into V_i. This gives an equitable tree-m-coloring of G by (c1). So $d_{V_i}(x) = 1$, which follows $d_{V_2'}(x) = 3$ and $d_{V_3}(x) = 0$. Note that $d_{V_1'}(x) \geq 2$.

Let $v \in V_3$. If $d_{V_2'}(v) \leq 1$, then we move w_2, x, v into V_1, V_3, V_2, respectively. This gives an equitable tree-m-coloring of G, a contradiction. If $d_{V_1'}(v) \leq 1$, then exchanging x and v also results in an equitable tree-m-coloring of G, a contradiction. Hence $d_{V_1'}(v) \geq 2$ and $d_{V_2'}(v) \geq 2$. If $d_{V_2}(v) \geq 3$, then $d_{V_i}(v) \leq 1$. We move w_2 into V_1, y_1, y_2 into V_2, x into V_3, and z, v into V_i. This gives an equitable tree-m-coloring of G by (c1) and (c2.1), a contradiction. Therefore, $d_{V_2}(v) = 2$ for every $v \in V_3$. If $vw_2 \in E(G)$, then $d_{V_2'}(v) = 1$. We move w_2, v, x into V_1, V_2, V_3, respectively, and obtain an equitable tree-m-coloring of G, a contradiction. Hence $vw_2 \notin E(G)$ and $d_{V_2'}(v) = 2$.

We now count the number f of exchangeable pairs between V_2' and V_3.

Let $v \in V_3$ and let $u \in V_2'$ such that $uv \in E(G)$. If $d_{V_1'}(u) \leq 1$, then we move u, v, x into V_1, V_2, V_3, respectively. If $d_{V_2'}(u) = 0$, then we move w_2, v, x into V_1, V_2, V_3, respectively. In either case we obtain an equitable tree-m-coloring of G, a contradiction. So $d_{V_1'}(u) \geq 2$, $d_{V_2'}(u) \geq 1$, and thus $d_{V_3}(u) \leq \Delta - (2 + 1) \leq 3$. If $d_{V_3}(u) = 3$, then u has two neighbors u_1 and u_2 in V_3 such that they are not adjacent. Move w_2 into V_1, u_1, u_2 into V_2, and u, x into V_3. This gives an equitable tree-m-coloring of G by (c1) and by the fact that $d_{V_3}(x) = 0$. Hence $d_{V_3}(u) \leq 2$ and thus (u, v) forms an exchangeable pair between V_2' and V_3. This implies $f \geq 2|V_3| = 2t$.

Now we count f in another direction. Let $u \in V_2'$. If there are three vertices $u_1, u_2, u_3 \in V_3$ such that (u, u_j) forms an exchangeable pair between V_2' and V_3 for each $j \in [3]$, then we assume, without loss of generality, that $u_1u_2 \notin E(G)$. We move w_2 into V_1, u_1, u_2 into V_2, and u, x into V_3. Since $G[V_2 \cup \{u_1, u_2\} \setminus \{u, w_2\}]$ is a forest by (c1) and $d_{V_3}(x) = 0$, this gives an equitable tree-m-coloring of G, a contradiction. So there are at most two vertices in V_3 forming exchangeable pairs with u. This gives $f \leq 2|V_2'| = 2(t - 1)$, a contradiction. □

Lemma 2.6 *There exists $y \in V_i$ for some $4 \le i \le m$ such that $d_{V_2}(y) \ge 3$ and $d_{V_3}(y) \le 1$.*

Proof Arbitrarily fix an integer i so that $4 \le i \le m$. Suppose $d_{V_2}(v) \le 2$ for every vertex $v \in V_i$.

Let $v \in V_i$. If $vw_2 \in E(G)$, then $d_{V_2'}(v) \le 1$. We move w_2, v, x into V_1, V_2, V_i, respectively, and obtain an equitable tree-m-coloring of G, a contradiction. Hence $vw_2 \notin E(G)$. If $d_{V_2'}(v) \le 1$, then we obtain an equitable tree-m-coloring by moving w_2, v, x into V_1, V_2, V_i, respectively. Hence $2 \ge d_{V_2}(v) \ge d_{V_2'}(v) \ge 2$. Therefore, for every vertex $v \in V_i$, we have $vw_2 \notin E(G)$ and $d_{V_2'}(v) = 2$.

We count the number f of exchangeable pairs between V_2' and V_i as follows. For each $v \in V_i$ and each $z \in N_{V_2'}(v)$, (z, v) is an exchangeable pair by Lemma 2.5(b), which implies $f \ge 2|V_i| = 2t$.

On the other hand, let $z \in V_2'$. If there are vertices $y_1, y_2, y_3 \in V_i$ such that (z, y_j) forms an exchangeable pair between V_2' and V_i for each $j \in [3]$, then we assume, without loss of generality, that $y_1 y_2 \notin E(G)$. Since $d_{V_i}(x) \le 1$, $G[V_2 \cup \{y_1, y_2\} \setminus \{z, w_2\}]$ is a forest by Lemma 2.5(c1). If we further have that $G[V_i \cup \{z, x\} \setminus \{y_1, y_2\}]$ is a forest, then we obtain an equitable tree-m-coloring of G by moving w_2 into V_1, y_1, y_2 into V_2, and z, x into V_i. If we come to the case that $G[V_i \cup \{z, x\} \setminus \{y_1, y_2\}]$ has a cycle, then $zx \in E(G)$ and $d_{V_i \setminus \{y_1, y_2\}}(x) = 1$. It follows that $xy_1 \notin E(G)$. If $d_{V_2'}(x) \le 2$, then we move w_2 into V_1, x, y_1 into V_2, and z into V_i. This gives an equitable tree-m-coloring of G. Hence $d_{V_2'}(x) \ge 3$, and thus $d_{V_2'}(x) + d_{V_i}(x) \ge 3 + 1 = 4$, contradicting Lemma 2.5(c2.2) (note that all conditions needed by this lemma are satisfied as $vw_2 \notin E(G)$ for every vertex $v \in V_i$). So there are at most two vertices in V_i forming exchangeable pairs with z, and therefore, $f \le 2|V_2'| = 2(t-1)$, a contradiction.

This proves that there is a vertex $y \in V_i$ such that $d_{V_2}(y) \ge 3$. Finally, since $d_{V_1'}(y) \ge 2$ by Lemma 2.2, $d_{V_3}(y) \le \Delta - (2+3) \le 1$. □

In the following, we set y be a vertex in V_i with $4 \le i \le m$ such that $d_{V_3}(y) \le 1$, which exists by Lemma 2.6.

Lemma 2.7 *$d_{V_2'}(v) \ge 2$ for every vertex $v \in V_3$.*

Proof If there exists $v \in V_3$ such that $d_{V_2'}(v) \le 1$, then we move w_2, v, y, x into V_1, V_2, V_3, V_i, respectively, and obtain an equitable tree-m-coloring of G, a contradiction. □

Let

$$\begin{aligned} A &= \{v \mid v \in \cup_{i=3}^m V_i, d_{V_2}(v) = 2\}, \\ A' &= \{v \mid v \in \cup_{i=3}^m V_i, d_{V_2'}(v) = 2\}, \\ B &= A' \setminus N_{\cup_{i=3}^m V_i}(w_2), \\ S &= N_{V_2}(A), \\ S' &= N_{V_2'}(B). \end{aligned}$$

Lemma 2.8 *$d_{V_1'}(z) \ge 2$ for every $z \in S$.*

Proof Suppose, to the contrary, that $d_{V_1'}(z) \leq 1$ for some $z \in S$. It follows that there is a vertex $v \in V_j$ for some $3 \leq j \leq m$ such that $zv \in E(G)$.

If $j = 3$, then we move z, v, y, x into V_1, V_2, V_3, V_i $(4 \leq i \leq m)$, respectively. If $4 \leq j \leq m$, then we move z, v, x into V_1, V_2, V_j, respectively. In either case we obtain an equitable tree-m-coloring of G, a contradiction. □

Lemma 2.9 $S' \subseteq S$.

Proof Let $s \in S'$. It follows that there is a vertex $b \in B$ such that $bs \in E(G)$. Since $b \notin N_{\cup_{i=3}^m V_i}(w_2)$, $d_{V_2}(b) = d_{V_2'}(b) = 2$. Therefore, $b \in A$ and thus $s \in S$. □

We are ready to complete the proof of Theorem 1.1. Since $d_{\cup_{i=3}^m V_i}(z) \leq \Delta - 2 \leq 4$ for every $z \in S'$ by Lemmas 2.8 and 2.9, $2|B| = e(B, S') \leq 4|S'|$ by the definition of B, implying

$$|B| \leq 2|S'|. \tag{2.2}$$

Also, we have

$$\begin{aligned} 3(m-2)t - |A'| &= 2|A'| + 3(|\cup_{i=3}^m V_i| - |A'|) \\ &\overset{\text{Lemmas 2.4 and 2.7}}{\leq} e(V_2', \cup_{i=3}^m V_i) \\ &\overset{\text{Lemmas 2.8 and 2.9}}{\leq} (\Delta - 2)|S'| + \Delta(|V_2' \setminus S'|) \\ &= \Delta(t-1) - 2|S'|. \end{aligned} \tag{2.3}$$

Since $\Delta \leq 6$,

$$|A'| - |B| \leq 6. \tag{2.4}$$

Combing (2.2), (2.3) and (2.4) together, we deduce

$$\begin{aligned} 6(t-1) - 2|S'| &\overset{(2.2)}{\leq} 6t - (|B| + 6) \overset{(2.4)}{\leq} 6t - |A'| \\ &\overset{(2.3)}{\leq} e(V_2', \cup_{i=3}^m V_i) \overset{(2.3)}{\leq} 6(t-1) - 2|S'| \end{aligned} \tag{2.5}$$

as $m \geq 4$ and $\Delta \leq 6$. This implies that all qualities in (2.5) holds and therefore

- $d_{\cup_{i=3}^m V_i}(z) = \Delta - 2$ for every vertex $z \in S'$, and
- $d_{\cup_{i=3}^m V_i}(z) = \Delta$ for every vertex $z \in V_2' \setminus S'$.

Combining this with Lemmas 2.8 and 2.9, we conclude that $d_{V_2}(z) = 0$ for every $z \in V_2'$, i.e., V_2' is an independent set. Exchanging x and w_2, we finally receive an equitable tree-m-coloring of G, a contradiction.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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