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Note

Every subcubic graph is packing $(1, 1, 2, 2, 3)$-colorable

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ABSTRACT

For a sequence $S = (s_1, \dots, s_k)$ of non-decreasing integers, a packing S-coloring of a graph G is a partition of its vertex set $V(G)$ into $V_1, \dots, V_k$ such that for every pair of distinct vertices $u, v \in V_i$, where $1 \le i \le k$, the distance between u and v is at least $s_i + 1$. The packing chromatic number, $\chi_p(G)$, of a graph G is the smallest integer k such that G has a packing $(1, 2, \dots, k)$-coloring. Gastineau and Togni asked an open question "Is it true that the 1-subdivision ($D(G)$) of any subcubic graph G has packing chromatic number at most 5?" and later Brešar, Klavžar, Rall, and Wash conjectured that it is true.
In this paper, we prove that every subcubic graph has a packing $(1, 1, 2, 2, 3)$-coloring and it is sharp due to the existence of subcubic graphs that are not packing $(1, 1, 2, 2)$-colorable. As a corollary of our result, $\chi_p(D(G)) \le 6$ for every subcubic graph G, improving a previous bound (8) due to Balogh, Kostochka, and Liu in 2019, and we are now just one step away from fully solving the conjecture.

1. Introduction

For a sequence $S = (s_1, \dots, s_k)$ of non-decreasing integers, a packing S-coloring of a graph G is a partition of its vertex set $V(G)$ into $V_1, \dots, V_k$ such that for every pair of distinct vertices $u, v \in V_i$, where $1 \le i \le k$, the distance between u and v is at least $s_i + 1$. The packing chromatic number (PCN), $\chi_p(G)$, of a graph G is defined to be the smallest integer k such that G has a packing $(1, 2, \dots, k)$-coloring. The first time of studying a packing S-coloring can be traced back to the paper of Goddard, Hedetniemi, Hedetniemi, Harris, and Rall [15]. The concept of packing S-coloring was first formally introduced by Goddard and Xu [16] and is now a very popular topic in graph coloring. Moreover, its edge counterpart was recently studied by Gastineau and Togni [13], Hocquard, Lajou, and Lužar [17], Liu, Santana, and Short [20], as well as Liu and Yu [21].

The notion of packing chromatic number was introduced by Goddard, Hedetniemi, Hedetniemi, Harris, and Rall [15] in 2008 under the name broadcast chromatic number, and it was motivated by a frequency assignment problem in broadcast networks. The concept has drawn the attention of many researchers recently (e.g., see [1–4,6,7,10,14,18,19,22]). In particular, Fiala and Golovach [9] proved that finding the PCN of a graph is NP-complete even in the class of trees. Sloper [23]

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showed that the infinite complete ternary tree (every vertex has 3 child vertices) has unbounded PCN. Brešar, Gastineau and Togni [5] proved that the PCN of any 2-connected bipartite subcubic outerplanar graph is bounded by 7. The question whether every cubic graph has a bounded packing chromatic number was first asked by Goddard et al. [15] and discussed in many papers (e.g., see [1,6,7,14]). Balogh, Kostochka, and Liu [1] answered the question in the negative using the probabilistic method and later Brešar and Ferme [3] provided an explicit construction.

The 1-subdivision of a graph G, denoted by $D(G)$, is obtained from G by replacing every edge with a path of two edges. Gastineau and Togni [14] asked the open question whether it is true that the subdivision of any subcubic graph is packing $(1,2,3,4,5)$-colorable and later Brešar, Klavžar, Rall, and Wash [7] conjectured it is true.

Conjecture 1.1 *(Brešar, Klavžar, Rall, and Wash [7]). The* 1*-subdivision of every subcubic graph is packing* $(1,2,3,4,5)$*-colorable.*

Balogh, Kostochka, and Liu [2] proved that the packing chromatic number of the 1-subdivision of subcubic graphs is bounded by 8. Furthermore, Conjecture 1.1 has been confirmed for many subclasses of subcubic graphs. In particular, Brešar, Klavžar, Rall, and Wash [7] proved it for generalized prism of a cycle, Liu, Liu, Rolek, and Yu [19] showed it for subcubic planar graphs with girth at least 8, Kostochka and Liu [18] confirmed it for subcubic outerplanar graphs, and Mortada and Togni [22] recently extended this class by including each subcubic 3-saturated graph that has no adjacent heavy vertices.

Gastinue and Togni [14] proved the following statement, which is invaluable for proving Conjecture 1.1.

Proposition 1 *(Gastineau and Togni [14]). Let G be a graph and $(s_1,\dots,s_k)$ be a sequence of non-decreasing positive integers. If G is packing $(s_1,\dots,s_k)$-colorable, then $D(G)$ is packing $(1,2s_1+1,\dots,2s_k+1)$-colorable.*

Gastineau and Togni [14] showed that the Petersen graph has no packing $(1,1,k,k')$-colorings when $k,k' \geq 2$. Indeed, the maximum size of the union of two independent sets in the Petersen graph is 7 and the diameter of the Petersen graph is 2. Brešar, Klavžar, Rall, and Wash [7] proved that the 1-subdivision of the Petersen graph is packing $(1,2,3,4,5)$-colorable. By Proposition 1, if one can show every subcubic graph except the Petersen graph has a packing $(1,1,2,2)$-coloring, then Conjecture 1.1 is confirmed.

Much other packing S-colorings have also been studied. In particular, Gastineau and Togni [14] proved that every subcubic graph is packing $(1,1,2,2,2)$-colorable and packing $(1,2,2,2,2,2,2)$-colorable. Balogh et al. [2] showed that every subcubic graph has a packing $(1,1,2,2,3,3,k)$-coloring with color $k \geq 4$ used at most once and every 2-degenerate subcubic graph has a packing $(1,1,2,2,3,3)$-coloring. Cranston and Kim [8] showed that every cubic graph except the Petersen graph has a packing $(2,2,2,2,2,2,2,2)$-coloring. Thomassen [24] and independently Hartke, Jahanbekam, and Thomas [11] proved that every cubic planar graph is packing $(2,2,2,2,2,2,2)$-colorable.

In this paper, we prove that every subcubic graph has a packing $(1,1,2,2,3)$-coloring.

Theorem 1.2. *Every subcubic graph G has a packing* $(1,1,2,2,3)$*-coloring.*

Our result is also sharp due to the fact that the Petersen graph is not packing $(1,1,2,2)$-colorable. By Theorem 1.2 and Proposition 1, a packing $(1,1,2,2,3)$-coloring of G implies a packing $(1,3,3,5,5,7)$-coloring of $D(G)$. Therefore, $\chi_p(D(G)) \leq 6$ for every subcubic graph G, improving the previous bound (8) of Balogh et al. [2], and we are now just one step away from fully solving Conjecture 1.1.

Corollary 1.3. *Let G be a subcubic graph. Then $\chi_p(D(G)) \leq 6$.*

2. Proof of Theorem 1.2

We can assume G is connected since otherwise we apply the argument to each component of G. We may also assume that G is cubic since every subcubic graph is a proper subgraph of some larger cubic graph. Take two disjoint independent sets I_1 and I_2 such that

$$|I_1| + |I_2| \text{ is maximum among all choices of } I_1, I_2. \tag{2.1}$$

Among those sets I_1, I_2 satisfying Condition (2.1), we further take I_1, I_2 such that

$$\text{the number of connected components in } G - I_1 - I_2 \text{ is minimum.} \tag{2.2}$$

Let $G' = G[V(G) - I_1 - I_2]$ and define the graph H_{I_1,I_2} to be the graph H with $V(H) = V(G) - I_1 - I_2$ and $E(H) = \{v_1v_2 \mid d_G(v_1,v_2) \leq 2, v_1, v_2 \in V(H)\}$. We use the abbreviation H to denote H_{I_1,I_2} if the sets I_1 and I_2 are clear from the context. Note that $V(G') = V(H)$, G' is the induced subgraph on $V(G) - I_1 - I_2$, and H adds edges between vertices in G' of distance two. An example of G, G', H is shown in Fig. 1. The corresponding graph $G(H)$ of H in G is the graph with $V(G(H)) = V(H) \cup \{u \mid u \in V(G) \setminus V(H),\ v_1, v_2 \in V(H),\ \text{and } uv_1, uv_2 \in E(G)\}$ and $E(G(H)) = \{v_1v_2 \mid v_1v_2 \in E(G), v_1, v_2 \in$

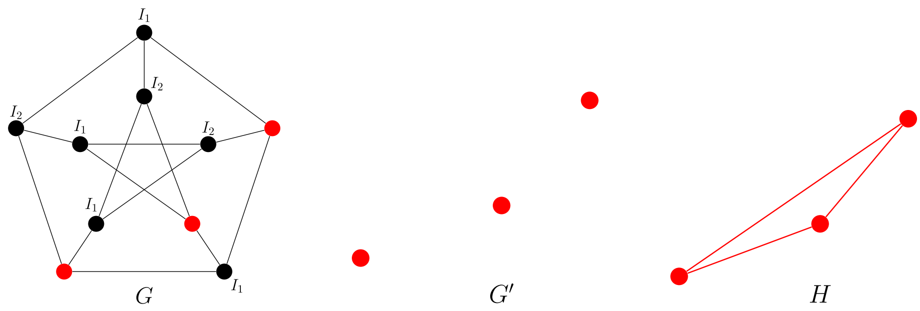

Fig. 1. An example of G, G', and H. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$V(H)\} \cup \{uv_1, uv_2 \mid v_1v_2 \notin E(G), uv_1, uv_2 \in E(G), u \in V(G) \setminus V(H), v_1, v_2 \in V(H)\}$. Essentially, $G(H)$ shows how each component of H is connected in G. Denote G' the *red graph* and each vertex in $I_1 \cup I_2$ *black*. For the addition and subtraction of subscripts within the set $\{1, 2, \dots, k\}$, we perform these operations modulo k. Note that the notation $N(u)$ refers to the neighborhood in G unless otherwise stated.

Lemma 2.1. $\Delta(G') \leq 1$.

Proof. Suppose not, i.e., there is a vertex u of degree at least two in G'. Let $\{u_1, u_2\} \subseteq N_{G'}(u)$. If $N_{G'}(u) \setminus \{u_1, u_2\} = \emptyset$, then we may assume $N_G(u) \setminus \{u_1, u_2\} = \{u_3\}$ and $u_3 \in I_1$. Since $\{u_1, u_2\} \cap (I_1 \cup I_2) = \emptyset$, no matter whether $N_{G'}(u) \setminus \{u_1, u_2\}$ is empty or not, we can add u to I_2 to obtain a contradiction with Condition (2.1). □

By Lemma 2.1, we have the following corollary.

Corollary 2.2. *A connected component in G' is either a P_1 or a P_2.*

We now pay attention to the structures between two P_2 s (of G') in G. By Lemma 2.1, two red P_2 s cannot be adjacent via an edge in G. Furthermore, we show at most one P_2 can be included in a connected component of H.

Lemma 2.3. *At most one red P_2 can be included in a connected component of H.*

Proof. We first show two red P_2 s cannot be connected by a vertex in $I_1 \cup I_2$ (i.e., at distance one in H).

Claim 2.4. *Two red P_2 s cannot be connected by a vertex in $I_1 \cup I_2$ (i.e., adjacent in H).*

Proof. Suppose not, i.e., two red P_2 s, u_1u_2 and v_1v_2, are connected by a vertex $w_1 \in I_1$ with $u_2w_1, v_1w_1 \in E(G)$. Let $N(u_2) = \{u_1, w_1, u_2'\}$ and $N(v_1) = \{w_1, v_2, v_1'\}$. By Lemma 2.1, $u_2', v_1' \in I_1 \cup I_2$. We may assume $u_2', v_1' \in I_2$ since otherwise we can add u_2 or v_1 to I_2, which contradicts Condition (2.1). We remove w_1 from I_1 and add u_2, v_1 to I_1 to increase the size of $I_1 \cup I_2$, which is again a contradiction with Condition (2.1). □

We are ready to prove the lemma by induction on the distance between two red P_2 s. The base case is already shown in Claim 2.4. Assume two red P_2 s cannot be at distance at most $k-1$ in H, where $k \geq 2$. We now show two red P_2 s cannot be at distance k in H. Let u_1u_2, v_1v_2 be two P_2 s in H that are at distance k. Let $x_1, \dots, x_{k-1} \in V(G')$ and $w_1, \dots, w_k \in I_1 \cup I_2$ such that $u_1u_2w_1x_1w_2 \dots x_{k-1}w_kv_1v_2$ is a path in G. We may assume $w_1 \in I_1$. Let $N(u_2) = \{u_1, w_1, u_2'\}$. We know $u_2' \in I_2$ since otherwise we can add u_2 to I_2 to increase the size of $I_1 \cup I_2$, which is a contradiction with Condition (2.1). We remove w_1 from I_1 and add u_2 to I_1, creating a new red P_2 w_1x_1 which is at distance $k-1$ from v_1v_2 in H. This is a contradiction with the inductive hypothesis. □

We turn our attention to the structures between red P_1 s (of G') in G.

Lemma 2.5. *If three red vertices are joined to the same vertex in $I_1 \cup I_2$ (we call such a configuration C_1), then they form a triangle component by themselves in H (see Fig. 2 left picture).*

Proof. Let u_1, v_1, w_1 be three red vertices in $V(G')$ with their common neighbor $u \in I_1$. We first note that $u_1v_1 \notin E(G)$. Otherwise, say $N(u_1) = \{u, v_1, u_2\}$ and $N(v_1) = \{u, u_1, v_2\}$. If u_2 or v_2 belongs to I_1, then we add u_1 or v_1 to I_2 respectively, which is a contradiction with Condition (2.1). Thus, we assume $u_2, v_2 \in I_2$. Now we move u from I_1 to I_2, and add v_1 to

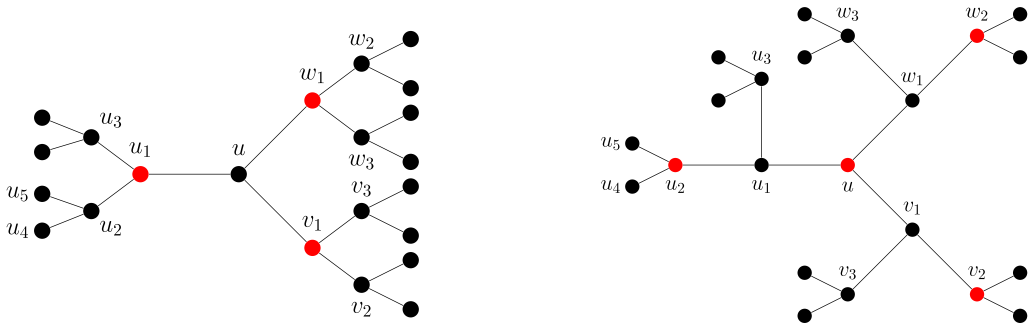

Fig. 2. Configurations C_1 and C_2.

I_1, which again contradicts Condition (2.1). Similarly, $u_1w_1, v_1w_1 \notin E(G)$. Let $N(u_1) = \{u, u_2, u_3\}$, $N(v_1) = \{u, v_2, v_3\}$, and $N(w_1) = \{u, w_2, w_3\}$. We may assume that $|\{u_2, u_3, v_2, v_3, w_2, w_3\}| = 6$. Otherwise, suppose $u_2 = v_2 \in I_1$ and we assign u to I_1, then both u_3, v_3 must be in I_2. However, we delete v_2, u from I_1, add v_1 to I_1, and add u to I_2. This contradicts Condition (2.2). Suppose first that u_2 is red. If $u_3 \in I_1$, then we add u_1 to I_2. If $u_3 \in I_2$, then we move u from I_1 to I_2, and add u_1 to I_1. In each case we obtain contradiction with Condition (2.1). Hence $u_2 \in I_1 \cup I_2$, and by symmetry $u_3, v_2, v_3, w_2, w_3 \in I_1 \cup I_2$.

We may assume $u_2 \in I_1$ and $u_3 \in I_2$ since otherwise, say both $u_2, u_3 \in I_2$, we move u from I_1 to I_2, and add u_1 to I_1, which is a contradiction with Condition (2.1). Similarly, we assume $v_2, w_2 \in I_1$ and $v_3, w_3 \in I_2$. Let $N(u_2) = \{u_1, u_4, u_5\}$. By symmetry, we only need to show $u_4, u_5 \in I_1 \cup I_2$. We first know u_4, u_5 cannot be both in $V(G')$ since otherwise we move u, u_2 from I_1 to I_2, and add u_1 to I_1. This contradicts Condition (2.1). Therefore, we may assume $u_4 \in V(G')$ and $u_5 \in I_2$. However, we move u_2 to $V(G')$, move u to I_2, and add u_1 to I_1. The number of components in G' is decreased by one, which contradicts Condition (2.2). □

Let C_2 be the configuration that three red P_1 s are joining to the same red P_1 via a vertex in $I_1 \cup I_2$ (see Fig. 2 right picture). We show that configuration C_2 does not exist in G.

Lemma 2.6. *Configuration C_2 does not exist in G.*

Proof. Suppose to the contrary that there is a configuration C_2 in G (see Fig. 2 right picture). We have three red P_1 s, u_2, v_2, w_2, joining to the same red P_1, u, via vertices $u_1, v_1, w_1 \in I_1 \cup I_2$ respectively. By Lemma 2.5, $u_3, v_3, w_3 \in I_1 \cup I_2$. If all of $u_1, v_1, w_1 \in I_1$ (or I_2), then we can add u to I_2 and the size of $I_1 \cup I_2$ is increased, which is a contradiction with Condition (2.1). Therefore, we may assume $u_1 \in I_1$ and $v_1, w_1 \in I_2$. However, we move u_1 to $V(G')$ and add u to I_1, which is a contradiction with Condition (2.2) as the number of components in G' is decreased by one. □

We show the maximum degree of H is bounded and prove some additional properties of H.

Lemma 2.7. $\Delta(H) \leq 3$. *Furthermore, if u_1 is a 3-vertex in H, then it must belong to a red P_2 (cannot be a red P_1), say u_1u_2, in G and u_2 must be a 1-vertex in H.*

Proof. Let $u_1 \in V(G')$. By Corollary 2.2, u_1 is either an endpoint of a red P_2 or a P_1 by itself.

In the former case, say this red P_2 is u_1u_2. Let $N(u_1) = \{u_2, u_3, u_4\}$ and $N(u_2) = \{u_1, u_5, u_6\}$. Note that $|\{u_3, u_4, u_5, u_6\}| = 4$. Otherwise, say $u_3 = u_5 \in I_1$ and u_1 has degree three in H. Let $N(u_3) = \{u_1, u_2, u_7\}$. Then $u_4, u_7 \in I_2$, since otherwise we can either add u_1 to I_2 or reassign u_3 to I_2 and add u_1 to I_1, which is a contradiction with Condition (2.1). However, this implies all three neighbors of u_4 must be in $V(G')$ and it is a contradiction with Lemma 2.5. By Lemma 2.1, $u_3, u_4, u_5, u_6 \in I_1 \cup I_2$. Let $N(u_3) = \{u_1, u_7, u_8\}$, $N(u_4) = \{u_1, u_9, u_{10}\}$, $N(u_5) = \{u_2, u_{11}, u_{12}\}$, $N(u_6) = \{u_2, u_{13}, u_{14}\}$. Note that the following proof still works if some of $u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}$ are the same vertex. By Lemma 2.5, u_7 and u_8 cannot be both in G', and u_9 and u_{10} cannot be both in G'. Thus, u_1 has degree at most 3 in H. We may assume that $u_3 \in I_1$ and $u_4 \in I_2$ since otherwise, say $u_3, u_4 \in I_1$, we can add u_1 to I_2 and it contradicts Condition (2.1). Similarly, we may assume $u_5 \in I_1$ and $u_6 \in I_2$.

Claim 2.8. *If $u_7 \in V(G')$, then both u_{13} and u_{14} must be in $I_1 \cup I_2$. By symmetry, if $u_9 \in V(G')$, then both u_{11} and u_{12} must be in $I_1 \cup I_2$.*

Proof. Suppose to the contrary that $u_{13} \in V(G')$. By Lemma 2.3, each of u_7 and u_{13} is a red P_1. By Lemma 2.5, $u_{14} \in I_1$. We move u_3 and u_6 to $V(G')$, and add u_1 to I_1 and u_2 to I_2. This is a contradiction with Condition (2.2) since the number of components in G' is dropped by one. □

Suppose now that u_1 is a 3-vertex in H. It follows that $|\{u_7, u_8\} \cap V(G')| = |\{u_9, u_{10}\} \cap V(G')| = 1$. Assume, without loss of generality, that $u_7, u_9 \in V(G')$. Claim 2.8 implies that u_2 has degree 1 in H.

In the latter case, say $N(u_1) = \{u_2, u_3, u_4\}$ with $u_2, u_3, u_4 \in I_1 \cup I_2$. Let $N(u_2) = \{u_1, u_5, u_6\}$, $N(u_3) = \{u_1, u_7, u_8\}$, and $N(u_4) = \{u_1, u_9, u_{10}\}$. By Lemma 2.5, $|\{u_5, u_6, u_7, u_8, u_9, u_{10}\} \cap V(G')| \leq 3$ and thus u_1 has degree at most three in H. If u_1 has degree three, then Lemma 2.5 implies $|\{u_5, u_6\} \cap V(G')| = |\{u_7, u_8\} \cap V(G')| = |\{u_9, u_{10}\} \cap V(G')| = 1$. Assume, without loss of generality, that $u_5, u_7, u_9 \in V(G')$. We then have a configuration C_2 and it contradicts Lemma 2.6. □

The proof of Claim 2.8 actually implies the following lemma.

Lemma 2.9. *Let $N(u_1) = \{u_2, u_3, u_4\}$ and $N(u_2) = \{u_1, u_5, u_6\}$. If u_1u_2 is a red P_2 with $u_3, u_5 \in I_1$ and $u_4, u_6 \in I_2$, then it is impossible to have two distinct red P_1 s so that one is in $N(u_3) \setminus \{u_1\}$ and the other is in $N(u_6) \setminus \{u_2\}$.* □

Lemma 2.10. *No component of H is isomorphic to a cycle with one vertex adjacent to a leaf.*

Proof. Suppose not, i.e., H has a cycle with one vertex adjacent to a leaf. Let the cycle be $u_1u_2 \ldots u_ku_1$ and let x_1 be the leaf adjacent to u_1, where $k \geq 3$. By Lemma 2.7, x_1u_1 is a red P_2 in G. Since there are no other red P_2 s in the cycle by Lemma 2.3, each u_i with $2 \leq i \leq k$ is a red P_1 in G. Now the definition of H implies that there is a cycle, say $u_1w_1u_2w_2 \cdots u_kw_ku_1$, in G. Let $N(u_i) = \{w_{i-1}, w_i, x_i\}$ and $N(w_i) = \{u_i, u_{i+1}, y_i\}$, where $1 \leq i \leq k$. By Lemma 2.3 and 2.5, each x_i with $i \neq 1$ and each y_i are in $I_1 \cup I_2$. We may assume $w_1 \in I_1$ and $w_k \in I_2$ since otherwise we can add u_1 to I_1 or I_2, which contradicts Condition (2.1). It follows that there is an i with $2 \leq i \leq k$ such that $w_{i-1} \in I_1$ and $w_i \in I_2$. If $x_i \in I_1$, then let $I_2 := I_2 \cup \{u_i\} \setminus \{w_i\}$. If $x_i \in I_2$, then let $I_1 := I_1 \cup \{u_i\} \setminus \{w_{i-1}\}$. In each case we obtain a contradiction with Condition (2.2). □

By Lemmas 2.7 and 2.10, we conclude that each component of H is a tree or an even cycle or an odd cycle. Clearly, H is 3-colorable. Let h be a proper 3-coloring of H using colors A, B, C such that each tree and even cycle component of H is colored using colors A and B, and

(i) the color C is used exactly once on each odd cycle component of H, and more precisely,

(ii) if u_1u_2 is an edge of an odd cycle component such that $u_1, u_2 \in V(G')$, then we arbitrarily choose a vertex from u_1 and u_2, and color its other neighbor on the cycle with C.

We now complete the proof of Theorem 1.2. Since each vertex of G is either in $I_1 \cup I_2$ or colored with A, B, or C, we construct a coloring f of G by assigning vertices in I_1 with color 1_a, vertices in I_2 with color 1_b, vertices colored by A with color 2_a, vertices colored by B with color 2_b, and vertices colored by C with color 3. Since I_1 and I_2 are independent sets in G, vertices with color 1_a or 1_b forms a 1-independent set respectively. By the definitions of H and the colorings h, vertices with color 2_a or 2_b forms a independent set in H and therefore forms a 2-independent set in G respectively. At last, it is sufficient to show that vertices with color 3 forms a 3-independent set, and therefore, f is a packing $(1, 1, 2, 2, 3)$-coloring of G.

Suppose not, i.e., there are two vertices u, v with $f(u) = f(v) = 3$ and $d_G(u, v) \leq 3$ (denoted by a 3-3 *conflict*). By the coloring assignment rules of h and f, u and v are in different components of H. Let S_1, S_2 be two components of H such that $u \in S_1$ and $v \in S_2$. Since the color 3 (C) is only used on the odd cycle components of H, we may assume that S_1 and S_2 are odd cycles. Let the cycle S_1 be $u_1u_2 \ldots u_ku_1$, $k \geq 3$. We discuss different cases where a 3-3 conflict can occur.

Case 1: The corresponding cycle of S_1 in G has a red P_2, say, u_1u_2.

The cycle S_1 of H corresponds to a cycle of G, say, $u_1u_2w_2 \cdots u_kw_ku_1$. Let $N(u_1) = \{u_2, w_k, x_1\}$, $N(u_2) = \{u_1, w_2, x_2\}$, $N(u_i) = \{w_{i-1}, w_i, x_i\}$ for $3 \leq i \leq k$, and $N(w_i) = \{u_i, u_{i+1}, y_i\}$ for $2 \leq i \leq k$. By Corollary 2.2, Lemma 2.3 and 2.5, each x_i, y_i, and w_i, where $1 \leq i \leq k$, is in $I_1 \cup I_2$. By the rule **(ii)** of h and by the definition of f, we may assume $f(u_3) = 3$, i.e., $u := u_3$.

By Condition (2.1), we assume $w_k \in I_1$ and $x_1 \in I_2$. If $k = 3$, then we claim $w_2 \in I_1$, and consequently (by Condition (2.1)) $x_2 \in I_2$ and $x_3 \in I_2$. Suppose not, i.e., $w_2 \in I_2$. Assume, without loss of generality, that $x_3 \in I_2$. Now we can reassign $I_1 := I_1 \cup \{u_1, u_3\} \setminus \{w_3\}$ and $I_2 := I_2$. This is a contradiction with Condition (2.1). If $k \geq 5$, then by Lemma 2.9 we have $w_2 \in I_1$ and $x_2 \in I_2$.

We claim for every $2 \leq i \leq k$, $w_i \in I_1, x_i, y_i \in I_2$, and the cycle $u_1u_2w_2 \cdots u_kw_ku_1$ has no chord. This is true for $k = 3$ and thus we assume $k \geq 5$. Suppose that there is an i with $w_{i-1} \in I_1$ and $w_i \in I_2$, where $3 \leq i \leq k-2$. If $x_i \in I_1$, then we reassign $I_1 := I_1$ and $I_2 := I_2 \cup \{u_i\} \setminus \{w_i\}$. If $x_i \in I_2$, then we reassign $I_1 := I_1 \cup \{u_i\} \setminus \{w_{i-1}\}$ and $I_2 := I_2$. In either case we obtain a contradiction with Condition (2.2), since the number of components in H is dropped by one. Hence, $w_i \in I_1$ for every $2 \leq i \leq k$. It follows that $x_i, y_i \in I_2$ for every $2 \leq i \leq k$, and there is no chord in the cycle.

Claim 2.11. *$N(x_i) \setminus \{u_i\} \subseteq I_1$ and $N(y_i) \setminus \{w_i\} \subseteq I_1$, where $1 \leq i \leq k$.*

Proof. Since S_1 is a component of H, for each $1 \leq i \leq k$, vertices in $N(x_i) \setminus \{u_i\}$ cannot be red. Thus, $N(x_i) \setminus \{u_i\} \subseteq I_1$ since each x_i, where $1 \leq i \leq k$, is in I_2. For each w_iu_{i+1} with $2 \leq i \leq k$, it can become a red P_2 via reassigning $I_1 := I_1 \cup \{u_2, \ldots, u_i\} \setminus \{w_2, \ldots, w_i\}$ and $I_2 := I_2$. Note that this switch operation does not violate Conditions (2.1) and (2.2), but after this operation we can apply Lemma 2.9 to show that $N(y_i) \setminus \{w_i\} \subseteq I_1$. □

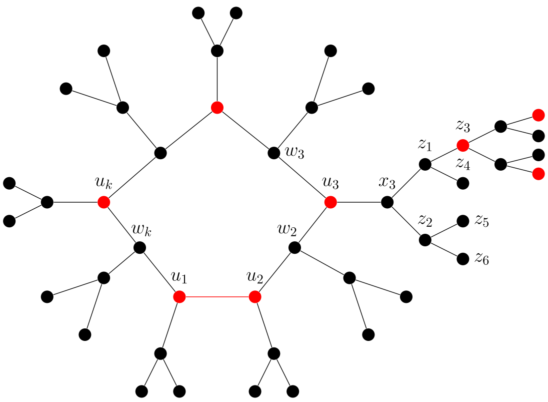

Fig. 3. Case 1 and Case 2.

Now we lock the position of the vertex v. Recall that v and u (actually u_3) form a 3-3 conflict. Let $N(x_3) = \{u_3, z_1, z_2\}$, $N(z_1) = \{x_3, z_3, z_4\}$, and $N(z_2) = \{x_3, z_5, z_6\}$. Since $d_G(u_3, v) \le 3$, $v \in \{z_3, z_4, z_5, z_6\}$ by Claim 2.11. Without loss of generality, assume $v = z_3$. Next we show at most two of z_3, z_4, z_5, z_6 are in G'.

Claim 2.12. *At most two of z_3, z_4, z_5, z_6 are in G'.*

Proof. If $\{z_3, z_4, z_5, z_6\} \subseteq V(G')$, then we reassign $I_1 := I_1 \cup \{x_3\} \setminus \{z_1, z_2\}$ and $I_2 := I_2 \cup \{z_1, z_2, u_3\} \setminus \{x_3\}$, which contradicts Condition (2.1). If three of z_3, z_4, z_5, z_6 are in G', say, $z_3, z_4, z_5 \in G'$ and $z_6 \in I_2$, then we reassign $I_1 := I_1 \cup \{x_3\} \setminus \{z_1, z_2\}$ and $I_2 := I_2 \cup \{z_1, u_3\} \setminus \{x_3\}$. This drops the number of compoments in H by one and thus violates Condition (2.2). Hence, $|\{z_3, z_4, z_5, z_6\} \cap V(G')| \le 2$. $\square$

According to Claim 2.12, we only need to consider two cases up to symmetry, i.e., $z_3, z_4 \in G'$ and $z_5, z_6 \in I_2$, or $z_3, z_5 \in G'$ and $z_4, z_6 \in I_2$.

If $z_3, z_4 \in G'$ and $z_5, z_6 \in I_2$, then we claim that z_3z_4 itself is a component of H, which implies $v \ne z_3$, a contradiction. We reassign $I_1 := I_1$ and $I_2 := I_2 \cup \{u_3\} \setminus \{x_3\}$. This operation adds x_3 to G', and does not violate Conditions (2.1) and (2.2). Now applying Lemma 2.5 we come to the required conclusion.

Suppose now that $z_3, z_5 \in G'$ and $z_4, z_6 \in I_2$. Since z_3 (v) is a vertex of S_2 and it is not part of a P_2 in G' according to rule **(ii)** of the coloring h, $N(z_3) \setminus \{z_1\} \subseteq I_1 \cup I_2$, and furthermore, each vertex of $N(z_3) \setminus \{z_1\}$ has a neighbor, besides z_3, in G' (see Fig. 3 left picture). In this case, we reassign $I_1 := I_1$ and $I_2 := I_2 \cup \{u_3\} \setminus \{x_3\}$. This operation adds x_3 to G', and does not violate Conditions (2.1) and (2.2), but forms a configuration C_2, contradicting Lemma 2.6.

Case 2: The corresponding cycles of both S_1 and S_2 in G have no red P_2.

The cycle S_1 of H corresponds to a cycle $u_1w_1u_2w_2 \dots u_kw_ku_1$ in G, where $u_i \in V(G')$ for $1 \le i \le k$. For $1 \le i \le k$, let $N(u_i) = \{w_i, w_{i-1}, x_i\}$ and $N(w_i) = \{u_i, u_{i+1}, y_i\}$. We assume $f(u_3) = 3$, i.e., $u = u_3$, and $w_1 \in I_1$.

We claim for every $1 \le i \le k$, $w_i \in I_1$ and $x_i, y_i \in I_2$, and there is no chord in the cycle $u_1w_1u_2w_2 \dots u_kw_ku_1$. Suppose not, i.e., there is an i with $w_{i-1} \in I_1$ and $w_i \in I_2$, where $2 \le i \le k$. If $x_i \in I_1$, then we switch u_i, w_i between in I_2 and in G'. This is a contradiction with Condition (2.2). If $x_i \in I_2$, then we switch u_i, w_{i-1} between in I_1 and in G'. This is again a contradiction with Condition (2.2). Therefore, $w_i \in I_1$ and $y_i \in I_2$ for each $1 \le i \le k$. Furthermore, $x_i \in I_2$, where $1 \le i \le k$, since otherwise we add u_i to I_2, which violates Condition (2.1). By Lemma 2.5, the cycle $u_1w_1u_2w_2 \dots u_kw_ku_1$ is chordless.

Claim 2.13. *$N(x_i) \setminus \{u_i\} \subseteq I_1$ and $N(y_i) \setminus \{w_i\} \subseteq I_1$, where $1 \le i \le k$.*

Proof. By Lemma 2.6, for each $1 \le i \le k$ we have $N(x_i) \setminus \{u_i\} \subseteq I_1$. Moreover, we reassign $I_1 := I_1 \cup \{u_1, \dots, u_k\} \setminus \{w_1, \dots, w_k\}$ and $I_2 := I_2$. This operation does not violate Conditions (2.1) and (2.2), but now we can apply Lemma 2.6 to conclude that $N(y_i) \setminus \{w_i\} \subseteq I_1$ for each $1 \le i \le k$. $\square$

Recall that v and u (actually u_3) form a 3-3 conflict. Let $N(x_3) = \{u_3, z_1, z_2\}$, $N(z_1) = \{x_3, z_3, z_4\}$, and $N(z_2) = \{x_3, z_5, z_6\}$ (see Fig. 3 right picture). Since $d_G(u_3, v) \le 3$, $v \in \{z_3, z_4, z_5, z_6\}$ by Claim 2.13. Applying almost the same proof with Case 1 starting from Claim 2.12, we complete the proof.

3. Concluding remarks

The study of packing (1, 1, 2, 2)-coloring of subcubic graphs is a crucial approach to prove Conjecture 1.1. We observe that a packing $(1, \dots, 1, 2, \dots, 2)$-coloring can be viewed as an intermediate coloring between a proper coloring (packing

$(1, \ldots, 1)$-coloring) and a square coloring (packing $(2, \ldots, 2)$-coloring). Methods used in proving results of proper coloring and square coloring (e.g., see [10,12,24]) can be useful.

We feel one might approach the problem "every subcubic graph except the Petersen graph has a packing $(1, 1, 2, 2)$-coloring" by providing a more detailed analysis on the odd cycle components. Furthermore, adding a condition regarding the odd cycles in H, such as "the number of odd cycles in H is minimized", maybe helpful. We conclude this paper by posting the following conjecture.

Conjecture 3.1. *Every subcubic graph except the Petersen graph has a packing* $(1, 1, 2, 2)$*-coloring.*

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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