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# Conflict-free Incidence Coloring of Outer-1-planar Graphs

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**Abstract** An incidence of a graph $G$ is a vertex-edge pair $(v, e)$ such that $v$ is incidence with $e$. A conflict-free incidence coloring of a graph is a coloring of the incidences in such a way that two incidences $(u, e)$ and $(v, f)$ get distinct colors if and only if they conflict each other, i.e., (i) $u = v$, (ii) $uv$ is $e$ or $f$, or (iii) there is a vertex $w$ such that $uw = e$ and $vw = f$. The minimum number of colors used among all conflict-free incidence colorings of a graph is the conflict-free incidence chromatic number. A graph is outer-1-planar if it can be drawn in the plane so that vertices are on the outer-boundary and each edge is crossed at most once. In this paper, we show that the conflict-free incidence chromatic number of an outer-1-planar graph with maximum degree $\Delta$ is either $2\Delta$ or $2\Delta + 1$ unless the graph is a cycle on three vertices, and moreover, all outer-1-planar graphs with conflict-free incidence chromatic number $2\Delta$ or $2\Delta + 1$ are completely characterized. An efficient algorithm for constructing an optimal conflict-free incidence coloring of a connected outer-1-planar graph is given.

**Keywords** outer-1-planar graph; incidence coloring; combinatorial algorithm; channel assignment problem
**2020 MR Subject Classification** 05C10; 05C15

## 1 Introduction

For groups of geographically separated people who need to keep in continuous voice communication, such as aircraft pilots and air traffic controllers, two-way radios are widely used. This motivates us to investigate how to design a two-way radio network efficiently and economically.

In a two-way radio network, each node represents a two-way radio that can both transmit and receive radio waves and there is a link between two nodes if and only if they may contact each other. Waves can transmit between two linked two-way radios in two different directions simultaneously. For a link $L$ connecting two nodes $N_i$ and $N_j$ in a two-way radio network, it is usually assigned with two channels $C(N_i, N_j)$ and $C(N_j, N_i)$. The former one is used to transmit waves from $N_i$ to $N_j$ and the later one is used to transmit waves from $N_j$ to $N_i$. The *associated channel box* $B(N_i)$ of a node $N_i$ in a two-way radio network is a multiset of channels $C(N_i, N_j)$ and $C(N_j, N_i)$ such that $N_i$ is linked to $N_j$. An efficient way to avoid possible interference is to assign channels to links so that every radio receives a rainbow associated channel box (in other words, every two channels in $B(N_i)$ for every node $N_i$ in the network are apart). For the sake of economy, while assigning channels to a two-way radio network, the fewer channels are used, the better. This can be modeled by the conflict-free incidence coloring of graphs.

From now on, we use the language of graph theory and then define conflict-free incidence coloring. We consider finite graphs and use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph $G$. The *degree* $d_G(v)$ of a vertex $v$ in a graph $G$ is the number of edges incident with $v$ in $G$. We use $d(v)$ instead of $d_G(v)$ whenever the graph $G$ is clear from the content. We call $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$ and $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ the

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*maximum degree* and the *minimum degree* of a graph $G$. Other undefined notation is referred to [4].

Let $v$ be a vertex of $G$ and $e$ be an edge incident with $v$. We call the vertex-edge pair $(v,e)$ an *incidence* of $G$. For an edge $e=uv\in E(G)$, let $\mathrm{Inc}(e)=\{(u,e),(v,e)\}$, and for a vertex $v\in V(G)$, let $\mathrm{Inc}(v)=\cup_{e\ni v}\mathrm{Inc}(e)$. For a subset $U\subseteq E(G)$, let $\mathrm{Inc}(U)=\{\mathrm{Inc}(e)\mid e\in U\}$. Two incidences $(u,e)$ and $(v,f)$ are *conflicting* if (i) $u=v$, (ii) $uv$ is $e$ or $f$, or (iii) there is a vertex $w$ such that $uw=e$ and $vw=f$. In other words, two incidences are conflicting if and only if there is a vertex $w$ such that both of them belong to $\mathrm{Inc}(w)$.

A *conflict-free incidence $k$-coloring* of a graph $G$ is a coloring of the incidences using $k$ colors in such a way that every two conflicting incidences get distinct colors. The minimum integer $k$ such that $G$ has a conflict-free incidence $k$-colorable is the *conflict-free incidence chromatic number* of $G$, denoted by $\chi_i^c(G)$. For a conflict-free incidence coloring $\varphi$ of a graph $G$ and an edge $e=uv\in E(G)$, we use $\varphi\big(\mathrm{Inc}(e)\big)$ to denote the set $\{\varphi(u,e),\varphi(v,e)\}$. For a subset $U\subseteq E(G)$, let $\varphi\big(\mathrm{Inc}(U)\big)=\{\varphi\big(\mathrm{Inc}(e)\big)\mid e\in U\}$.

We look back into the channel assignment problem of two-way radio networks and explain why the conflict-free incidence coloring of graphs can model it. Let $G$ be the graph representing the two-way radio network and let $L=N_iN_j$ be an arbitrary link, i.e, $L\in E(G)$. Assigning two channels $C(N_i,N_j)$ and $C(N_j,N_i)$ to $L$ is now equivalent to coloring the incidences $(N_i,L)$ and $(N_j,L)$. The goal of assigning every radio $N_i$ a rainbow associated channel box is translated to coloring the incidences of $G$ so that every two incidences in $\mathrm{Inc}(N_i)$ receive distinct colors. This is exactly what we shall do while constructing a conflict-free incidence coloring of $G$.

From a theoretical point of view, one may be interested in a fact that the conflict-free incidence coloring relates to the *$b$-fold edge-coloring*, which is an assignment of sets of size $b$ to edges of a graph so that adjacent edges receive disjoint sets. An *$(a:b)$-edge-coloring* is a $b$-fold edge coloring out of $a$ available colors. The *$b$-fold chromatic index* $\chi'_b(G)$ is the least integer $a$ such that an $(a:b)$-edge-coloring of $G$ exists. It is not hard to check that $\chi_i^c(G)=\chi'_2(G)$ for every graph $G$. However, there are hard problems related to $\chi'_2(G)$, among which the most famous one is the Berge-Fulkerson conjecture[9], which states that every bridgeless cubic graph has a collection of six perfect matchings that together cover every edge exactly twice. This is equivalent to conjecture that every bridgeless cubic graph $G$ has a $(6:2)$-edge-coloring, i.e., $\chi'_2(G)\le 6$. This conjecture is still widely open[8, 10, 13, 15] and was generalized by Seymour[18] to $\gamma$-graphs.

The structure of this paper organizes as follows. In Section 2, we establish fundamental results for the conflict-free incidence chromatic number of graphs. In Section 3, we investigate the conflict-free incidence coloring of outer-1-planar graphs by showing that $2\Delta\le\chi_i^c(G)\le 2\Delta+1$ for outer-1-planar graphs $G$ with maximum degree $\Delta$ unless $G\cong C_3$, and moreover, characterizing outer-1-planar graphs $G$ with $\chi_i^c(G)$ equal to $2\Delta$ or $2\Delta+1$. An efficient algorithm for constructing an optimal conflict-free incidence coloring of a connected outer-1-planar graph is also given. We end this paper with an interesting open problem relative to the complexity in Section 4.

## 2 Fundamental Results

Let $\chi'(G)$ be the *chromatic index* of $G$, the minimum integer $k$ such that $G$ admits an edge $k$-coloring so that adjacent edges receive distinct colors. The following is an interesting relationship between $\chi_i^c(G)$ and $\chi'(G)$.

**Proposition 2.1.** $2\Delta(G)\le\chi_i^c(G)\le 2\chi'(G)$.

*Proof.* Since $|\mathrm{Inc}(v)|=2\Delta(G)$ for a vertex $v$ with maximum degree, $\chi_i^c(G)\ge 2\Delta(G)$ for every graph $G$. If $\varphi$ is a proper edge coloring of $G$ using the colors $\{1,2,\cdots,\chi'(G)\}$, then one can construct a conflict-free incidence $2\chi'(G)$-coloring of $G$ such that $\varphi(\mathrm{Inc}(e)) = \{\varphi(e), \varphi(e) + \chi'(G)\}$ for every edge $e \in E(G)$. It follows that $\chi_i^c(G) \leq 2\chi'(G)$. □

The well-known Vizing's theorem (see [4, p128]) states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every simple graph $G$. This divides simple graphs into two classes. A simple graph $G$ belongs to *class one* if $\chi'(G) = \Delta(G)$, and belongs to class two if $\chi'(G) = \Delta(G) + 1$. The following are immediate corollaries of Proposition 2.1.

**Algorithm 1. COLOR-CYCLE**($n$)

*This algorithm constructs an optimal conflict-free incidence coloring of $C_n$ in linear time.*

**Input:** The length $n$ of a cycle $C_n$;
**Output:** A conflict-free incidence $\chi_i^c(C_n)$-coloring $\varphi$ of $C_n$.

```
{Vertices of C_n are v_1, v_2, ..., v_n in this ordering.}
if n = 3 then
  φ(Inc(v_1 v_2)) ← {1, 2};
  φ(Inc(v_2 v_3)) ← {3, 4};
  φ(Inc(v_3 v_1)) ← {5, 6};
  return
end if
p ← the quotient of n divided by 2;
r ← the remainder of n divided by 2;
if r = 0 then
  v_{2p+1} ← v_1;
  for i = 1 to 2p do
    if i ≡ 1 (mod 2) then
      φ(Inc(v_i v_{i+1})) ← {1, 2};
    else
      φ(Inc(v_i v_{i+1})) ← {3, 4};
    end if
  end for
else
  for i = 1 to 2p − 2 do
    if i ≡ 1 (mod 2) then
      φ(Inc(v_i v_{i+1})) ← {1, 2};
    else
      φ(Inc(v_i v_{i+1})) ← {3, 4};
    end if
  end for
  φ(Inc(v_{2p−1} v_{2p})) ← {1, 5};
  φ(Inc(v_{2p} v_{2p+1})) ← {2, 3};
  φ(Inc(v_{2p+1} v_1)) ← {4, 5};
end if
```

**Proposition 2.2.** *If $G$ is a class one graph, then $\chi_i^c(G) = 2\Delta(G)$.*

**Proposition 2.3.** *If $G$ is simple graph, then $\chi_i^c(G) \in \{2\Delta(G), 2\Delta(G) + 1, 2\Delta(G) + 2\}$.*

The well-known Kőnig's theorem (see [4, p127]) states that every bipartite graph is of class 1. So the following is immediate by Proposition 2.2.

**Theorem 2.4.** *If $G$ is a bipartite graph, then $\chi_i^c(G) = 2\Delta(G)$.*

Now that we have Proposition 2.2, it would be worth determining the conflict-free incidence chromatic number of a certain class of graphs of class two. We first look into a cycle $C_n$ of length $n$.

**Theorem 2.5.**

$$\chi_i^c(C_n) = \begin{cases} 4, & \text{if } n \text{ is even}, \\ 5, & \text{if } n \geq 5 \text{ is odd}, \\ 6, & \text{if } n = 3. \end{cases} \tag{2.1}$$

*Proof.* One can easily see that $C_n$ admits neither a conflict-free incidence 3-coloring for any integer $n \geq 3$, and nor a conflict-free incidence 4-coloring for any odd $n \geq 3$. Moreover, $C_3$ does not admit a conflict-free incidence 5-coloring. Hence Algorithm 1 outputs a conflict-free incidence coloring of $C_n$ using the least number of colors in linear time and the result follows. □

We now pay attention to the $n$-order complete graph $K_n$. The famous result of Fiorini and Wilson[6] states that $K_n$ is of class 1 provided $n$ is even. Hence Proposition 2.2 directly imply the following.

**Proposition 2.6.** $\chi_i^c(K_{2n}) = 2\Delta(K_{2n}) = 4n - 2$.

Fiorini and Wilson[6] also showed that $K_n$ is of class 2 provided $n$ is odd, and thus Proposition 2.2 cannot be applied to such a $K_n$. Nevertheless, we can determine the conflict-free incidence chromatic number of $K_n$ with $n$ being odd from another view of point.

**Proposition 2.7.** *If $G$ is the graph derived from $K_{2n+1}$ by removing less than $n/2$ edges, then $\chi_i^c(G) = 2\Delta(G) + 2 = 4n + 2$.*

*Proof.* We first show that $\chi_i^c(G) \geq 4n+2$. Suppose for a contradiction that $\varphi$ is a conflict-free incidence $(4n+1)$-coloring of $G$. Since $G$ totally has more than $4n^2 + 2n - n = (4n+1)n$ incidences, there is a color of $\varphi$, say 1, that has been used at least $n+1$ times. Since every two strong incidences of a vertex are differently colored, there are $n+1$ vertices of $G$, say $v_1, v_2, \cdots, v_{n+1}$, such that for each $1 \leq i \leq n+1$, $\varphi(v_i, v_iu_i) = 1$, where $u_i$ is one neighbor of $v_i$. Since every two weak incidences of a vertex are also differently colored, each $u_i$ is different from every $u_j$ with $j \neq i$. If $u_i$ coincides with some $v_j$ with $j \neq i$, then $\varphi(v_i, v_iu_i) = \varphi(u_i, u_iu_j)$, a contradiction as $(v_i, v_iu_i)$ conflicts $(u_i, u_iu_j)$. Hence each $u_i$ is different from every $v_j$ with $j \neq i$. It follows that $V(G) \supseteq \bigcup_{i=1}^{n+1} \{u_i, v_i\}$ and thus $|V(G)| \geq 2n+2$, a contradiction. To show the equality, we apply proposition 2.3 to $G$. It follows that $\chi_i^c(G) \leq 2\Delta(G) + 2 = 4n+2$, as desired. □

Combining Propositions 2.6 and Propositions 2.7 together, we conclude the following.

**Theorem 2.8.**

$$\chi_i^c(K_n) = \begin{cases} 2n-2, & \text{if } n \text{ is even}, \\ 2n, & \text{if } n \text{ is odd}. \end{cases} \tag{2.2}$$

We use the polygon method to construct an optimal conflict-free incidence coloring of $K_n$ by Algorithm 2. To analyze the complexity of the algorithm, we need look into its lines 4 and 13. If $n$ is even, then for each $1 \leq i \leq n-1$, $E_i = \{v_{i-j}v_{i+j} | j = 1, \cdots, \frac{n-2}{2}\} \cup \{v_iv_n\}$ by line 4, where the subscripts are taken module $n$ and $v_0$ is recognized as $v_{n-1}$. If $n$ is odd, then for each $1 \leq i \leq n$, $E_i = \{v_{i-j}v_{i+j+1} | j = 0, 1, \cdots, \frac{n-3}{2}\}$ according to line 13, where the subscripts are taken module $n$ and $v_0$ is recognized as $v_n$. It follows that the complexity of Algorithm 2 is $O((n-1)n/2) = O(n^2)$.

**Algorithm 2. COLOR-COMPLETE-GRAPH**$(n)$

*This algorithm constructs an optimal conflict-free incidence coloring of $K_n$ in quadratic time.*

**Input:** The order $n$ of a complete graph $K_n$;

**Output:** A conflict-free incidence $\chi_i^c(K_n)$-coloring $\varphi$ of $K_n$.

{*Vertices of $K_n$ are $v_1, v_2, \cdots, v_n$.*}

**if** $n \equiv 0 \pmod 2$ **then**

  $G \leftarrow$ an $(n-1)$-sided regular polygon formed by placing $v_1, v_2, \cdots, v_{n-1}$ on a circle, with $v_n$ at the center of the circle, and connecting every pair of vertices by straight line;
  {*$G$ now is a special drawing of $K_n$ in the plane.*}

  **for** $i = 1$ to $n-1$ **do**

    $E_i \leftarrow$ the set of all edges that lie on lines perpendicular to $v_iv_n$ in $G$ along with the edge $v_iv_n$ itself;

    **for** each edge $e \in E_i$ **do**

      $\varphi\big(\mathrm{Inc}(e)\big) \leftarrow \{2i-1, 2i\}$;

    **end for**

  **end for**

**else**

  $G \leftarrow$ an $n$-sided regular polygon formed by placing $v_1, v_2, \cdots, v_n$ on a circle and connecting every pair of vertices by straight line;

  $v_{n+1} \leftarrow v_1$;

  **for** $i = 1$ to $n$ **do**

    $E_i \leftarrow$ the set of all edges that lie on lines parallel to $v_iv_{i+1}$ in $G$ along with the edge $v_iv_{i+1}$ itself;

    **for** each edge $e \in E_i$ **do**

      $\varphi\big(\mathrm{Inc}(e)\big) \leftarrow \{2i-1, 2i\}$;

    **end for**

  **end for**

**end if**

## 3 Outer-1-planar Graphs

In this section we determine the conflict-free incidence chromatic numbers of outer-1-planar graphs, a subclass of planar partial 3-trees[1], which serve many applications ranging from network reliability to machine learning. Formally speaking, a graph is *outer-1-planar* if it can be drawn in the plane so that vertices are on the outer-boundary and each edge is crossed at most once. The notion of outer-1-planarity was first introduced by Eggleton[5] and outer-1-planar graphs are also known as *outerplanar graphs with edge crossing number one*[5] and *pseudo-outerplanar graphs*[19, 21, 26]. The coloring of outer-1-planar graphs were investigated by many authors including [3, 12, 14, 16, 19, 21–26].

The most popular result on the edge coloring of planar graphs is that planar graphs with maximum degree at least 7 is of class one[17, 20]. Since there exist class two planar graphs with maximum degree $\Delta$ for each $\Delta \leq 5$, the remaining problem is to determine whether every planar graph with maximum degree 6 is of class one, and this is still quite open (see survey [2]). Therefore, investigating the edge coloring of subclasses of planar graphs is natural and interesting. Fiorini[7] showed that every outerplanar graph is of class one if and only if it is not an odd cycle, and this conclusion had been generalized to the class of series-parallel graphs by Juvan, Mohar, and Thomas[11]. Zhang, Liu, and Wu[26] showed that outer-1-planar graphs with maximum degree at least 4 are of class one. The chromatic indexes of outer-1-planar graphs with maximum degree at most 3 was completely determined by Zhang[22].

We restate Zhang's definition[22] as follows. Let $G_2, G_4, G_8$, and $H_t$ be configurations defined by Figure 3.1. For any solid vertex $v$ of a configuration and any graph $G$ containing such a configuration, the degree of $v$ in $G$ is exactly the number of edges that are incident with $v$ in the picture.

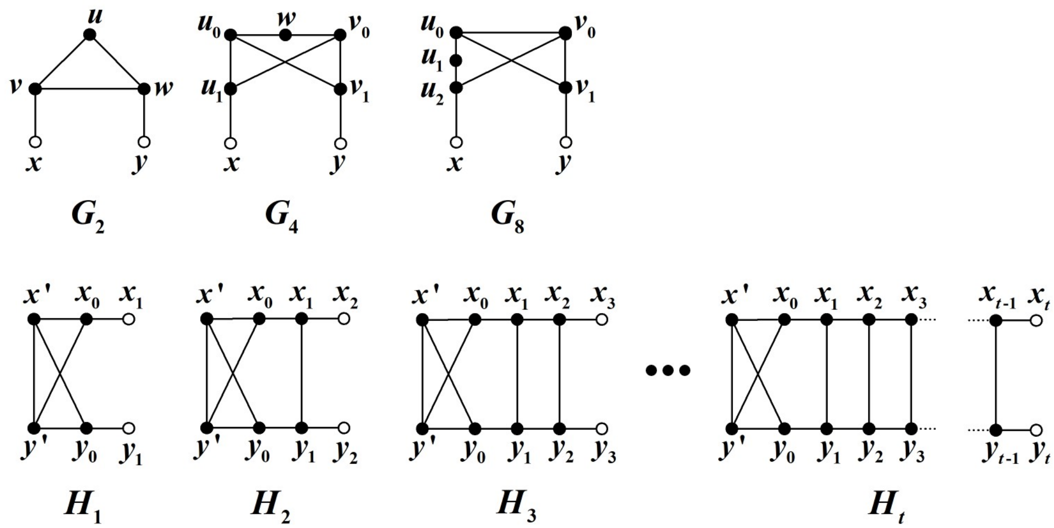


**Figure 3.1.** The configurations $G_2, G_4, G_8$ and $H_t$

A graph belongs to the class $\mathcal{P}$, if it is isomorphic to $K_4^+$ (equal to $K_4$ with one edge subdivided) or derived from a graph $G \in \mathcal{P}$ by one of the following operations:

Let $\mathcal{P}^+$ be the class of connected outer-1-planar graphs with maximum degree 3 that contains some graph in $\mathcal{P}$ as a subgraph. Now we summarize the result of Zhang[22] and Zhang, Liu, and Wu[26] as follows.

**Theorem 3.1.**

$$\chi'(G) = \begin{cases} \Delta(G), & \text{if } G \notin \mathcal{P}^+ \text{ and } G \text{ is not an odd cycle,} \\ \Delta(G) + 1, & \text{otherwise,} \end{cases} \tag{3.1}$$

*if $G$ is a connected outer-1-planar graph.*

**$G \sqcup_z G_t$ with $t = 2, 4, 8$.** remove a vertex $z$ of degree two from $G$, and then paste a copy of $G_2$, or $G_4$, or $G_8$ on the current graph accordingly, by identifying $x$ and $y$ with $z_1$ and $z_2$, respectively, where $z_1$ and $z_2$ are the neighbors of $z$ (see Figure 3.2 for an example);

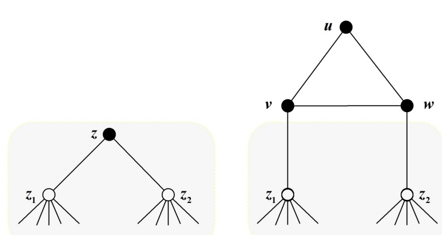


**Figure 3.2.** The graph on the left shows $G$ and the one on the right shows $G \sqcup_z G_2$

$\boldsymbol{G \vee_{z_1 z_2} H_t}$ **with** $\boldsymbol{t \geq 1}$**.** remove an edge $z_1 z_2$ from $G$, and then paste a copy of $H_t$ on the current graph by identifying $x_t$ and $y_t$ with $z_1$ and $z_2$, respectively (see Figure 3.3 for an example).

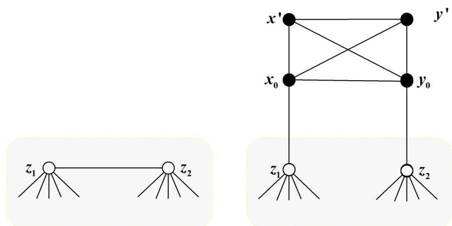


**Figure 3.3.** The graph on the left shows $G$ and the one on the right shows $G \vee_{z_1 z_2} H_1$

**Remark on Theorem 3.1.** Zhang[22] claimed that every connected outer-1-planar graph with maximum degree 3 is of class one if and only if $G \notin \mathcal{P}$. However, this statement is incorrect. Indeed, Zhang showed that every graph in $\mathcal{P}$ is of class two. This further implies that every outer-1-planar graph with maximum degree 3 that contains some graph in $\mathcal{P}$ is of class two. In other words, every graph in $\mathcal{P}^+$ is of class two. Using the same proof of Theorem 3.3 in[22], one can show that if $G$ is a connected outer-1-planar graph with maximum degree 3 not in $\mathcal{P}^+$ then it is of class one (note that the minimal counterexample to this statement is 2-connected and thus Zhang's original proof works now). Conclusively, every connected outer-1-planar graph with maximum degree 3 is of class one if and only if $G \notin \mathcal{P}^+$. Combining this with the result of Zhang, Liu, and Wu[26] that every outer-1-planar graph with maximum degree at least 4 is of class one, we have Theorem 3.1.

The following is an immediate corollary of Theorem 3.1 and Proposition 2.1.

**Theorem 3.2.** *If $G$ is a connected outer-1-planar graph such that $G \notin \mathcal{P}^+$ and $G$ is not an odd cycle, then $\chi_i^c(G) = 2\Delta(G)$.*

The next goal of this section is to prove $\chi_i^c(G) = 2\Delta(G) + 1$ if $G \in \mathcal{P}^+$ or $G$ is an odd cycle unless $G \cong C_3$. Theorem 2.5 supposes this conclusion while $G$ is an odd cycle of length at least 5. Hence in the following we assume that $G \in \mathcal{P}^+$. Note that $K_4^+$ is the smallest graph (in terms of the order) in $\mathcal{P}^+$. Now we prove $\chi_i^c(G) = 7$ for every graph $G \in \mathcal{P}^+$ by a series of lemmas.

**Lemma 3.3.** $\chi_i^c(K_4^+) = 7$.

*Proof.* Figure 3.4 shows a conflict-free incidence 7-colorable of $K_4^+$, so it is sufficient to show that 6 colors are not enough to create a conflict-free incidence coloring of $K_4^+$.

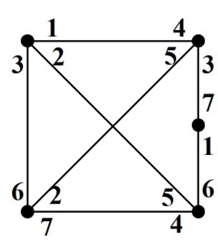


**Figure 3.4.** A conflict-free incidence 7-colorable of $K_4^+$

Suppose for a contradiction that $\varphi$ is a conflict-free incidence 6-coloring of $K_4^+$. Since $K_4^+$ has 7 edges and 14 incidences, there is a color, say 1, such that $\varphi(x_1, x_1x_1') = \varphi(x_2, x_2x_2') =$

$\varphi(x_3, x_3x_3') = 1$. If $x_i = x_j$ or $x_i' = x_j'$ or $x_i = x_j'$ for some $1 \leq i < j \leq 3$, then $(x_i, x_ix_i')$ and $(x_j, x_jx_j')$ are conflicting and thus they cannot in a same color. Hence $|\{x_1, x_2, x_3, x_1', x_2', x_3'\}| = 6$, contradicting the fact that $|K_4^+| = 5$. □

From now on, if we say coloring a graph or a configuration we mean coloring its incidences so that every two conflicting ones receive distinct colors.

**Lemma 3.4.** *If the configuration $G_2$ is colored with 6 colors under $\varphi$, then $\varphi\big(\mathrm{Inc}(vx)\big) \cap \varphi\big(\mathrm{Inc}(wy)\big) = \emptyset$.*

*Proof.* If $\varphi$ is a conflict-free incidence 6-coloring of $G_2$, then $\varphi(u, uv), \varphi(v, uv), \varphi(u, uw), \varphi(w, uw), \varphi(v, vw)$ and $\varphi(w, vw)$ are pairwise distinct, so we assume, without loss of generality, that they are $1, 2, 3, 4, 5$, and 6, respectively. This forces that $\varphi\big(\mathrm{Inc}(vx)\big) = \{3, 4\}$ and $\varphi\big(\mathrm{Inc}(wy)\big) = \{1, 2\}$, as desired. □

**Lemma 3.5.** *If the configuration $G_4$ is colored with 6 colors under $\varphi$, then $\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$.*

*Proof.* If $\varphi$ is a conflict-free incidence 6-coloring of $G_4$, we have three cases: $\varphi\big(\mathrm{Inc}(u_1x)\big) = \varphi\big(\mathrm{Inc}(v_1y)\big)$, or $\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, or $|\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big)| = 1$. If $\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, then we win. So it is sufficient to show contradictions for another two cases. Without loss of generality, we assume $\varphi\big(\mathrm{Inc}(u_1x)\big) = \{1, 2\}$, $\varphi\big(\mathrm{Inc}(u_1v_0)\big) = \{3, 4\}$, and $\varphi\big(\mathrm{Inc}(u_0u_1)\big) = \{5, 6\}$.

**Case 1.** $\varphi\big(\mathrm{Inc}(u_1x)\big) = \varphi\big(\mathrm{Inc}(v_1y)\big)$.

Now $\varphi\big(\mathrm{Inc}(v_1y)\big) \cup \varphi\big(\mathrm{Inc}((u_0u_1)\big) = \{1, 2, 5, 6\}$ and $\varphi\big(\mathrm{Inc}(v_1y)\big) \cup \varphi\big(\mathrm{Inc}(u_1v_0)\big) = \{1, 2, 3, 4\}$ forces $\varphi\big(\mathrm{Inc}(u_0v_1)\big) = \{3, 4\}$ and $\varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{5, 6\}$, respectively. It follows $\varphi\big(\mathrm{Inc}(u_0u_1, u_0v_1)\big) = \varphi\big(\mathrm{Inc}(u_1v_0, v_0v_1)\big) = \{3, 4, 5, 6\}$ and thus $\varphi\big(\mathrm{Inc}(u_0w)\big) = \varphi\big(\mathrm{Inc}(v_0w)\big) = \{1, 2\}$, which is impossible.

**Case 2.** $|\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big)| = 1$.

Assume, by symmetry, that $\varphi\big(\mathrm{Inc}(v_1y)\big) = \{1, a\}$, where $a \in \{3, 4\}$. It follows that $\varphi\big(\mathrm{Inc}(v_1y)\big) \cup \varphi\big(\mathrm{Inc}(u_0u_1)\big) = \{1, a, 5, 6\}$, forcing $\varphi\big(\mathrm{Inc}(u_0v_1)\big) = \{2, b\}$, $b \in \{3, 4\} \setminus \{a\}$. Now $\varphi\big(\mathrm{Inc}(u_0u_1, u_0v_1)\big) = \{2, b, 5, 6\}$ and $\varphi\big(\mathrm{Inc}(v_1y)\big) \cup \varphi\big(\mathrm{Inc}(u_0v_1)\big) = \{1, 2, 3, 4\}$, which implies $\varphi\big(\mathrm{Inc}(u_0w)\big) = \{1, a\}$ and $\varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{5, 6\}$, respectively. It follows that $\varphi\big(\mathrm{Inc}(u_1v_0, v_0v_1, u_0w)\big) = \{1, 3, 4, 5, 6\}$ and thus $\mathrm{Inc}(wv_0)$ have to be colored with 2, which is impossible. □

**Lemma 3.6.** *If the configuration $G_8$ is colored with 6 colors under $\varphi$, then $\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$.*

*Proof.* If $\varphi$ is a conflict-free incidence 6-coloring of $G_8$, we have three cases: $\varphi\big(\mathrm{Inc}(u_2x)\big) = \varphi\big(\mathrm{Inc}(v_1y)\big)$, or $\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, or $|\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big)| = 1$. If $\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, then we win. So it is sufficient to show contradictions for another two cases. Without loss of generality, we assume $\varphi\big(\mathrm{Inc}(v_1y)\big) = \{1, 2\}$, $\varphi\big(\mathrm{Inc}(u_0v_1)\big) = \{3, 4\}$, and $\varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{5, 6\}$.

**Case 1.** $\varphi\big(\mathrm{Inc}(u_2x)\big) = \varphi\big(\mathrm{Inc}(v_1y)\big)$.

Now $\varphi\big(\mathrm{Inc}(u_2x)\big) \cup \varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{1, 2, 5, 6\}$ and $\varphi\big(\mathrm{Inc}(u_0v_1, v_0v_1)\big) = \{3, 4, 5, 6\}$ forces $\varphi\big(\mathrm{Inc}(u_2v_0)\big) = \{3, 4\}$ and $\varphi\big(\mathrm{Inc}(u_0v_0)\big) = \{1, 2\}$, respectively. It follows $\varphi\big(\mathrm{Inc}(u_0v_0, u_0v_1)\big) = \varphi\big(\mathrm{Inc}(u_2x)\big) \cup \varphi\big(\mathrm{Inc}(u_2v_0)\big) = \{1, 2, 3, 4\}$ and thus $\varphi\big(\mathrm{Inc}(u_0u_1)\big) = \varphi\big(\mathrm{Inc}(u_1u_2)\big) = \{5, 6\}$, which is impossible.

**Case 2.** $|\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big)| = 1$.

Assume, by symmetry, that $\varphi\big(\mathrm{Inc}(u_2x)\big) = \{1, a\}$, where $a \in \{5, 6\}$. It follows that $\varphi\big(\mathrm{Inc}(u_0v_1)\big) \cup \varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{3, 4, 5, 6\}$ and $\varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{5, 6\}$, forcing $\varphi\big(\mathrm{Inc}(u_0v_0)\big) = \{1, 2\}$ and $\varphi\big(\mathrm{Inc}(u_2v_0)\big) = \{3, 4\}$. Now $\varphi\big(\mathrm{Inc}(u_0v_0, u_0v_1)\big) = \{1, 2, 3, 4\}$ which implies $\varphi\big(\mathrm{Inc}(u_0u_1)\big) = \{5, 6\}$. It follows that $\varphi\big(\mathrm{Inc}(u_0u_1, u_2x, u_2v_0)\big) = \{1, 3, 4, 5, 6\}$ and thus $\mathrm{Inc}(u_1u_2)$ have to be colored with 2, which is impossible. □

**Lemma 3.7.** *If the configuration $H_t$ with some $t \geq 1$ is colored with 6 colors under $\varphi$, then $\varphi\big(\mathrm{Inc}(x_{t-1}x_t)\big) = \varphi\big(\mathrm{Inc}(y_{t-1}y_t)\big)$.*

*Proof.* We prove it by induction on $t$. If $\varphi$ is a conflict-free incidence 6-coloring of $H_1$, then we assume, without loss of generality, $\varphi(x', x'y'), \varphi(y', x'y'), \varphi(x', x'y_0), \varphi(y_0, x'y_0), \varphi(x', x'x_0)$, and $\varphi(x_0, x'x_0)$ are $1, 2, 3, 4, 5$, and 6, respectively. Since $\varphi\big(\mathrm{Inc}(x'y', x'x_0)\big) = \{1, 2, 5, 6\}$ and $\varphi\big(\mathrm{Inc}(x'y', x'y_0)\big) = \{1, 2, 3, 4\}$, we have $\varphi\big(\mathrm{Inc}(x_0y')\big) = \{3, 4\}$ and $\varphi\big(\mathrm{Inc}(y'y_0)\big) = \{5, 6\}$, which imply $\varphi\big(\mathrm{Inc}(x_0x_1)\big) = \varphi\big(\mathrm{Inc}(y_0y_1)\big) = \{1, 2\}$. This completes the proof of the base case. Now suppose that the lemma holds for $H_{t-1}$ with some $t \geq 2$ and prove that it also holds for $H_t$. By the induction hypothesis, $\varphi\big(\mathrm{Inc}(x_{t-2}x_{t-1})\big) = \varphi\big(\mathrm{Inc}(y_{t-2}y_{t-1})\big)$. This implies $\varphi\big(\mathrm{Inc}(x_{t-1}x_t)\big) = \{1, 2, 3, 4, 5, 6\} \setminus \{\varphi\big(\mathrm{Inc}(x_{t-2}x_{t-1})\big) \cup \varphi\big(\mathrm{Inc}(x_{t-1}y_{t-1})\big)\}$ and $\varphi\big(\mathrm{Inc}(y_{t-1}y_t)\big) = \{1, 2, 3, 4, 5, 6\} \setminus \{\varphi\big(\mathrm{Inc}(y_{t-2}y_{t-1})\big) \cup \varphi\big(\mathrm{Inc}(x_{t-1}y_{t-1})\big)\}$, and thus $\varphi\big(\mathrm{Inc}(x_{t-1}x_t)\big) = \varphi\big(\mathrm{Inc}(y_{t-1}y_t)\big)$, as desired. □

**Lemma 3.8.** *If $\varphi$ is a partial incidence coloring of the configuration $G_2$ such that $\varphi\big(\mathrm{Inc}(vx)\big) \cap \varphi\big(\mathrm{Inc}(wy)\big) = \emptyset$, then $\varphi$ can be extended to a conflict-free incidence 6-coloring of the configuration $G_2$.*

*Proof.* Suppose $\varphi\big(\mathrm{Inc}(vx)\big) = \{1, 2\}$ and $\varphi\big(\mathrm{Inc}(wy)\big) = \{3, 4\}$. It is easy to see that we can extend $\varphi$ to a conflict-free incidence 6-coloring of $G_2$ by coloring $\mathrm{Inc}(uv, uw, vw)$ so that $\varphi\big(\mathrm{Inc}(uv)\big) = \{3, 4\}$, $\varphi\big(\mathrm{Inc}(uw)\big) = \{1, 2\}$, and $\varphi\big(\mathrm{Inc}(vw)\big) = \{5, 6\}$. □

**Lemma 3.9.** *If $\varphi$ is a partial incidence coloring of the configuration $G_4$ such that $\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, then $\varphi$ can be extended to a conflict-free incidence 6-coloring of the configuration $G_4$.*

*Proof.* Suppose $\varphi\big(\mathrm{Inc}(u_1x)\big) = \{1, 2\}$ and $\varphi\big(\mathrm{Inc}(v_1y)\big) = \{3, 4\}$. It is easy to see that we can extend $\varphi$ to a conflict-free incidence 6-coloring of $G_4$ by coloring $\mathrm{Inc}(u_0v_1, v_0w, u_0w, u_1v_0, u_0u_1, v_0v_1)$ so that $\varphi\big(\mathrm{Inc}(u_0v_1)\big) = \varphi\big(\mathrm{Inc}(v_0w)\big) = \{1, 2\}$, $\varphi\big(\mathrm{Inc}(u_0w)\big) = \varphi\big(\mathrm{Inc}(u_1v_0)\big) = \{3, 4\}$, and $\varphi\big(\mathrm{Inc}(u_0u_1)\big) = \varphi\big(\mathrm{Inc}(v_0v_1)\big) = \{5, 6\}$. □

**Lemma 3.10.** *If $\varphi$ is a partial incidence coloring of the configuration $G_8$ such that $\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, then $\varphi$ can be extended to a conflict-free incidence 6-coloring of the configuration $G_8$.*

*Proof.* Suppose $\varphi\big(\mathrm{Inc}(u_2x)\big) = \{1, 2\}$ and $\varphi\big(\mathrm{Inc}(v_1y)\big) = \{3, 4\}$. We can extend $\varphi$ to a conflict-free incidence 6-coloring of $G_4$ by coloring the incidences on $v_0v_1, u_0u_1, u_0v_0, u_1u_2, u_0v_1$, and $u_2v_0$ so that $\varphi\big(\mathrm{Inc}(v_0v_1)\big) = \varphi\big(\mathrm{Inc}(u_0u_1)\big) = \{1, 2\}$, $\varphi\big(\mathrm{Inc}(u_0v_0)\big) = \varphi\big(\mathrm{Inc}(u_1u_2)\big) = \{3, 4\}$, and $\varphi\big(\mathrm{Inc}(u_0v_1)\big) = \varphi\big(\mathrm{Inc}(u_2v_0)\big) = \{5, 6\}$.

□

**Lemma 3.11.** *If $\varphi$ is a partial incidence coloring of the configuration $H_t$ with some $t \geq 1$ such that $\varphi\big(\mathrm{Inc}(x_{t-1}x_t)\big) = \varphi\big(\mathrm{Inc}(y_{t-1}y_t)\big)$, then $\varphi$ can be extended to a conflict-free incidence 6-coloring of the configuration $H_t$.*

*Proof.* We prove it by induction on $t$. If $\varphi$ is a partial incidence coloring of the configuration $H_1$ such that $\varphi\big(\mathrm{Inc}(x_0x_1)\big) = \varphi\big(\mathrm{Inc}(y_0y_1)\big) = \{1,2\}$, then $\varphi$ can be extended to a conflict-free incidence 6-coloring of $H_t$ by coloring $\mathrm{Inc}(x'y', x'y_0, x_0y', x'x_0, y'y_0)$ so that $\varphi\big(\mathrm{Inc}(x'y')\big) = \{1,2\}$, $\varphi\big(\mathrm{Inc}(x'y_0)\big) = \varphi\big(\mathrm{Inc}(x_0y')\big) = \{3,4\}$, and $\varphi\big(\mathrm{Inc}(x'x_0)\big) = \varphi\big(\mathrm{Inc}(y'y_0)\big) = \{5,6\}$. This completes the proof of the base case. Now suppose that the lemma holds for $H_{t-1}$ with some $t \geq 2$ and prove that it also holds for $H_t$. Assume, without loss of generality, that $\varphi\big(\mathrm{Inc}(x_{t-1}x_t)\big) = \varphi\big(\mathrm{Inc}(y_{t-1}y_t)\big) = \{1,2\}$. We extend $\varphi$ by coloring $\mathrm{Inc}(x_{t-2}x_{t-1}, y_{t-2}y_{t-1}, x_{t-1}y_{t-1})$ so that $\varphi\big(\mathrm{Inc}(x_{t-2}x_{t-1})\big) = \varphi\big(\mathrm{Inc}(y_{t-2}y_{t-1})\big) = \{3,4\}$ and $\varphi\big(\mathrm{Inc}(x_{t-1}y_{t-1})\big) = \{5,6\}$. This constructs a partial incidence coloring of the configuration $H_{t-1} = H_t - \{x_{t-1}y_{t-1}, x_{t-1}x_t, y_{t-1}y_t\}$ such that $\varphi\big(\mathrm{Inc}(x_{t-2}x_{t-1})\big) = \varphi\big(\mathrm{Inc}(y_{t-2}y_{t-1})\big)$. Since any incidence of $\mathrm{Inc}(x_{t-1}y_{t-1}, x_{t-1}x_t, y_{t-1}y_t)$ is conflict-free to any incidence of $\mathrm{Inc}(H_{t-1})$, by the induction hypothesis, the extended $\varphi$ can be further extended to a conflict-free incidence 6-coloring of the configuration $H_t$. □

**Proposition 3.12.** *If $G \in \mathcal{P}$, then $\chi_i^c(G) = 7$.*

*Proof.* We proceed by induction on $|G|$. Since the smallest graph in $\mathcal{P}$ is $K_4^+$, and $\chi_i^c(K_4^+) = 7$ by Lemma 3.3, the proof of the base case has been done. Now assume $|G| > 5$. By the construction of $\mathcal{P}$, we meet four cases. Here and elsewhere, once $G$ contains a configuration as shown in Figure 3.1, we use the same labelling of any vertex appearing on the configuration as the one marked in the corresponding picture.

**Case 1.** There is a graph $G' \in \mathcal{P}$ and a degree 2 vertex $z$ of $G'$ such that $G = G' \sqcup_z G_2$ (or $G = G' \sqcup_z G_4$, or $G = G' \sqcup_z G_8$, respectively).

By the induction hypothesis, $\chi_i^c(G') = 7$. Let $z_1, z_2$ be two neighbors of $z$ in $G'$ and let $\varphi$ be a conflict-free incidence 7-coloring of $G'$. Clearly, $\varphi\big(\mathrm{Inc}(zz_1)\big) \cap \varphi\big(\mathrm{Inc}(zz_2)\big) = \emptyset$. We construct a conflict-free incidence 7-coloring $\phi$ of $G$ as follows. Let $\phi\big(\mathrm{Inc}(vx)\big) = \varphi\big(\mathrm{Inc}(zz_1)\big)$ and $\phi\big(\mathrm{Inc}(wy)\big) = \varphi\big(\mathrm{Inc}(zz_2)\big)$ (or $\phi\big(\mathrm{Inc}(u_1x)\big) = \varphi\big(\mathrm{Inc}(zz_1)\big)$ and $\phi\big(\mathrm{Inc}(v_1y)\big) = \varphi\big(\mathrm{Inc}(zz_2)\big)$ , or $\phi\big(\mathrm{Inc}(u_2x)\big) = \varphi\big(\mathrm{Inc}(zz_1)\big)$ and $\phi\big(\mathrm{Inc}(v_1y)\big) = \varphi\big(\mathrm{Inc}(zz_2)\big)$ , respectively). This makes a partial incidence coloring of the configuration $G_2$ (or $G_4$, or $G_8$, respectively) such that $\varphi\big(\mathrm{Inc}(vx)\big) \cap \varphi\big(\mathrm{Inc}(wy)\big) = \emptyset$ (or $\varphi\big(\mathrm{Inc}(u_1x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, or $\varphi\big(\mathrm{Inc}(u_2x)\big) \cap \varphi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, respectively). By Lemma 3.8 (or Lemma 3.9, or Lemma 3.10, respectively), $\varphi$ can be extended to a conflict-free incidence 7-coloring of the configuration $G_2$ (or $G_4$, or $G_8$, respectively) and thus any two conflicting incidences of $I(E(G) \setminus E(G'))$ receive distinct colors. Now for every edge $e \in E(G) \cap E(G')$, let $\phi\big(\mathrm{Inc}(e)\big) = \varphi\big(\mathrm{Inc}(e)\big)$. This completes a 7-coloring of the incidences of $G$ and it is easy to check that this coloring is conflict-free.

On the other hand, we show that $G$ admits no conflict-free incidence 6-coloring. Suppose, for a contradiction, that $\phi$ is a conflict-free incidence 6-coloring of $G$. By Lemma 3.4 (or Lemma 3.5, or Lemma 3.6, respectively), $\phi\big(\mathrm{Inc}(vx)\big) \cap \phi\big(\mathrm{Inc}(wy)\big) = \emptyset$ (or $\phi\big(\mathrm{Inc}(u_1x)\big) \cap \phi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, or $\phi\big(\mathrm{Inc}(u_2x)\big) \cap \phi\big(\mathrm{Inc}(v_1y)\big) = \emptyset$, respectively). This makes us possible to construct a conflict-free incidence 6-coloring $\varphi$ of $G'$ by setting $\varphi\big(\mathrm{Inc}(zz_1)\big) = \phi\big(\mathrm{Inc}(vx)\big)$, $\varphi\big(\mathrm{Inc}(zz_2)\big) = \phi\big(\mathrm{Inc}(wy)\big)$, (or $\varphi\big(\mathrm{Inc}(zz_1)\big) = \phi\big(\mathrm{Inc}(u_1x)\big)$, $\varphi\big(\mathrm{Inc}(zz_2)\big) = \phi\big(\mathrm{Inc}(v_1y)\big)$, or $\varphi\big(\mathrm{Inc}(zz_1)\big) = \phi\big(\mathrm{Inc}(u_2x)\big)$, $\varphi\big(\mathrm{Inc}(zz_2)\big) = \phi\big(\mathrm{Inc}(v_1y)\big)$, respectively) and $\varphi\big(\mathrm{Inc}(e)\big) = \phi\big(\mathrm{Inc}(e)\big)$ for every edge $e \in E(G') \setminus E(G)$. This is a contradiction.

**Case 2.** There is a graph $G' \in \mathcal{P}$ and an edge $z_1z_2$ of $G'$ such that $G = G' \vee_{z_1z_2} H_i$.

By the induction hypothesis, $\chi_i^c(G') = 7$. Let $\varphi$ be a conflict-free incidence 7-coloring of $G'$. We construct a conflict-free incidence 7-coloring $\phi$ of $G$ as follows. Let $\phi\big(\mathrm{Inc}(x_{i-1}x_i)\big) = \phi\big(\mathrm{Inc}(y_{i-1}y_i)\big) = \varphi\big(\mathrm{Inc}(z_1z_2)\big)$. This makes a partial incidence coloring of the configuration $H_i$ such that $\phi\big(\mathrm{Inc}(x_{i-1}x_i)\big) = \phi\big(\mathrm{Inc}(y_{i-1}y_i)\big)$. By Lemma 3.11, $\phi$ can be extended to a conflict-free incidence 7-coloring of the configuration $H_i$. Now for every edge $e \in E(G) \cap E(G')$, let

$\phi\big(\mathrm{Inc}(e)\big) = \varphi\big(\mathrm{Inc}(e)\big)$. This completes a 7-coloring of the incidences of $G$ and it is easy to check that this coloring is conflict-free.

On the other hand, we show that $G$ admits no conflict-free incidence 6-coloring. Suppose, for a contradiction, that $\phi$ is a conflict-free incidence 6-coloring of $G$. By Lemma 3.7, $\phi\big(\mathrm{Inc}(x_{i-1}x_i)\big) = \phi\big(\mathrm{Inc}(y_{i-1}y_i)\big)$. This makes us possible to construct a conflict-free incidence 6-coloring $\varphi$ of $G'$ by setting $\varphi\big(\mathrm{Inc}(z_1z_2)\big) = \phi\big(\mathrm{Inc}(x_{i-1}x_i)\big)$ and $\varphi\big(\mathrm{Inc}(e)\big) = \phi\big(\mathrm{Inc}(e)\big)$ for every edge $e \in E(G') \setminus E(G)$. This is a contradiction. □

Algorithm 3 summarises the idea of proving Theorem 3.12, showing how we can construct a conflict-free incidence 7-coloring of a graph in $\mathcal{P}$ efficiently. Now we are ready to prove a more general result as follows.

**Algorithm 3. COLOR-CLASS-P**$(G)$

**Input:** A graph $G \in \mathcal{P}$;
**Output:** A conflict-free incidence 7-coloring $\varphi$ of $G$.

$i \leftarrow 0$;
$G_0 \leftarrow G$;
**while** $G_i \ncong K_4^-$ **do**
  **if** there is a graph $G' \in \mathcal{P}$ with a degree 2 vertex $z$ such that $G_i = G' \sqcup_z G_t$ for some $t \in \{2, 4, 8\}$ **then**
    $G_{i+1} \leftarrow G'$;
    $\mathrm{sign}_i \leftarrow t$;
  **else**
    Find a graph $G' \in \mathcal{P}$ with an edge $z_1z_2$ such that $G_i = G' \vee_{z_1z_2} H_t$ for some integer $t$;
    $G_{i+1} \leftarrow G'$;
    $\mathrm{sign}_i \leftarrow 0$;
  **end if**
  $i \leftarrow i + 1$;
**end while**
{ *We obtain a series* $G_0, G_1, \cdots, G_i$ *of graphs in* $\mathcal{P}$ *where* $G_0 = G$ *and* $G_i = K_4^-$. }
Construct a conflict-free 7-coloring $\varphi_i$ of $G_i$ by Lemma 3.3;
**for** $j = i - 1$ to $0$ **do**
  Extend $\varphi_{j+1}$ to a conflict-free 7-coloring $\varphi_j$ of $G_j$ by Lemma 3.8, 3.9, 3.10, or 3.11 whenever $\mathrm{sign}_j$ equals to 2, 4, 8, or 0, respectively;
**end for**
$\varphi \leftarrow \varphi_0$;

**Theorem 3.13.** *If $G \in \mathcal{P}^+$, then $\chi_i^c(G) = 7$.*

*Proof.* We proceed by induction on $|G|$. Note that the base case is supported by Lemma 3.3. By the definition of $\mathcal{P}$, every graph in $\mathcal{P}$ has exactly one vertex of degree 2, besides which all vertices are of degree 3. By Proposition 3.12, we assume $G \in \mathcal{P}^+ \setminus \mathcal{P}$.

Suppose that $G$ contains a graph $H \in \mathcal{P}$ as a proper subgraph. Let $u$ be the unique vertex of degree 2 of $H$ and let $v$ and $w$ be the two neighbors of $u$ in $H$. Since $\Delta(G) \leq 3$ and $G$ is connected, the degree of $u$ in $G$ must be 3. Let $x$ be the third neighbor of $u$ in $G$. Since every vertex in $V(H) \setminus \{u\}$ has degree 3 in $H$ (and thus in $G$), $u$ is a cut-vertex of $G$.

Let $H'$ be the subgraph of $G$ containing $u$ such that $V(H') \cap V(H) = \{u\}$ and $V(H') \cup V(H) = V(G)$. Since $u$ has degree 1 in $H'$, $H'$ is not an odd cycle. Therefore, if $H' \in \mathcal{P}^+$, then $\chi_i^c(H') = 7$ by the induction hypothesis, and if $H' \notin \mathcal{P}^+$, then $\chi'(H') = \Delta(H') \leq 3$ by Theorem 3.1 and thus $\chi_i^c(H') \leq 6$ by Proposition 2.1. In each case, there is a conflict-free incidence 7-coloring $\phi'$ of $H'$.

**Algorithm 4. COLOR-CLASS-P-PLUS($G$)**

**Input:** A graph $G \in \mathcal{P}^+$;
**Output:** A conflict-free incidence 7-coloring $\varphi$ of $G$.
1: **if** $G \in \mathcal{P}$ **then**
2:   **COLOR-CLASS-P**($G$);
    {*The coloring outputted by line 2 is denoted by* $\varphi$.}
3: **else**
4:   Find a subgraph $H \in \mathcal{P}$ of $G$ with a vertex $u$ that has exactly two neighbors $v$ and $w$ in $H$;
5:   $H' \leftarrow$ the graph with vertex set $V(G) \setminus (V(H) \setminus \{u\})$ and edge set $(E(G) \setminus E(H)) \cup \{ux\}$;
6:   $x \leftarrow$ the unique neighbor of $u$ in $H'$;
7:   **if** $H' \in \mathcal{P}^+$ **then**
8:     **COLOR-CLASS-P-PLUS**($H'$);
      {*The coloring outputted by line 8 is denoted by* $\phi'$.}
9:   **else**
10:     Find a proper edge 3-coloring $\varphi'$ of $H'$ by Theorem 3.1;
11:     **for** each edge $e \in H'$ **do**
12:       $\phi'\big(\mathrm{Inc}(e)\big) \leftarrow \{\varphi'(e), \varphi'(e)+3\}$;
13:     **end for**
14:   **end if**
15:   **COLOR-CLASS-P**($H$);
    {*The coloring outputted by line 15 is denoted by* $\phi$.}
16:   Exchange (if necessary) the colors of $\phi$ so that $\phi\big(\mathrm{Inc}(uv)\big)$, $\phi\big(\mathrm{Inc}(uw)\big)$, and $\phi'\big(\mathrm{Inc}(ux)\big)$ are pairwise disjoint;
17:   $\varphi \leftarrow$ the coloring obtained via combing $\phi'$ with $\phi$;
18: **end if**

Since $H \in \mathcal{P}$, there is a conflict-free incidence 7-coloring $\phi$ of $H$ by Proposition 3.12. We permute (if necessary) the colors of $\phi$ so that $\phi\big(\mathrm{Inc}(uv)\big)$, $\phi\big(\mathrm{Inc}(uw)\big)$, and $\phi'\big(\mathrm{Inc}(ux)\big)$ are pairwise disjoint, and then obtain a conflict-free incidence 7-coloring of $G$ by combining $\phi'$ with $\phi$. This implies $\chi_i^c(G) \le 7$.

On the other hand, $\chi_i^c(G) \ge \chi_i^c(H) = 7$. Hence $\chi_i^c(G) = 7$. □

Algorithm 4 shows the idea of constructing a conflict-free incidence 7-coloring of a give graph in $\mathcal{P}^+$. Now that we have Theorems 2.5, 3.2, and 3.13, the conflict-free incidence chromatic number of connected outer-1-planar graphs (and thus all outer-1-planar graphs) can be completely determined by Theorem 3.14. Algorithm 5 shows an approach to efficiently construct a conflict-free incidence $\chi_i(G)$-coloring $\varphi$ of a connected out-1-planar graph $G$.

**Theorem 3.14.**

$$\chi_i^c(G) = \begin{cases} 6, & \text{if } G \cong C_3, \\ 2\Delta(G), & \text{if } G \notin \mathcal{P}^+ \text{ and } G \text{ is not an odd cycle}, \\ 2\Delta(G)+1, & \text{otherwise} \end{cases}$$

*for every connected outer-1-planar graph $G$.*

**Algorithm 5. COLOR-O1P**($G$)

*This algorithm constructs an optimal conflict-free incidence coloring of a connected outer-1-planar graph $G$.*

**Input:** A connected out-1-planar graph $G$;
**Output:** A conflict-free incidence $\chi_i(G)$-coloring $\varphi$ of $G$.

```
if G is a cycle then
  COLOR-CYCLE(|G|);
else
  if G ∈ P^+ then
    COLOR-CLASS-P-PLUS(G);
  else
    Find a proper edge Δ(G)-coloring ϕ of G by Theorem 3.1;
    for each edge e ∈ G do
      φ(Inc(e)) ← {ϕ(e), ϕ(e) + Δ(G)};
    end for
  end if
end if
```

## 4 Open Problem

To end this paper, we leave an open problem relative to the complexity of the conflict-free incidence coloring. As one can know from Proposition 2.3 that $\chi_i^c(G) \in \{2\Delta(G), 2\Delta(G)+1, 2\Delta(G)+2\}$ for every simple graph $G$, an interesting problem is to investigate the complexity of the following question. We conjecture that **CFICP** is NP-Complete.

**Conflict-free incidence coloring Problem (CFICP)**
Input: A graph $G$ and a positive integer $k$.
Question: Is there a conflict-free incidence $k$-coloring of $G$?

## Conflict of Interest

The authors declare no conflict of interest.

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