Discrete Applied Mathematics 357 (2024) 74–80

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Improper odd coloring of IC-planar graphs[☆]

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ARTICLE INFO

Article history:
Received 5 December 2023
Received in revised form 23 May 2024
Accepted 30 May 2024
Available online xxxx

Keywords:
IC-planar graph
Odd coloring
Conflict-free coloring
Hypergraph

ABSTRACT

An IC-planar graph is a graph that can be drawn in the plane in such a way that each edge is crossed at most once and each vertex is incident with at most one crossed edge. In this paper, we show that every IC-planar graph can be colored with nine colors so that for every non-isolate vertex there exists a color occurring odd times in its neighbors.

1. Introduction

A *k-coloring* of a graph $G = (V, E)$ is a mapping c from its vertex set V to $[k]$, where $[k]$ denotes the set of integers $\{1, 2, \dots, k\}$. A coloring c is *proper* if adjacent vertices receive distinct colors under c. A proper coloring c is *odd* if for every non-isolate vertex v there exists a color occurring odd times in its neighbors. The notion of odd coloring was introduced in 2022 by Petruševski and Škrekovski [20], and this topic has attracted the interest of many research groups including [3,4,6–9,11,13,15,16,19,21,22].

A *hypergraph* is a generalization of a graph, where an edge can connect any number of vertices, instead of just two vertices like in a traditional graph. In a hypergraph, each edge is called a *hyperedge*. In 2012, Cheilaris, Keszegh, and Pálvölgyi [5] defined odd colorings for hypergraphs. Precisely, an *odd coloring of a hypergraph* $\mathcal{H}$ with k colors is a function c from its vertex set to $[k]$, such that for each hyperedge of $\mathcal{H}$ there is a color that occurs an odd number of times on the vertices of this hyperedge. Under this notion, an odd k-coloring of a 2-uniform hypergraph (i.e., a traditional graph) is equivalent to its proper k-coloring, instead of its proper odd k-coloring, however, let us forget this confusion now.

Given a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is the set of its adjacent vertices in G, denoted by $N_G(v)$. We define a hypergraph $\mathcal{H} = (X, S)$ based on G such that $X = V$ and $S = \{N_G(v) \mid v \in V\}$. We call such a hypergraph $\mathcal{H}$ an *open-neighborhood hypergraph* of G, denoted by $\mathcal{H}_{ON}(G)$. In this sense, an odd coloring of $\mathcal{H}_{ON}(G)$ is equivalent to the *improper odd coloring* of G, i.e., a coloring of G, not necessarily proper, such that for every non-isolate vertex there exists a color occurring odd times in its neighbors.

In 2018, a relative notion with the improper odd coloring is given by Abel et al. [1]. A vertex coloring of a graph is *open-neighborhood conflict-free* if for every non-isolated vertex there is a color appearing exactly once in its open neighborhood. It is easy to see that every open-neighborhood conflict-free coloring is an improper odd coloring.

A *minor* of a graph G is a graph obtained from G by means of a sequence of vertex and edge deletions and edge contractions. A graph G is *minor-k-colorable* if every minor of G has a proper k-coloring. Huang, Guo, and Yuan [10] showed

☆ Supported in part by the Fundamental Research Funds for the Central Universities, China (No. ZYTS24076) and the Natural Science Basic Research Program of Shaanxi, China (Nos. 2023-JC-YB-001, 2023-JC-YB-054).

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https://doi.org/10.1016/j.dam.2024.05.048
0166-218X/

that every minor-k-colorable connected graph is open-neighborhood conflict-free k-colorable for each integer $k \geq 2$. Since every minor of a planar graph is still planar, and thus it is 4-colorable by Four Color Theorem, the result of Huang, Guo, and Yuan implies that every planar graph is open-neighborhood conflict-free 4-colorable, and therefore is improperly odd 4-colorable.

On graphs that are not minor-closed, the result of Huang, Guo, and Yuan cannot be applied. In view of this, we focus on IC-planar graphs in this paper. A graph is *IC-planar* if it has an embedding in the plane in such a way that each edge is crossed at most once and each vertex is incident with at most one crossed edge. The structures and colorings on IC-planar graphs were extensively investigated by various of research groups, see [2,12,14,17,23] for example. In particular, Pan, Wang, and Liu [18] showed that IC-planar graphs are properly odd 10-colorable. Note that the class of IC-planar graphs is not minor-closed. Actually, given any graph M, we can subdivide each edge sufficiently many times (i.e., replace each edge with a sufficiently long path where each interior vertex has degree exactly two) so that the resulting graph G is IC-planar, but now G has an M-minor.

The main result of this paper is as follows.

Theorem 1.1. *Every IC-planar graph has an improper odd* 9*-coloring. In other words, the hypergraph $\mathcal{H}_{ON}(G)$ has an odd* 9*-coloring if G is IC-planar.*

Notations. Basic notations follow any standard textbook of graph theory so we only mention unusual ones. A k-, k^+-, and k^--vertex (resp. face) of a plane is a vertex (resp. face) of degree k, at least k, and at most k, respectively. Given an IC-plane graph G (i.e., an embedded IC-planar graph so that each vertex is incident with at most one crossed edge), we turn all its crossings into new vertices of degree 4, and the resulting graph is said to be the *associated plane graph* of G, denoted by G^*. A vertex of G^* is *true* if $v \in V(G)$, and *false* otherwise. Observe that $d_{G^*}(v) = d_G(v)$ if v is true and $d_{G^*}(v) = 4$ if v is false. A face of G^* is *false* if it is incident with at least one false vertex, and *true* otherwise. For an odd (not necessarily proper) coloring c of a subgraph H induced by $S \subseteq V(G)$ and a vertex $v \in V(G)$ (note that v may not be colored under c), if there is an unique color, say α, that appears an odd number of times in $N_G(v)$, then we set $c_o(v) = \alpha$, and otherwise we set $c_o(v) = 0$.

2. Reducibilities

Let G be a counterexample to Theorem 1.1 with the minimum number of vertices. We could assume that G is embedded in a plane so that the number of crossings is as few as possible. For each $f \in F(G^*)$, we denote $n_k(f)$ to be the number of true k-vertices incident with f. Since G is IC-planar, two false vertices are not adjacent in G^* and any true vertex is adjacent to at most one false vertex in G^*.

A k_t-face of G^* is a k-face incident with exactly t 2-vertices. A true 4-vertex u is v*-special* if v is true and each face incident with uv in G^* is either a 3-face or a 4_1-face. For each $v \in V(G)$, $\mathrm{sp}(v)$ denotes the number of v-special vertices, and $m_3'(v)$ denotes the number of 3_1-faces incident with v.

Lemma 2.1. *G has no k-vertices with $k = 1, 3, 5, 7$.*

Proof. Suppose that G has a k-vertex v with $N_G(v) = \{v_1, v_2, \ldots, v_k\}$. Since $G' = G - v$ has fewer vertices than G, G' has an improper odd 9-coloring c by the minimality of G. Color each vertex other than v in G with the same color in G' and color v with a color in $[9] \setminus \{c_o(v_1), c_o(v_2), \ldots, c_o(v_k)\}$. Since the degree of v is odd, the oddness of v is satisfied naturally. Now, the improper odd 9-coloring of G' can return back to G, a contradiction. □

Lemma 2.2. *8^--vertices are not adjacent in G.*

Proof. Suppose that two 8^--vertices u and v are adjacent. Let $N_G(u) = \{v, u_1, u_2, \ldots, u_{i-1}\}$ and $N_G(v) = \{u, v_1, v_2, \ldots, v_{j-1}\}$, where $i, j \leq 8$. By the minimality of G, $G' = G - \{u, v\}$ has an improper odd 9-coloring c. Color each vertex other than u and v in G with the same color in G', and then color u with a color in $[9] \setminus \{c_o(v), c_o(u_1), c_o(u_2), \ldots, c_o(u_{i-1})\}$ and v with a color in $[9] \setminus \{c_o(u), c_o(v_1), c_o(v_2), \ldots, c_o(v_{i-1})\}$. This extends c to an improper odd 9-coloring of G, a contradiction. □

Lemma 2.3. *For each face f of G^*, we have $n_2(f) + n_4(f) \leq \lfloor \frac{d_{G^*}(f)}{2} \rfloor$.*

Proof. This is an immediate corollary of Lemma 2.2. □

Lemma 2.4. *If uvw is a boundary path of some face of G^*, then at most one of u and w is v-special.*

Proof. Let f be the face of G^* such that uvw is its boundary path. Suppose on the contrary that u and w are both v-special. Since v-special vertices are 4-vertices by the definition, v is not a 2-vertex and thus f is neither a 3-face nor a 4_1-face by Lemma 2.2. This contradicts the definition of v-special vertex. □

Lemma 2.5. *If u is v-special, then uv is not incident with a 3_1-face.*

Proof. Assume that uv is incident with a 3-face $[vuw]$. If this face is a 3_1-face, then w is a 2-vertex adjacent to the true 4-vertex u, contradicting Lemma 2.2. □

Lemma 2.6. *Any false 3-face f of G^* is not incident with a 2-vertex.*

Proof. Let $f = [uvw]$ be a false 3-face such that u is a 2-vertex and w is false. Redraw G by re-embedding u into the other face incident with vw which is not f. This avoids forming the crossing w, and the resulting drawing has less crossings, a contradiction to our assumption. □

Lemma 2.7. *Any 3_1-face is not incident with a 9-vertex in G^*.*

Proof. Let $f = [uvw]$ be a 3_1-face such that v is a 9-vertex with neighbors $u, w, v_1, v_2, v_3, v_4, v_5, v_6, v_7$, and u is a 2-vertex. By the minimality of G, $G' = G - \{u, v\}$ has an improper odd 9-coloring c. Note that w is true by Lemma 2.6. We color v with $\alpha \in [9] \setminus \{c_o(v_1), c_o(v_2), c_o(v_3), c_o(v_4), c_o(v_5), c_0(v_6), c_0(v_7), c(w)\}$ and denote the resulting coloring of $G - u$ still by c. Now we extend c to an improper odd 9-coloring of G by coloring u with a color in $[9] \setminus \{c_o(v), c_o(w)\}$. Note that the oddness of u is satisfied as $\alpha \neq c(w)$. □

Lemma 2.8. *A 4-face is incident with at most one 2-vertex in G^*.*

Proof. Suppose that $[v_1v_2v_3v_4]$ is a 4-face incident with at least two 2-vertices. By Lemma 2.2, we may assume without loss of generality that v_1 and v_3 are 2-vertices. If v_2 or v_4 is false, then we can rotate the 4-face on the axis of the line segment connecting v_1 and v_3 to avoid at least one crossing while drawing this graph, a contradiction to our assumption. Hence v_2 and v_4 are true. By the minimality of G, $G' = G - v_1$ has an improper odd 9-coloring c. Color each vertex other than v_1 in G with the same color in G', and then color v_1 with a color in $[9] \setminus \{c_o(v_2), c_o(v_4)\}$. Since $c(v_2) \neq c(v_4)$ as v_3 is a 2-vertex in G', the oddness of v_1 is satisfied. Now we obtain an improper odd 9-coloring of G, a contradiction. □

Lemma 2.9. *For each 9^+-vertex v of G^*, $m_3'(v) \leq d_{G^*}(v) - 9$.*

Proof. Let S be the set of 2-vertices such that for each $u \in S$, uv is incident with a 3_1-face $[uvw]$, where w is represented by γ_u. Let $T = \{\gamma_u \mid u \in S\}$ and $Y = N_G(v) \setminus (S \cup T)$.

Suppose for a contradiction that $m_3'(v) \geq d_{G^*}(v) - 8$. By the minimality of G, $G - v$ has an improper odd 9-coloring c. We erase the color of each vertex in S, and then color v with a color not in $\{c(x) \mid x \in T\} \cup \{c_o(x) \mid x \in Y\}$, which has size at most $|T| + |Y| = d_{G^*}(v) - |S| \leq d_{G^*}(v) - (d_{G^*}(v) - 8) = 8$. Note that for each $u \in S$, $c_o(u) = c(\gamma_u)$. Arbitrarily choose a vertex $s \in S$, and color each vertex $u \in S \setminus \{s\}$ with a color different from $c_o(\gamma_u)$. Now we come to a coloring of $G - s$, still denoted by c. Finally we complete an improper odd 9-coloring of G by coloring s with a color different from $c_o(v)$ and $c_o(\gamma_s)$. □

3. Proof of Theorem 1.1

We apply a discharging argument on G^* to accomplish the proof. We assign an initial charge $\mu(v) = d_{G^*}(v) - 6$ to each $v \in V(G^*)$ and $\mu(f) = 2d_{G^*}(f) - 6$ to each $f \in F(G^*)$. By Euler's Formula, we have $\sum_{v \in V(G^*)} \mu(v) + \sum_{f \in F(G^*)} \mu(f) < 0$. In the following, we design appropriate discharging rules to redistribute the charges, obtaining a final charging function μ^* on $V(G^*) \cup F(G^*)$ such that $\mu^*(x) \geq 0$ for each $x \in V(G^*) \cup F(G^*)$. Since the total sum of charges is uncharged in the discharging procedure, this is a contradiction implying Theorem 1.1.

The discharging rules are defined as follows.

R1. Every 4^+-face f sends 2 to each of its incident 2-vertices, and $\frac{2d_{G^*}(f) - 6 - 2n_2(f)}{n_4(f)}$ to each of its adjacent true 4-vertices.

R2. Every 9^+-vertices v sends 1 to each of its incident 3_1-faces, $\frac{1}{2}$ to each v-special vertex, and $d_{G^*}(v) - 6 - m_3'(v) - \frac{1}{2}\mathrm{sp}(v)$ to its adjacent false 4-vertex.

R3. Every 3_1-face sends 2 to its incident 2-vertex.

R4. Every false 4-vertex v with $\alpha(v) \geq 2$ and $n_4(v) > 0$ sends $\frac{\alpha(v) - 2}{n_4(v)}$ to each of its adjacent true 4-vertices, where $\alpha(v)$ denotes the total amount of charges that v gets from its adjacent 9^+-vertices by **R2** and $n_4(v)$ denotes the number of true 4-vertices adjacent to v.

Remark 1. *If f is a 4-face, then $n_2(f) \leq 1$ by Lemma 2.8. If f is a 5^+-face, then $n_2(f) \leq \lfloor \frac{d_{G^*}(f)}{2} \rfloor$ by Lemma 2.2. In each case, we have $2d_{G^*}(f) - 6 - 2n_2(f) \geq 0$. Hence **R1** is valid.*

Remark 2. *If v is a 9^+-vertex, then by Lemmas 2.4 and 2.5, we have $d_{G^*}(v) \geq 2\mathrm{sp}(v) + m_3'(v)$. It follows*

$$\begin{aligned} d_{G^*}(v) - 6 - m_3'(v) - \frac{1}{2}\mathrm{sp}(v) &\geq d_{G^*}(v) - 6 - m_3'(v) - \frac{1}{4}\Big(d_{G^*}(v) - m_3'(v)\Big) \\ &\geq \frac{3}{4}\Big(d_{G^*}(v) - (d_{G^*}(v) - 9) - 8\Big) \\ &> 0 \end{aligned}$$

*by Lemma 2.9 and thus **R2** is valid too.*

We do not need consider k-vertices with $k = 1, 3, 5, 7$ by Lemma 2.1, and also we shall forget 6-vertices and 8-vertices as they are not involved in the discharging rules and their initial charges are nonnegative.

Let v be a 2-vertex. Since the face incident with v is either a 4^+-face or a 3_1-face, $\mu^*(v) = -4+2+2 = 0$ by **R1** and **R3**.

Let v be a true 4-vertex with neighbors v_1, v_2, v_3, v_4 lying in this ordering around v in G^*, and let f_1, f_2, f_3, f_4 be the face incident with the path v_1vv_2, v_2vv_3, v_3vv_4, v_4vv_1, respectively.

If f_i is a 5^+-face, then it sends

$$\frac{2d_{G^*}(f_i) - 6 - 2n_2(f_i)}{n_4(f_i)} \geq \frac{2d_{G^*}(f_i) - 6 - 2\lfloor \frac{d_{G^*}(f_i)}{2} \rfloor + 2n_4(f_i)}{n_4(f_i)} \geq 2$$

to v by **R1** and by Lemma 2.3.

If f_i is a 4_0-face, then it sends $\frac{2\times 4-6}{n_4(f_i)} \geq 1$ to v by **R1** as $n_4(f_i) \leq 2$ by Lemma 2.2.

Hence, if some of $f_1, f_2, f_3.f_4$ is a 5^+-face, or two of $f_1, f_2, f_3.f_4$ are 4_0-faces, then $\mu^*(v) \geq -2+2 = 0$. Next we assume that each f_i is a 4^--face. Note that each f_i is incident with at most one 2-vertex by Lemma 2.2 and Lemma 2.8.

Assume first that each v_i is true. By Lemma 2.2, each v_i is a 9^+-vertex. If v is incident with exactly one 4_0-face, say f_1, then each of f_2, f_3, and f_4 is either a 3-face or a 4_1-face, and therefore v is both v_3-special and v_4-special. It follows that $\mu^*(v) \geq -2+1+2\times\frac{1}{2} = 0$ by **R1** and **R2**. If v is not incident with any 4_0-face, then each f_i is either a 3-face or a 4_1-face, and thus v is v_i-special for each $i \in [4]$. This implies $\mu^*(v) \geq -2+4\times\frac{1}{2} = 0$ by **R2**.

Assume now that v_1 is false (note that only one of v_1, v_2, v_3, v_4 can be false). Let $u \in N_G(v) \setminus \{v_2, v_3, v_4\}$, i.e., uv is an edge of G passing the crossing v_1. Let w_1w_4 be an edge of G crossing uv such that w_i is on f_i with $i = 1, 4$.

Case 1. *v is incident with exactly one 4_0-face.*

By symmetry we consider two cases.

Suppose first that f_1 is a 4_0-face. This implies that f_2 and f_3 are 4_1-faces or 3-faces. If $n_4(f_1) \leq 1$, then f_1 sends $2\times 4-6 = 2$ to v by **R1**, and thus $\mu^*(v) \geq -2+2 = 0$. Hence we assume $n_4(f_1) \geq 2$. This implies that $n_4(f_1) = 2$, w_1 is a true 4-vertex, and w_4 is a 9^+-vertex by Lemma 2.2. Therefore, f_4 cannot be a 4_1-face, and thus it must be a 3-face. It follows that v is both v_3-special and v_4-special. Hence $\mu^*(v) \geq -2+\frac{2\times 4-6}{2}+2\times\frac{1}{2} = 0$ by **R1** and **R2**.

Suppose now that f_2 is a 4_0-face, denoted by $[vv_2w_2v_3]$. By **R1**, f_2 sends $\frac{2\times 4-6}{n_4(f_2)} \geq 1$ to v as $n_4(f_2) \leq 2$. By Lemma 2.2, f_1 and f_4 cannot simultaneously be 4_1-faces, as otherwise w_1 and w_4 are two adjacent 2-vertices in G. Let f_i' with $i = 1, 4$ be an face sharing the common edge v_1w_i with f_i. Let w_i with $i = 2, 3$ be a vertex on f_i such that $v_iw_i \in E(G)$. Note that it may happen that $w_3 = v_4$. In what follows, we consider three subcases.

Case 1.1. *f_1 is a 3-face and f_4 is a 4_1-face.*

In this case, $w_1 = v_2$ and w_4 is a 2-vertex. Now we calculate $\alpha(v_1)$.

Let u_1' be a vertex on f_1' such that $u_1'v_2 \in E(G)$. If u_1' is a true 4-vertex, then $u_1' \neq u$ and thus f_1' is not a 3-face, furthermore, f_1' is not a 4_1-face as u and v_2 are 9^+-vertices. It follows that u_1' is not v_2-special. Since f_2 is assumed to be a 4_0-face, v and w_2 are not v_2-special . Therefore, u_1', v_1, v, w_2 are four consecutive neighbors of v_2 in G^* that are not v_2-special. Since f_1', f_1, f_2 are not 3_1-faces by Lemma 2.6, by counting faces around v_2, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(v_2) \geq 3 + 2\mathrm{sp}(v_2) + m_3'(v_2).$$

Hence v_2 sends to v_1

$$\begin{aligned} d_{G^*}(v_2) - 6 - m_3'(v_2) - \frac{1}{2}\mathrm{sp}(v_2) &\geq d_{G^*}(v_2) - 6 - m_3'(v_2) - \frac{1}{4}\Big(d_{G^*}(v_2) - 3 - m_3'(v_2)\Big) \\ &= \frac{3}{4}\Big(d_{G^*}(v_2) - 7 - m_3'(v_2)\Big) \\ &\geq \frac{3}{4}\Big(d_{G^*}(v_2) - 7 - \big(d_{G^*}(v_2) - 9\big)\Big) \\ &= \frac{3}{2} \end{aligned}$$

by **R2** and Lemma 2.9.

Let x_i with $i = 1, 4$ be a vertex on f_i' such that $ux_i \in E(G)$. If x_1 is a true 4-vertex, then $x_1 \neq v_2$ and thus f_1' is not a 3-face, furthermore, f_1' is not a 4_1-face. This implies that x_1 is not u-special. If x_4 is a true 4-vertex, then $x_4 \neq w_4$ and thus f_4' is not a 3-face, furthermore, f_4' is not a 4_1-face as $x_4w_4 \notin E(G)$. This implies that x_4 is not u-special. Therefore, x_1, v_1, x_4 are three consecutive neighbors of u in G^* that are not u-special. Since f_1' and f_4' are not 3_1-faces by Lemma 2.6, by counting faces around u, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(u) \geq 2 + 2\mathrm{sp}(u) + m_3'(u).$$

Hence u sends to v_1

$$\begin{aligned} d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{2}\mathrm{sp}(u) &\geq d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{4}\Big(d_{G^*}(u) - 2 - m_3'(u)\Big) \\ &= \frac{3}{4}\Big(d_{G^*}(u) - \frac{22}{3} - m_3'(u)\Big) \\ &\geq \frac{3}{4}\Big(d_{G^*}(u) - \frac{22}{3} - \big(d_{G^*}(u) - 9\big)\Big) \\ &= \frac{5}{4} \end{aligned}$$

by **R2** and Lemma 2.9. This implies $\alpha(v_1) \geq \frac{3}{2} + \frac{5}{4} = \frac{11}{4}$. Since in this case v is the only true 4-vertex adjacent to v_1, v_1 gives v at least $\frac{11}{4} - 2 = \frac{3}{4}$ by **R4**.

Since f_4 is a 4_1-face, f_3 is a 3-face or 4_1-face, and v_4 is a 9^+-vertex, we conclude that v is v_4-special. This implies that v_4 sends $\frac{1}{2}$ to v by **R2**. Hence $\mu^*(v) \geq -2 + \frac{3}{4} + \frac{1}{2} + 1 > 0$ as f_2 sends at least 1 to v.

Case 1.2. *f_1 is a 4_1-face and f_4 is a 3-face.*

In this case, w_1 is a 2-vertex and $w_4 = v_4$. Now we calculate $\alpha(v_1)$.

Let u_4' be a vertex on f_4' such that $u_4'v_4 \in E(G)$. If u_4' is a true 4-vertex, then $u_4' \neq u$ and thus f_4' is not a 3-face, furthermore, f_4' is not a 4_1-face as u and v_4 are 9^+-vertices. It follows that u_4' is not v_4-special. Therefore, v_1 and u_4' are two consecutive neighbors of v_4 in G^* that are not v_4-special. Since f_4' is not a 3_1-face by Lemma 2.6, by counting faces around v_4, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(v_4) \geq 1 + 2\mathrm{sp}(v_4) + m_3'(v_4).$$

Hence v_4 sends to v_1

$$\begin{aligned} d_{G^*}(v_4) - 6 - m_3'(v_4) - \frac{1}{2}\mathrm{sp}(v_4) &\geq d_{G^*}(v_4) - 6 - m_3'(v_4) - \frac{1}{4}\Big(d_{G^*}(v_4) - 1 - m_3'(v_2)\Big) \\ &= \frac{3}{4}\Big(d_{G^*}(v_4) - m_3'(v_4) - \frac{23}{3}\Big) \\ &\geq \frac{3}{4}\Big(d_{G^*}(v_4) - (d_{G^*}(v_4) - 9) - \frac{23}{3}\Big) \\ &= 1 \end{aligned}$$

by **R2** and Lemma 2.9.

Let x_i with $i = 1, 4$ be a vertex on f_i' such that $ux_i \in E(G)$. If x_1 is a true 4-vertex, then $x_1 \neq w_1$ and thus f_1' is not a 3-face, furthermore, f_1' is not a 4_1-face as $x_1w_1 \notin E(G)$. This implies that x_1 is not u-special. If x_4 is a true 4-vertex, then $x_4 \neq v_4$ and thus f_4' is not a 3-face, furthermore, f_4' is not a 4_1-face. This implies that x_4 is not u-special. Therefore, x_1, v_1, x_4 are three consecutive neighbors of u in G^* that are not u-special. Since f_1' and f_4' are not 3_1-faces by Lemma 2.6, by counting faces around u, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(u) \geq 2 + 2\mathrm{sp}(u) + m_3'(u).$$

If $m_3'(u) = d_{G^*}(u) - 9$, then the above inequality implies $\mathrm{sp}(u) \leq 3$. So u sends v_1 at least $d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{2}\mathrm{sp}(u) \geq d_{G^*}(u) - 6 - (d_{G^*}(u) - 9) - \frac{1}{2} \times 3 = \frac{3}{2}$ by **R2**. If $m_3'(u) \leq d_{G^*}(u) - 10$, then u sends to v_1

$$\begin{aligned} d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{2}\mathrm{sp}(u) &\geq d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{4}\Big(d_{G^*}(u) - 2 - m_3'(u)\Big) \\ &= \frac{3}{4}\Big(d_{G^*}(u) - m_3'(u) - \frac{22}{3}\Big) \\ &\geq \frac{3}{4}\Big(d_{G^*}(u) - (d_{G^*}(u) - 10) - \frac{22}{3}\Big) \\ &= 2 \end{aligned}$$

by **R2** and Lemma 2.9. Therefore, v_1 gets at least $\min\{\frac{3}{2}, 2\} = \frac{3}{2}$ from u, and thus $\alpha(v_1) \geq \frac{3}{2} + 1 = \frac{5}{2}$. Since in this case v is the only true 4-vertex adjacent to v_1, v_1 gives v at least $\frac{5}{2} - 2 = \frac{1}{2}$ by **R4**.

Since f_4 is a 3-face, f_3 is a 3-face or 4_1-face, and v_4 is a 9^+-vertex, we conclude that v is v_4-special. This implies that v_4 sends $\frac{1}{2}$ to v by **R2**. Hence $\mu^*(v) \geq -2 + \frac{1}{2} + 1 + \frac{1}{2} \geq 0$ as f_2 sends at least 1 to v.

Case 1.3. *f_1 and f_4 are both 3-faces.*

Using same arguments as what we had done in Case 1.1 and Case 1.2, we conclude that v_2 and v_4 sends to v_1 at least $\frac{3}{2}$ and 1 to v_1, respectively. This implies that $\alpha(v_1) \geq \frac{3}{2} + 1 = \frac{5}{2}$. In this case, v is the only true 4-vertex adjacent to v_1. So v_1 sends at least $\frac{5}{2} - 2 = \frac{1}{2}$ to v by **R4**. Since f_4 is a 3-face and f_3 is a 3-face or 4_1-face, we conclude that v is v_4-special, and thus v_4 sends $\frac{1}{2}$ to v by **R2**. Hence $\mu^*(v) \geq -2 + \frac{1}{2} + 1 + \frac{1}{2} = 0$ as f_2 sends at least 1 to v.

Case 2. *v is incident only with 3-face or 4_1-face.*

In this case, v is a v_i-special for each $i \in \{2, 3, 4\}$. So each v_i sends $\frac{1}{2}$ to v by **R2**. In the following, we show that v_1 would send at least $\frac{1}{2}$ to v, and thus $\mu^*(v) \geq -2 + 4 \times \frac{1}{2} = 0$.

By Lemma 2.2, f_1 and f_4 cannot simultaneously be 4_1-faces, as otherwise w_1 and w_4 are two adjacent 2-vertices in G. Let f_i' with $i = 1, 4$ be an face sharing the common edge v_1w_i with f_i. Let w_i with $i = 2, 3$ be a vertex on f_i such that $v_iw_i \in E(G)$. Note that it may happen that $w_2 = v_3$ or $w_3 = v_4$. By symmetry, we distinguish two subcases.

Case 2.1. *f_1 is a 3-face and f_4 is a 4_1-face.*

In this case, w_4 is a 2-vertex and $w_1 = v_2$. Since v is the unique true 4-vertex adjacent to v_1, by **R4**, it is sufficient to show that $\alpha(v_1) \geq 2 + \frac{1}{2} = \frac{5}{2}$.

Let u_1' be a vertex on f_1' such that $u_1'v_2 \in E(G)$. If u_1' is a true 4-vertex, then $u_1' \neq u$ and thus f_1' is not a 3-face, furthermore, f_1' is not a 4_1-face as u and v_2 are 9^+-vertices. So u_1' is not v_2-special, and thus v_1 and u_1' are two consecutive neighbors of v_2 that are not v_2-special. Clearly, f_1' is not a 3_1-face by Lemma 2.6. By counting faces around v_2, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(v_2) \geq 1 + 2\text{sp}(v_2) + m_3'(v_2).$$

Hence v_2 sends to v_1

$$\begin{aligned}
d_{G^*}(v_2) - 6 - m_3'(v_2) - \frac{1}{2}\text{sp}(v_2) &\geq d_{G^*}(v_2) - 6 - m_3'(v_2) - \frac{1}{4}\Big(d_{G^*}(v_2) - 1 - m_3'(v_2)\Big) \\
&= \frac{3}{4}\left(d_{G^*}(v_2) - m_3'(v_2) - \frac{23}{3}\right) \\
&\geq \frac{3}{4}\left(d_{G^*}(v_2) - (d_{G^*}(v_2) - 9) - \frac{23}{3}\right) \\
&= 1
\end{aligned}$$

by **R2** and Lemma 2.9.

Let x_i with $i = 1, 4$ be a vertex on f_i' such that $ux_i \in E(G)$. If x_1 is a true 4-vertex, then $x_1 \neq v_2$ and thus f_1' is not a 3-face, furthermore, f_1' is not a 4_1-face as u and v_2 are 9^+-vertices. This implies that x_1 is not u-special. If x_4 is a true 4-vertex, then $x_4 \neq w_4$ and thus f_4' is not a 3-face, furthermore, f_4' is not a 4_1-face as $x_4w_4 \notin E(G)$. This implies that x_4 is not u-special. Therefore, x_1, v_1, x_4 are three consecutive neighbors of u that are not u-special. Since f_1' and f_4' are not 3_1-faces by Lemma 2.6, by counting faces around u, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(u) \geq 2 + 2\text{sp}(u) + m_3'(u).$$

If $m_3'(u) = d_{G^*}(u) - 9$, then the above inequality implies $\text{sp}(u) \leq 3$. So u sends v_1 at least $d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{2}\text{sp}(u) \geq d_{G^*}(u) - 6 - (d_{G^*}(u) - 9) - \frac{1}{2} \times 3 = \frac{3}{2}$ by **R2**. If $m_3'(u) \leq d_{G^*}(u) - 10$, then

$$\begin{aligned}
d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{2}\text{sp}(u) &\geq d_{G^*}(u) - 6 - m_3'(u) - \frac{1}{4}\Big(d_{G^*}(u) - 2 - m_3'(u)\Big) \\
&= \frac{3}{4}\left(d_{G^*}(u) - m_3'(u) - \frac{22}{3}\right) \\
&\geq \frac{3}{4}\left(d_{G^*}(u) - (d_{G^*}(u) - 10) - \frac{22}{3}\right) \\
&= 2
\end{aligned}$$

by **R2** and Lemma 2.9. Thus v_1 gets at least $\min\{\frac{3}{2}, 2\} = \frac{3}{2}$ from u. This implies $\alpha(v_1) \geq 1 + \frac{3}{2} = \frac{5}{2}$, as desired.

Case 2.2. *f_1 and f_4 are both 3-faces.*

With same or symmetry arguments as above, we conclude that each of v_2 and v_4 sends at least 1 to v_1. Similarly, we can show that u sends at least $\frac{3}{2}$ to v_1. This implies $\alpha(v_1) \geq 1 + 1 + \frac{3}{2} = \frac{7}{2} > \frac{5}{2}$, as desired.

Let v be a false 4-vertex such that v_1, v_2, v_3, v_4 are all its neighbors lying in this ordering. By Lemma 2.2, v is adjacent to at least two 9^+-vertices. Assume, without loss generality, that v_1 and v_2 are 9^+-vertices. Let f_1 be the face incident with the path v_1vv_2 and let x_1 be a vertex on f_1 such that $x_1v_1 \in E(G)$. Note that it may happen that $x_1 = v_2$.

If x_1 is a true 4-vertex, then $x_1 \neq v_2$ and thus f_1 is not a 3-face, furthermore, f_1 is not a 4_1-face as v_1 and v_2 are 9^+-vertices. This implies that x_1 is not v_1-special. Therefore, x_1 and v are two consecutive neighbors of v_1 that are not v_1-special. Since f_1 is clearly not 3_1-face by Lemma 2.6, by counting faces around v_1, we have, by Lemmas 2.4 and 2.5, that

$$d_{G^*}(v_1) \geq 1 + 2\mathrm{sp}(v_1) + m_3'(v_1).$$

Hence v_1 sends to v

$$\begin{aligned} d_{G^*}(v_1) - 6 - m_3'(v_1) - \frac{1}{2}\mathrm{sp}(v_1) &\geq d_{G^*}(v_1) - 6 - m_3'(v_1) - \frac{1}{4}\Big(d_{G^*}(v_1) - 1 - m_3'(v_1)\Big) \\ &\geq \frac{3}{4}\Big(d_{G^*}(v_1) - (d_{G^*}(v_1) - 9) - \frac{23}{3}\Big) \\ &= 1 \end{aligned}$$

by **R2** and Lemma 2.9. By the same reason, we can also show that v_2 sends at least 1 to v. Hence $\mu^*(v) \geq -2 + 1 \times 2 = 0$.

Let v be a 9^+-vertex. By **R2** and Remark 2, it is immediate that $\mu^*(v) \geq d_{G^*}(v) - 6 - m_3'(v) - \frac{1}{2}\mathrm{sp}(v) - (d_{G^*}(v) - 6 - m_3'(v) - \frac{1}{2}\mathrm{sp}(v)) = 0$.

Let f be a 3-face. If f is a 3_1-face, then f is incident with two 10^+-vertices by Lemmas 2.2 and 2.7, and thus $\mu^*(f) = 2 \times 3 - 6 + 2 \times 1 - 2 = 0$ by **R2** and **R3**. If f is not 3_1-face, then it is not involved in the discharging rules and thus $\mu^*(f) = 2 \times 3 - 6 = 0$.

Let f be a 4^+-face. By **R1** and Remark 1, we have $\mu^*(f) \geq 2d_{G^*}(f) - 6 - 2n_2(f) - \frac{2d_{G^*}(f)-6-2n_2(f)}{n_4(f)} \times n_4(f) = 0$.

Therefore, the final charge of every vertex and face of G^* is non-negative. This completes the proof.

Data availability

No data was used for the research described in the article.

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