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Incidence Coloring of Outer-1-planar Graphs

Meng-ke QI, Xin ZHANG†

School of Mathematics and Statistics, Xidian University, Xi'an 710071, China
(E-mail: mkqi@stu.xidian.edu.cn, †xzhang@xidian.edu.cn)

Abstract A graph is outer-1-planar if it can be drawn in the plane so that all vertices lie on the outer-face and each edge crosses at most one another edge. It is known that every outer-1-planar graph is a planar partial 3-tree. In this paper, we conjecture that every planar graph G has a proper incidence $(\Delta(G)+2)$-coloring and confirm it for outer-1-planar graphs with maximum degree at least 8 or with girth at least 4. Specifically, we prove that every outer-1-planar graph G has an incidence $(\Delta(G)+3, 2)$-coloring, and every outer-1-planar graph G with maximum degree at least 8 or with girth at least 4 has an incidence $(\Delta(G)+2, 2)$-coloring.

Keywords incidence coloring; outer-1-planar graph; planar graph
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1 Introduction

All graphs in this paper are simple and undirected. We denote by $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$ the vertex set, edge set, minimum degree, and maximum degree of a graph G. The degree of a vertex v in a graph G is denoted by $d_G(v)$, and we use $d(v)$ instead if the graph G is clear from the content. The *girth* of a non-acyclic graph G, denoted by $g(G)$, is the minimum length of a cycle in G. We refer the reads to [5] for other undefined but frequently used notations.

An *incidence* of G is a vertex-edge pair (v, e) with $v \in V(G)$ and $e \in E(G)$ such that v is an end-vertex of e. Two incidences (u, e) and (v, f) are *adjacent* if (i) $u = v$, or (ii) $e = f$, or (iii) $uv = e$ or $uv = f$. For a vertex $u \in V(G)$, the incidences (u, uv) (resp (v, uv)) are *strong incidences* (resp. *weak incidences*) of u, and the set of the strong (resp. weak) incidences is denoted by I_u (resp. A_u). A *proper incidence k-coloring* of a graph G is a coloring of the incidences using k colors in such a way that every two adjacent incidences get distinct colors. The minimum integer k such that G has a proper incidence k-coloring is the *incidence chromatic number* of G, denoted by $\chi_i(G)$. An *incidence* (k, ℓ)*-coloring* of G is a proper incidence k-coloring such that $|A_u| \le \ell$ for each $u \in V(G)$.

For a graph G, it is easy to see that $\chi_i(G) \ge \Delta(G)+1$, because we need at least $\Delta(G)+1$ colors to color the strong incidences and the weak incidences of a vertex of maximum degree. For the upper bound, Brualdi and Quinn Massey[6] raised the following conjecture in 1993.

Conjecture 1.1 (Incidence Coloring Conjecture (ICC)). *$\chi_i(G) \le \Delta(G)+2$ holds for every graph G.*

ICC had been verified for graphs with maximum degree at most 3 by Maydanskyi[12]. However, ICC is not completely correct. In 1997, Guiduli[7] found an interesting relationship between the incidence chromatic number and the directed star arboricity, and then used a

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†Corresponding author.

result of Algor and Alon[1] to show that Paley graphs have incidence chromatic number at least $\Delta(G)+\Omega(\log\Delta(G))$. More precisely, the following theorem is based on the results of [1, 7].

Theorem 1.2. *If G is a Paley graph and k is an integer such that $\Delta(G)\geq k^2\cdot 2^{2k-3}$, then*

$$\chi_i(G)>\Delta(G)+\frac{k}{2}. \tag{1.1}$$

Since Paley graphs on p vertices are $(p-1)/2$-regular, where p shall satisfy the condition that $p\equiv 1 \pmod 4$ is a prime[1], Palay graphs on at least 13 vertices are not planar (note that every planar graph contains a vertex of degree at most 5). To our best knowledge, all known counterexamples to ICC are Paley graphs with large maximum degree (see Theorem 1.2), and thus they are not planar. This makes us confident to make a weaker conjecture than ICC.

Conjecture 1.3 (Planar Incidence Coloring Conjecture (PICC))**.** $\chi_i(G)\leq\Delta(G)+2$ *holds for every planar graph G.*

We review some known results concerning PICC. First of all, Hosseini Dolama, Sopena, and Zhu[9] proved that every planar graph G is incidence $(\Delta(G)+7,7)$-colorable. Hence $\chi_i(G)\leq\Delta(G)+7$ holds for every planar graph G. This upper bound was improved to $\Delta(G)+5$ by Yang[16] in 2012.

Theorem 1.4[16]**.** $\chi_i(G)\leq\Delta(G)+5$ *holds for every planar graph G.*

For planar graphs with high girth, Hosseini Dolama and Sopena[8] proved that if G is a planar graph with $g(G)\geq 6$ and $\Delta(G)\geq 5$ then G has an incidence $(\Delta(G)+2,2)$-coloring. Kardoš et al. [10] showed that if G is a graph with maximum average degree less than 3 and $\Delta(G)\geq 4$ then $\chi_i(G)\leq\Delta(G)+2$. Since planar graphs with $g(G)\geq 6$ have maximum average degree less than 3, combining the results of those two groups and the fact that ICC holds for subcubic graphs[12], we conclude the following

Theorem 1.5. $\chi_i(G)\leq\Delta(G)+2$ *holds for every planar graph G with girth at least 6.*

Kardoš et al. [10] also proved that $\chi_i(G)\leq\Delta(G)+2$ if G is a planar graph with $g(G)\geq 5$ and $\Delta(G)\geq 8$. Bonamy, Lévêque, and Pinlou[4] showed that if G is a planar graph with $g(G)\geq 14$, then G has an incidence $(\Delta(G)+1,1)$-coloring, and thus $\chi_i(G)=\Delta(G)+1$.

Hosseini Dolama, Sopena, and Zhu[9] proved that every partial 2-tree G is incidence $(\Delta(G)+2,2)$-colorable and thus $\chi_i(G)\leq\Delta(G)+2$. Since series-parallel graphs are partial 2-trees and outerplanar graphs is a subclass of series-parallel graphs, the following is immediate.

Theorem 1.6. $\chi_i(G)\leq\Delta(G)+2$ *holds for every series-parallel (outerplanar) graph G.*

Moreover, Shiu and Sun[13] observed (actually as a corollary of a result of Wang and Lih[15]) that if G is an outerplanar with $g(G)\geq 7$, then G is incidence $(\Delta(G)+1,1)$-colorable.

A graph is *outer-1-planar* if it admits a drawing in the plane so that vertices lie on the outer-face and each edge is crossed at most once. A graph is *quasi-Hamiltonian* if its every block (i.e., maximal 2-connected component) is Hamiltonian. Zhang, Liu, and Wu[18] showed that the intersection of the class of quasi-Hamiltonian outer-1-planar graphs and the class of series-parallel graphs is exactly the class of outerplanar graphs, and Auer, et al.[2] showed that outer 1-planar graphs are planar and partial 3-trees. Since every 3-degenerate graph G is incidence $(\Delta(G)+4,3)$-colorable due to the result of Hosseini Dolama and Sopena[8], and partial 3-trees are clearly 3-degenerate, we have the following.

Theorem 1.7. $\chi_i(G)\leq\Delta(G)+4$ *holds for every outer-1-planar graph G.*

For other unmentioned results concerning the incidence coloring of graphs, a real-time online survey contributed by Éric Sopena[14] is highly recommended.

The goal of this paper is to improve the upper bound for $\chi_i(G)$ in Theorem 1.7 to $\Delta(G)+3$, and confirm PICC for outer-1-planar graphs G with $\Delta(G) \geq 8$ or $g(G) \geq 4$. Specifically, we prove the following theorems.

Theorem 1.8. *Every outer-1-planar graph G has an incidence $(\Delta(G)+3,2)$-coloring.*

Theorem 1.9. *Every outer-1-planar graph G with $\Delta(G) \geq 8$ has an incidence $(\Delta(G)+2,2)$-coloring.*

Theorem 1.10. *Every outer-1-planar graph G with $g(G) \geq 4$ has an incidence $(\Delta(G)+2,2)$-coloring.*

2 Reducible Configurations

Let Δ, p, q be positive integers and let G be a graph with $\Delta(G) \leq \Delta$. For a partial proper incidence $(\Delta+p)$-coloring φ of G, $\varphi(I_u)$ and $\varphi(A_u)$ denotes the set of colors that are used on the strong incidences and the weak incidences of u under φ, respectively. While extending φ to a proper incidence $(\Delta+p)$-coloring of G, the set of colors that cannot be used by a being colored incidence (u, uv) is therefore

$$F^{\varphi}(u, uv) = \varphi(I_u) \cup \varphi(A_u) \cup \varphi(I_v). \tag{2.1}$$

Formally, the set $F^{\varphi}(u, uv)$ is called the *forbidden set* for (u, uv), and the set

$$A^{\varphi}(u, uv) = C_p \setminus F^{\varphi}(u, uv) \tag{2.2}$$

is called the *available set* for (u, uv), where C_p denotes the set of colors $\{1, 2, \cdots, \Delta+p\}$.

A configuration of G is $(\Delta+p, q)$*-reducible* if it cannot occur in a minimal graph (in terms of the sum of the size and the order) which is not incidence $(\Delta+p, q)$-colorable.

Lemma 2.1. *A vertex v of degree 1 is $(\Delta+p,2)$-reducible for every integer $p \geq 2$.*

Proof. If G is an incidence non-$(\Delta+p,2)$-colorable minimal graph, then $G' = G - \{v\}$ is incidence $(\Delta+p,2)$-colorable. Let φ be an incidence $(\Delta+p,2)$-coloring of G' and let u be the neighbor of v. We extend φ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u, uv) and (v, uv) as follows.

First, since

$$|F^{\varphi}(u, uv)| \leq |\varphi(I_u)| + |\varphi(A_u)| + |\varphi(I_v)| \leq (\Delta-1)+2+0 = \Delta+1, \tag{2.3}$$

we can color (u, uv) with a color in $C_p \setminus F^{\varphi}(u, uv)$. This extended coloring of G is still denoted by φ and then we color (v, uv) with a color in $C_p \setminus F^{\varphi}(v, uv)$ if $|\varphi(A_u)| = 1$. This is possible since $F^{\varphi}(v, uv) = \varphi(I_u)$, which contains at most Δ colors. On the other hand, if $|\varphi(A_u)| = 2$, then color (v, uv) with a color in $\varphi(A_u)$. This results in an incidence $(\Delta+p,2)$-coloring of G as $\varphi(A_u) \cap \varphi(I_u) = \emptyset$. □

Lemma 2.2. *An edge uw with $d(u) = 2$ is*

(1) $(\Delta+2,2)$*-reducible for $\Delta \geq \gamma$ if $d(w) \leq \gamma - 1$, and*
(2) $(\Delta+3,2)$*-reducible.*

Proof. We set $p = 2$ or $p = 3$ while proving (1) or (2), respectively, and then use an uniform proof to prove both of them.

If G is an incidence non-$(\Delta + p, 2)$-colorable minimal graph, then $G' = G - \{u\}$ is incidence $(\Delta + p, 2)$-colorable. Let φ be an incidence $(\Delta + p, 2)$-coloring of G' and let v be the neighbor of u different from w. We extend φ to an incidence $(\Delta + p, 2)$-coloring of G by coloring (v, uv), (u, uv), (u, uw), and (w, uw) in such an ordering. In the following arguments, we always use φ to indicate the extended coloring at every stage .

First, since

$$|F^\varphi(v, uv)| \leq |\varphi(I_v)| + |\varphi(A_v)| + |\varphi(I_u)| \leq (\Delta - 1) + 2 + 0 = \Delta + 1, \tag{2.4}$$

we can color (v, uv) with $\varphi(v, uv) \in C_p \setminus F^\varphi(v, uv)$. Now we color (u, uv) as follows. Suppose first that $|\varphi(A_v)| = 1$. Since $\varphi(A_u) \subseteq \varphi(I_v)$,

$$|F^\varphi(u, uv)| \leq |\varphi(I_u)| + |\varphi(I_v)| \leq 0 + \Delta = \Delta. \tag{2.5}$$

It follows that there are at least $p \geq 2$ colors in $C_p \setminus F^\varphi(u, uv)$, from which we can choose one, say $\varphi(u, uv)$, to color (u, uv) so that $\{\varphi(v, uv), \varphi(u, uv)\} \neq \varphi(A_w)$. If $|\varphi(A_v)| = 2$, then $\varphi(A_v) \cap F^\varphi(u, uv) = \emptyset$ and thus we can choose one color, say $\varphi(u, uv)$, from $\varphi(A_v)$ to color (u, uv) such that $\{\varphi(v, uv), \varphi(u, uv)\} \neq \varphi(A_w)$.

Now we color (u, uw). If $|\varphi(A_w)| = 1$, then color (u, uw) with $\varphi(u, uw) \in C_p \setminus F^\varphi(u, uw)$. This is possible since

$$\begin{aligned} |F^\varphi(u, uw)| \leq& |\varphi(I_u)| + |\varphi(A_u)| + |\varphi(I_w)| \leq 1 + 1 + (d(w) - 1) \\ =& d(w) + 1 \leq \begin{cases} \Delta, & \text{if } \ p = 2, \\ \Delta + 1, & \text{if } \ p = 3 \end{cases} < \Delta + p. \end{aligned} \tag{2.6}$$

If $|\varphi(A_w)| = 2$, then

$$\begin{aligned} \varphi(A_w) \setminus F^\varphi(u, uw) &= \varphi(A_w) \setminus \big(\{\varphi(v, uv), \varphi(u, uv)\} \cup \varphi(I_w)\big) \\ &= \varphi(A_w) \setminus \{\varphi(v, uv), \varphi(u, uv)\} \neq \emptyset. \end{aligned} \tag{2.7}$$

Hence we can color (u, uw) with $\varphi(u, uw) \in \varphi(A_w) \setminus F^\varphi(u, uw)$.

Finally, since $\varphi(u, uw) \in \varphi(I_u) \cap \varphi(A_w)$,

$$\begin{aligned} |F^\varphi(w, uw)| \leq& |\varphi(I_w)| + |\varphi(A_w)| + |\varphi(I_u)| - 1 \leq (d(w) - 1) + 2 + 2 - 1 \\ =& d(w) + 2 \leq \begin{cases} \Delta + 1, & \text{if } \ p = 2, \\ \Delta + 2 & \text{if } \ p = 3 \end{cases} < \Delta + p. \end{aligned} \tag{2.8}$$

Hence we can complete an incidence $(\Delta + p, 2)$-coloring of G by coloring (u, uw) with $\varphi(w, uw) \in C_p \setminus F^\varphi(w, uw)$. □

Lemma 2.3. *A cycle $uxvyu$ with $d(u) = d(v) = 2$ is $(\Delta + p, 2)$-reducible for every integer $p \geq 2$.*

Proof. If G is an incidence non-$(\Delta + p, 2)$-colorable minimal graph, then $G' = G - \{u, v\}$ is incidence $(\Delta + p, 2)$-colorable. Let φ be an incidence $(\Delta + p, 2)$-coloring of G'. We aim to extend φ to an incidence $(\Delta + p, 2)$-coloring of G by coloring (u, ux), (u, uy), (v, vx), (v, vy), (x, ux), (x, vx), (y, uy), and (y, vy) properly.

Since $\varphi(I_u) = \varphi(I_v) = \emptyset$, $F^\varphi(x, ux) = F^\varphi(x, vx) = \varphi(I_x) \cup \varphi(A_x)$ and $A^\varphi(x, ux) = A^\varphi(x, vx)$. It follows that $|F^\varphi(x, ux)| \leq (\Delta - 2) + 2 = \Delta$ and thus both $A^\varphi(x, ux)$ and $A^\varphi(x, vx)$ have sizes at least $(\Delta + p) - \Delta = p \geq 2$. By symmetry, $A^\varphi(y, uy)$ and $A^\varphi(y, vy)$ are same and they have sizes at least 2. Without loss of generality, assume $A^{\varphi}(x,ux) = A^{\varphi}(x,vx) \supseteq \{\alpha_1,\alpha_2\}$ and $A^{\varphi}(y,uy) = A^{\varphi}(y,vy) \supseteq \{\beta_1,\beta_2\}$.

We extend φ to an incidence $(\Delta+p,2)$-coloring ϕ of G by coloring (x,ux), (x,vx), (y,uy), and (y,vy) with α_1, α_2, β_1, and β_2. By symmetry, we assume $|\varphi(A_x)| \le |\varphi(A_y)|$.

Case 1. There are two distinct colors a,b such that $\varphi(A_x) = \varphi(A_y) = \{a,b\}$.

Since $\{a,b\} \cap \{\alpha_1,\alpha_2,\beta_1,\beta_2\} \subseteq (\varphi(A_x) \cap (A^{\varphi}(x,ux)) \cup (\varphi(A_y) \cap A^{\varphi}(y,uy)) = \emptyset$, we can extend ϕ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u,ux) and (v,vx) with a, and (u,uy) and (v,vy) with b.

Case 2. There are three distinct colors a,b,c such that $\varphi(A_x) = \{a,c\}$ and $\varphi(A_y) = \{b,c\}$.

Without loss of generality, assume $a \ne \beta_1$. Since $\{a,c\} \cap \{\alpha_1,\alpha_2\} \subseteq \varphi(A_x) \cap A^{\varphi}(x,ux) = \emptyset$ and $\{b,c\} \cap \{\beta_1,\beta_2\} \subseteq \varphi(A_y) \cap A^{\varphi}(y,uy) = \emptyset$, we can extend ϕ to a partial incidence $(\Delta+p,2)$-coloring of G by coloring (v,vx) and (u,uy) with c, and (u,ux) with a. We now color (v,vy) with b. If $b \ne \alpha_2$, then we had already obtained an incidence $(\Delta+p,2)$-coloring of G, and if otherwise, then exchanging the colors of (x,ux) and (x,vx) also results in an incidence $(\Delta+p,2)$-coloring of G (note that in such a case $b \ne \alpha_1$ and $c \ne \alpha_2$).

Case 3. There are four distinct colors a,b,c,d such that $\varphi(A_x) = \{a,c\}$ and $\varphi(A_y) = \{b,d\}$.

Since $\{a,c\} \cap \{\alpha_1,\alpha_2\} \subseteq \varphi(A_x) \cap A^{\varphi}(x,ux) = \emptyset$ and $\{b,d\} \cap \{\beta_1,\beta_2\} \subseteq \varphi(A_y) \cap A^{\varphi}(y,uy) = \emptyset$, we can extend ϕ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u,ux) , (u,uy), (v,vx), and (v,vy) with a color in $\{a,c\} \setminus \{\beta_1\}$, $\{b,d\} \setminus \{\alpha_1\}$, $\{a,c\} \setminus \{\beta_2\}$, and $\{b,d\} \setminus \{\alpha_2\}$, respectively.

Case 4. There are three distinct colors a,b,c such that $\varphi(A_x) = \{a\}$ and $\varphi(A_y) = \{b,c\}$.

Without loss of generality, assume $a \ne \beta_1$ and $b \ne \alpha_1$. Since $\{a\} \cap \{\alpha_1,\alpha_2\} \subseteq \varphi(A_x) \cap A^{\varphi}(x,ux) = \emptyset$ and $\{b,c\} \cap \{\beta_1,\beta_2\} \subseteq \varphi(A_y) \cap A^{\varphi}(y,uy) = \emptyset$, we can extend ϕ to a partial incidence $(\Delta+p,2)$-coloring of G by coloring (u,uy) with b, (u,ux) with a, and (v,vy) with a color in $\{b,c\} \setminus \{\alpha_2\}$. We now color (v,vx) with a. If $a \ne \beta_2$, then we had already obtained an incidence $(\Delta+p,2)$-coloring of G. Hence we assume $a = \beta_2$. It follows $\alpha_2 \ne \beta_2$, and thus we recolor (v,vx) with α_2 and erase the colors of (x,vx) and (v,vy). The current partial incidence $(\Delta+p,2)$-coloring of G is denoted by ϕ'. Since $\alpha_2 \in \phi'(A_x) \cap \phi'(I_v)$ and $\alpha_1 \in \phi'(I_x)$,

$$|F^{\phi'}(x,vx)| \le |\phi'(I_x)| + |\phi'(A_x)| + |\phi'(I_v)| - 1 \le (\Delta-1) + 2 + 1 - 1 = \Delta + 1, \tag{2.9}$$

and thus we can color (x,vx) with a color $\alpha_3 \in C_p \setminus F^{\phi'}(x,vx)$ that is different from both α_1 and α_2. If $\{b,c\} \ne \{\alpha_2,\alpha_3\}$, then coloring (v,vy) with a color in $\{b,c\} \setminus \{\alpha_2,\alpha_3\}$ results in an incidence $(\Delta+p,2)$-coloring of G. On the other hand, if $\{b,c\} = \{\alpha_2,\alpha_3\}$, then exchange the colors of (x,ux) and (x,vx), color (v,vy) with α_3, and recolor (u,uy) with α_2 if necessary. This again gives an incidence $(\Delta+p,2)$-coloring of G.

Case 5. There are two distinct colors a,b such that $\varphi(A_x) = \{a\}$ and $\varphi(A_x) \cup \varphi(A_y) \subseteq \{a,b\}$.

Since

$$|F^{\varphi}(x,vx)| \le |\varphi(I_x)| + |\varphi(A_x)| + |\varphi(I_v)| \le (\Delta-2) + 1 + 0 = \Delta - 1, \tag{2.10}$$

$|A^{\varphi}(x,vx)| \ge (\Delta+p) - (\Delta-1) = p+1 \ge 3$, which implies that there is a new color α_3 such that $A^{\varphi}(x,vx) \supseteq \{\alpha_1,\alpha_2,\alpha_3\}$. It follows that $\{a\} \cap \{\alpha_1,\alpha_2,\alpha_3\} \subseteq \varphi(A_x) \cap A^{\varphi}(x,ux) = \emptyset$.

If $\varphi(A_y) = \{a,b\}$, then $\{a,b\} \cap \{\beta_1,\beta_2\} \subseteq \varphi(A_y) \cap A^{\varphi}(y,uy) = \emptyset$. Assume $b \notin \{\alpha_i,\alpha_j\}$ for some pair $\{i,j\} \subset \{1,2,3\}$. We can extend ϕ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u,ux) and (v,vx) with a, (u,uy) and (v,vy) with b, and then recoloring (x,xu) and (x,xv) with α_i and α_j if necessary, respectively.

If $\varphi(A_y) = \{a\}$ or $\varphi(A_y) = \{b\}$, then similarly, there is a new color β_3 such that $A^{\varphi}(y,vy) \supseteq \{\beta_1,\beta_2,\beta_3\}$. Without loss of generality, assume that $\alpha_i \ne \beta_i$ for $i = 1,2,3$, $a \notin \{\beta_k,\beta_l\}$ for some pair $\{k,l\} \subset \{1,2,3\}$, and $b \notin \{\alpha_i,\alpha_j\}$ for some pair $\{i,j\} \subset \{1,2,3\}$. Specifically, if $\varphi(A_y) = \{a\}$, then $\{a\} \cap \{\beta_1,\beta_2,\beta_3\} \subseteq \varphi(A_y) \cap A^{\varphi}(y,uy) = \emptyset$, and thus we can extend ϕ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u,ux), (u,uy), (v,vx), and (v,vy) with a, β_1, α_3, and a, and then recoloring (y,uy) and (y,vy) with β_2 and β_3, respectively. If $\varphi(A_y)=\{b\}$, then $\{b\}\cap\{\beta_1,\beta_2,\beta_3\}\subseteq\varphi(A_y)\cap A^{\varphi}(y,uy)=\emptyset$, and thus we can extend ϕ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u,ux) and (v,vx) with a, and (u,uy) and (v,vy) with b, and then recoloring (x,ux), (x,vx), (y,uy), and (y,vy) with α_i, α_j, β_k, and β_l if necessary, respectively. $\square$

Lemma 2.4. *A cycle $uxvyu$ with $d(u)=d(v)=3$ and $uv\in E(G)$ is*

(1) $(\Delta+2,2)$*-reducible for* $\Delta\geq 8$ *if* $d(y)\leq 7$*, and*
(2) $(\Delta+3,2)$*-reducible.*

Proof. We set $p=2$ or $p=3$ while proving (1) or (2), respectively, and then use an uniform proof to prove both of them.

If G is an incidence non-$(\Delta+p,2)$-colorable minimal graph, then $G'=G-\{u,v\}$ is incidence $(\Delta+p,2)$-colorable. Let φ be an incidence $(\Delta+p,2)$-coloring of G'. We aim to extend φ to an incidence $(\Delta+p,2)$-coloring of G by coloring (u,ux), (u,uy), (v,vx), (v,vy), (x,ux), (x,vx), (y,uy), (y,vy), (u,uv), and (v,uv) properly.

Since $\varphi(I_u)=\varphi(I_v)=\emptyset$, $F^{\varphi}(x,ux)=F^{\varphi}(x,vx)=\varphi(I_x)\cup\varphi(A_x)$ and $F^{\varphi}(y,uy)=F^{\varphi}(y,vy)=\varphi(I_y)\cup\varphi(A_y)$. It follows that $|F^{\varphi}(x,ux)|=|F^{\varphi}(x,vx)|\leq(\Delta-2)+2=\Delta$ and $A^{\varphi}(x,ux)=A^{\varphi}(x,vx)$ has size at least $(\Delta+p)-\Delta=p\geq 2$. If $p=2$, then $|F^{\varphi}(y,uy)|=|F^{\varphi}(y,vy)|\leq 5+2=7\leq\Delta-1$ (note that $d(y)\leq 7$ and $\Delta\geq 8$ in this case), and if $p=3$, then $|F^{\varphi}(y,uy)|=|F^{\varphi}(y,vy)|\leq(\Delta-2)+2=\Delta$. Hence $|F^{\varphi}(y,uy)|=|F^{\varphi}(y,vy)|\leq\Delta-3+p$ in each case and thus $A^{\varphi}(y,uy)=A^{\varphi}(y,vy)$ has size at least $(\Delta+p)-(\Delta-3+p)=3$. Without loss of generality, assume $A^{\varphi}(x,ux)=A^{\varphi}(x,vx)\supseteq\{\alpha_1,\alpha_2\}$ and $A^{\varphi}(y,uy)=A^{\varphi}(y,vy)\supseteq\{\beta_1,\beta_2,\beta_3\}$.

Case 1. $\{\alpha_1,\alpha_2\}\subset\{\beta_1,\beta_2,\beta_3\}$, in which case we assume $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$.

Subcase 1.1. There are colors $a\in\varphi(A_x)$ and $b\in\varphi(A_y)$ such that $a\neq b$.

Since $\{a\}\cap\{\alpha_1,\alpha_2\}\subseteq\varphi(A_x)\cap A^{\varphi}(x,ux)=\emptyset$ and $\{b\}\cap\{\alpha_1,\alpha_2\}\subseteq\varphi(A_y)\cap A^{\varphi}(y,uy)=\emptyset$, $\{a,b\}\cap\{\alpha_1,\alpha_2\}=\emptyset$. Hence we can extend φ to an incidence $(\Delta+p,2)$-coloring of G by coloring (x,ux), (y,uy), and (v,uv) with α_1, (x,vx), (y,vy), and (u,uv) with α_2, (u,ux) and (v,vx) with a, and (u,uy) and (v,vy) with b.

Subcase 1.2. There is a color a such that $\varphi(A_x)=\varphi(A_y)=\{a\}$.

Since $\{a\}\cap\{\alpha_1,\alpha_2\}\subseteq\varphi(A_x)\cap A^{\varphi}(x,ux)=\emptyset$, we can extend φ to a partial incidence $(\Delta+p,2)$-coloring ϕ of G by coloring (x,ux), (y,uy), and (v,uv) with α_1, (x,vx), (y,vy), and (u,uv) with α_2, and (u,ux) and (v,vy) with a. Since $\phi(I_u)\cup\phi(A_u)=\{a,\alpha_1,\alpha_2\}$ and $\{\alpha_1,\alpha_2\}\subseteq\phi(I_y)$,

$$\begin{aligned}|F^{\phi}(u,uy)|&\leq|\phi(I_u)|+|\phi(A_u)|+|\phi(I_y)|-2\leq 2+1+d(y)-2\\&=d(y)+1\leq\begin{cases}\Delta, & \text{if }\ p=2\\ \Delta+1, & \text{if }\ p=3\end{cases}<\Delta+p,\end{aligned}\tag{2.11}$$

This implies that we can color (u,uy) with a color in $C_p\setminus F^{\phi}(u,uy)$. Similarly, we can color (v,vx) with a color in $C_p\setminus F^{\phi}(v,vx)$, which is also nonempty, and then obtain an incidence $(\Delta+p,2)$-coloring of G.

Case 2. $\big|\{\alpha_1,\alpha_2\}\cap\{\beta_1,\beta_2,\beta_3\}\big|=1$, in which case we assume $\alpha_1=\beta_1$.

Subcase 2.1. There are colors $a\in\varphi(A_x)$ and $b\in\varphi(A_y)$ such that $a\neq b$.

Assume, without loss of generality, that $a\neq\beta_2$. Since $a\notin\{\alpha_1,\alpha_2\}$, $\alpha_1=\beta_1\neq b$, and $\beta_2\notin\{\alpha_1,b\}$, we can extend φ to a partial incidence $(\Delta+p,2)$-coloring ϕ of G by coloring (y,uy) and (v,uv) with α_1, (u,ux) and (v,vx) with a, (u,uy) with b, and (u,uv) and (y,vy) with β_2. If $\alpha_2\neq b$, then we can complete an incidence $(\Delta+p,2)$-coloring of G by coloring (x,ux) with α_1, (x,vx) with α_2, and (v,vy) with b. Hence we may assume $\alpha_2=b$.

If $|\varphi(A_x)| = 1$, then $|F^\varphi(x, ux)| = |F^\varphi(x, vx)| \le (\Delta - 2) + 1 = \Delta - 1$ and $A^\varphi(x, ux) = A^\varphi(x, vx)$ has size at least $(\Delta + p) - (\Delta - 1) = p + 1 \ge 3$. Hence there is a color $\alpha_3 \notin \{\alpha_1, \alpha_2\}$ such that $A^\varphi(x, ux) = A^\varphi(x, vx) \supseteq \{\alpha_1, \alpha_3\}$. If $\{\alpha_1, \alpha_3\} \subset \{\beta_1, \beta_2, \beta_3\}$, then we immediately come back to Case 1, and if $\{\alpha_1, \alpha_3\} \not\subset \{\beta_1, \beta_2, \beta_3\}$, then $\big|\{\alpha_1, \alpha_3\} \cap \{\beta_1, \beta_2, \beta_3\}\big| = 1$, in which case we may have $\alpha_3 = b$ by the same arguments as the above paragraph. However, this is impossible since $\alpha_3 \ne \alpha_2 = b$. Hence we may assume $|\varphi(A_x)| = 2$.

If $|\varphi(A_y)| = 1$, then we first extend ϕ to a partial incidence $(\Delta + p, 2)$-coloring ϕ' of G by coloring (x, ux) with α_1 and (x, vx) with α_2. Since $\alpha_1 \in \phi'(I_v) \cap \phi'(I_y)$ and $\beta_2 \in \phi'(A_v) \cup \phi'(I_y)$,

$$\begin{aligned} |F^{\phi'}(v, vy)| &\le |\phi'(I_v)| + |\phi'(A_v)| + |\phi'(I_y)| - 2 \le 2 + 2 + d(y) - 2 \\ &= d(y) + 2 \le \begin{cases} \Delta + 1, & \text{if } p = 2, \\ \Delta + 2, & \text{if } p = 3 \end{cases} < \Delta + p, \end{aligned} \tag{2.12}$$

we can complete an incidence $(\Delta + p, 2)$-coloring of G by coloring (v, vy) with a color in $C_p \setminus F^{\phi'}(v, vy)$ (note that $\Delta \ge 8$ if $p = 2$). Hence we may assume $|\varphi(A_y)| = 2$.

If $\varphi(A_y) \ne \{a, b\}$, then there is a color $c \in \varphi(A_y) \setminus \{a, b\}$. It is clear that coloring (x, ux) with α_1, (x, vx) with α_2, and (v, vy) with c results in an incidence $(\Delta + p, 2)$-coloring of G. Note that $c \notin \{a, \beta_1, b, \beta_2\} = \{a, \alpha_1, \alpha_2, \beta_2\}$. Hence we may further assume $\varphi(A_y) = \{a, b\}$.

Since $|\varphi(A_x)| = 2$, there is a color $c \in \varphi(A_x)$ such that $c \notin \{a, b\}$ (recall that $b = \alpha_2 \in A^\varphi(x, xu)$ and thus $b \notin \varphi(A_x)$). Since $\{\alpha_1, \alpha_2\} \subseteq A^\varphi(x, xu)$ and $\varphi(A_x) \cap A^\varphi(x, xu) = \emptyset$, $c \notin \{\alpha_1, \alpha_2\}$. We color (x, ux) with α_1, (x, vx) with α_2, (v, vy) with a, and recolor (v, vx) with c. This gives an incidence $(\Delta + p, 2)$-coloring of G provided $c \ne \beta_2$. Whereafter, if $c = \beta_2$, then $c \ne \beta_3$. Since $\beta_3 \ne \beta_1 = \alpha_1$ and $\beta_3 \notin \varphi(A_y) = \{a, b\}$, recoloring (u, uv) and (y, vy) with β_3 results in an incidence $(\Delta + p, 2)$-coloring of G.

Subcase 2.2. There is a color a such that $\varphi(A_x) = \varphi(A_y) = \{a\}$.

Since $\{a\} \cap \{\alpha_1, \alpha_2, \beta_2\} \subseteq \varphi(A_x) \cap (A^\varphi(x, ux) \cup A^\varphi(y, uy)) = \emptyset$, we can extend φ to a partial incidence $(\Delta + p, 2)$-coloring ϕ of G by coloring (x, ux), (y, uy), and (v, uv) with α_1, (x, vx) and (u, uv) with α_2, (u, uy) and (v, vx) with a, and (y, vy) with β_2.

Since $\phi(I_v) \cup \phi(A_v) = \{a, \alpha_1, \alpha_2, \beta_2\}$ and $\{\alpha_1, \beta_2\} \subseteq \phi(I_y)$,

$$\begin{aligned} |F^{\phi}(v, vy)| &\le |\phi(I_v)| + |\phi(A_v)| + |\phi(I_y)| - 2 \le 2 + 2 + d(y) - 2 \\ &= d(y) + 2 \le \begin{cases} \Delta + 1, & \text{if } p = 2, \\ \Delta + 2, & \text{if } p = 3 \end{cases} \\ &< \Delta + p. \end{aligned} \tag{2.13}$$

Since $\phi(I_u) \cup \phi(A_u) = \{a, \alpha_1, \alpha_2\}$ and $\{\alpha_1, \alpha_2\} \subseteq \phi(I_x)$,

$$|F^{\phi}(u, ux)| \le |\phi(I_u)| + |\phi(A_u)| + |\phi(I_x)| - 2 \le 2 + 1 + \Delta - 2 = \Delta + 1. \tag{2.14}$$

Hence we can respectively color (v, vy) and (u, ux) with a color in $C_p \setminus F^\phi(v, vy)$ and $C_p \setminus F^\phi(u, ux)$, completing an incidence $(\Delta + p, 2)$-coloring of G.

Case 3. $\big|\{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2, \beta_3\}\big| = 0$.

Subcase 3.1. There are colors $a \in \varphi(A_x)$ and $b \in \varphi(A_y)$ such that $a \ne b$.

Assume, without loss of generality, that $a \notin \{\beta_1, \beta_2\}$ and $b \ne \alpha_1$. Since $\{a, b\} \cap \{\alpha_1, \beta_1, \beta_2\} = (\{a\} \cap \{\alpha_1\}) \cup (\{b\} \cap \{\beta_1, \beta_2\}) \subseteq (\varphi(A_x) \cap A^\varphi(x, ux)) \cup (\varphi(A_y) \cap A^\varphi(y, uy)) = \emptyset$, we can extend φ to a partial incidence $(\Delta + p, 2)$-coloring ϕ of G by coloring (x, ux) with α_1, (x, vx) with α_2, (y, vy) and (u, uv) with β_1, (y, uy) and (v, uv) with β_2, (u, ux) and (v, vx) with a, and (u, uy) with b. If $\alpha_2 \ne b$, then we can obtain an incidence $(\Delta+p, 2)$-coloring of G by coloring (v, vy) with b. Hence we may assume $\alpha_2 = b$. Since $\phi(I_v) \cup \phi(A_v) = \{a, \alpha_2, \beta_1, \beta_2\}$ and $\{\beta_1, \beta_2\} \subseteq \phi(I_y)$,

$$|F^{\phi}(v, vy)| \le |\phi(I_v)| + |\phi(A_v)| + |\phi(I_y)| - 2 \le 2 + 2 + d(y) - 2$$

$$= d(y) + 2 \leq \begin{cases} \Delta + 1, & \text{if } p = 2, \\ \Delta + 2, & \text{if } p = 3 \end{cases}$$
$$< \Delta + p. \tag{2.15}$$

So it is possible to color (v, vy) with a color in $C_p \setminus F^{\phi}(v, vy)$ to complete an incidence $(\Delta+p, 2)$-coloring of G.

Subcase 3.2. There is a color a such that $\varphi(A_x) = \varphi(A_y) = \{a\}$.

Since $\varphi(I_u) = \varphi(I_v) = \emptyset$, $F^{\varphi}(x, ux) = F^{\varphi}(x, vx) = \varphi(I_x) \cup \varphi(A_x)$. It follows that $|F^{\varphi}(x, ux)| = |F^{\varphi}(x, vx)| \leq (\Delta - 2) + 1 = \Delta - 1$ and $A^{\varphi}(x, ux) = A^{\varphi}(x, vx)$ has size at least $(\Delta + p) - (\Delta - 1) = p + 1 \geq 3$. Assume $A^{\varphi}(x, ux) = A^{\varphi}(x, vx) \supseteq \{\alpha_1, \alpha_2, \alpha_3\}$, and assume further, without loss of generality, that $\alpha_3 \neq \beta_2$. Since $\{a\} \cap \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \subseteq (\varphi(A_x) \cap (A^{\varphi}(x, ux)) \cup (\varphi(A_y) \cap A^{\varphi}(y, uy)) = \emptyset$, we can extend φ to an incidence $(\Delta + p, 2)$-coloring of G by coloring (x, ux) and (v, uv) with α_1, (x, vx) with α_2, (v, vx) with α_3, (y, uy) with β_1, (y, vy) and (u, uv) with β_2, (u, ux) and (v, vy) with a, and (u, uy) with β_3. □

3 Proofs of the Main Theorems

Before proving Theorem 1.8, Theorem 1.9, and Theorem 1.10, we import some interesting structural lemmas for outer-1-planar graphs. The following two lemmas are direct corollaries of [17].

Lemma 3.1[17]**.** *Every outer-1-planar graph G contains*

- **(a3.1)** *a vertex v of degree at most 2, or*
- **(b3.1)** *a cycle $uxvyu$ with $d(u) = d(v) = 3$ and $uv \in E(G)$.*

Lemma 3.2[17]**.** *Every outer-1-planar graph G with $g(G) \geq 4$ contains*

- **(a3.2)** *a vertex v of degree 1, or*
- **(b3.2)** *an edge uv with $d(u) = d(v) = 2$, or*
- **(c3.2)** *a cycle $uxvyu$ with $d(u) = d(v) = 2$.*

Recently, Li and Zhang[11] improved Lemma 3.1 to the following.

Lemma 3.3[11]**.** *Every outer-1-planar graph G contains*

- **(a3.3)** *a vertex v of degree 1, or*
- **(b3.3)** *an edge uv with $d(u) = 2$ and $d(v) \leq 5$, or*
- **(c3.3)** *a cycle $uxvyu$ with $d(u) = d(v) = 2$, or*
- **(d3.3)** *a cycle $uxvyu$ with $d(u) = d(v) = 3$, $uv \in E(G)$, and $d(y) \leq 7$.*

Instead of proving Theorem 1.8, Theorem 1.9, and Theorem 1.10 directly, we prove slightly stronger results as follows.

Theorem 3.4. *Every outer-1-planar graph G with $\Delta(G) \leq \Delta$ has an incidence $(\Delta + 3, 2)$-coloring.*

Proof. Suppose, for a contradiction, that there is an incidence non-$(\Delta+3, 2)$-colorable minimal (in terms of $|V(G)| + |E(G)|$) outer-1-planar graph G with $\Delta(G) \leq \Delta$. By Lemma 3.1, G contains (as an outer-1-planar graph) the configuration **(a3.1)** or **(b3.1)**. However, **(a3.1)** is $(\Delta+3, 2)$-reducible by Lemma 2.1 and **(2)** of Lemma 2.2, and **(b3.1)** is $(\Delta+3, 2)$-reducible by **(2)** of Lemma 2.4, a contradiction. □

Theorem 3.5. *Every outer-1-planar graph G with $\Delta(G) \leq \Delta$ and $\Delta \geq 8$ has an incidence $(\Delta+2,2)$-coloring.*

Proof. Suppose, for a contradiction, that there is an incidence non-$(\Delta+2,2)$-colorable minimal (in terms of $|V(G)|+|E(G)|$) outer-1-planar graph G with $\Delta(G) \leq \Delta$. By Lemma 3.3, G contains at least one configuration among **(a**3.3**)**, **(b**3.3**)**, **(c**3.3**)**, and **(d**3.3**)**. However, **(a**3.3**)**, **(b**3.3**)**, **(c**3.3**)**, and **(d**3.3**)** are respectively $(\Delta+2,2)$-reducible by Lemma 2.1, **(1)** of Lemma 2.2, Lemma 2.3, and **(1)** of Lemma 2.4, a contradiction. □

Theorem 3.6. *Every outer-1-planar graph G with $g(G) \geq 4$ and $\Delta(G) \leq \Delta$ has an incidence $(\Delta+2,2)$-coloring.*

Proof. If $\Delta \leq 2$, then the result is trivial, since every cycle is incidence $(4,2)$-colorable and every path is incidence $(3,2)$-colorable. So we assume $\Delta \geq 3$. Suppose, for a contradiction, that there is an incidence non-$(\Delta+2,2)$-colorable minimal (in terms of $|V(G)|+|E(G)|$) outer-1-planar graph G with $\Delta(G) \leq \Delta$. By Lemma 3.2, G contains at least one configuration among **(a**3.2**)**, **(b**3.2**)**, and **(c**3.2**)**. However, **(a**3.2**)**, **(b**3.2**)**, and **(c**3.2**)** are respectively $(\Delta+2,2)$-reducible by Lemma 2.1, **(1)** of Lemma 2.2, and Lemma 2.3, a contradiction. □

Conflict of Interest

The authors declare no conflict of interest.

References

[1] Algor, I., Alon, N. The star arboricity of graphs. *Discrete Math.*, 75: 11–22 (1989)

[2] Auer, C., Bachmaier, C., Brandenburg, F.J., Gleißner, A., Hanauer, K., Neuwirth, D., Reislhuber, J. Recognizing outer 1-planar graphs in linear time. In: S. Wismath, A. Wolff (eds.), Graph Grawing, GD 2013, Lecture Notes in Comput. Sci., 2013, 8242: 107–118

[3] Auer, C., Bachmaier, C., Brandenburg, F.J., Gleißner, A., Hanauer, K., Neuwirth, D., Reislhuber, J. Outer-1-planar graphs. *Algorithmica*, 74: 1293–1320 (2016)

[4] Bonamy, M., Lévêque, B., and Pinlou, A. 2-distance coloring of sparse graphs. *J. Graph Theory*, 77(3): 190–218 (2014)

[5] Bondy, J.A., Murty, U.S.R. Graph Theory. Springer, 2008

[6] Brualdi, R.A., Quinn Massey, J.J. Incidence and strong edge colorings of graphs. *Discrete Math.*, 122: 51–58 (1993)

[7] Guiduli, B. On incidence coloring and star arboricity of graphs. *Discrete Math.*, 163: 275–278 (1997)

[8] Hosseini Dolama, M., Sopena, É. On the maximum average degree and the incidence chromatic number of a graph. *Discrete Math. Theoret. Comput. Sci.*, 7: 203–216 (2005)

[9] Hosseini Dolama, M., Sopena, É., Zhu, X. Incidence coloring of k-degenerated graphs. *Discrete Math.*, 283(1-3): 121–128 (2004)

[10] Kardoš, F., Maceková, M., Mockovčiaková, M., Sopena, É., Soták, R. Incidence coloring - Cold cases. *Discuss. Math. Graph Theory*, 40: 345–354 (2020)

[11] Li, Y., Zhang, X. The structure and the list 3-dynamic coloring of outer-1-planar graphs. *Discrete Math. Theoret. Comput. Sci.*, 23(3): #4 (2021)

[12] Maydanskiy, M. The incidence coloring conjecture for graphs of maximum degree three. *Discrete Math.*, 292: 131–141 (2005)

[13] Shiu, W.C. Sun, P.K. Invalid proofs on incidence coloring. *Discrete Math.*, 308(24): 6575–6580 (2008)

[14] Sopena, É. www.labri.fr/perso/sopena/TheIncidenceColoringPage.

[15] Wang, W.F., Lih, K.W. Labeling planar graphs with conditions on girth and distance two. *SIAM J. Discrete Math.*, 17(2): 264–275 (2003)

[16] Yang, D. Fractional incidence coloring and star arboricity of graphs. *Ars Combin.*, 105: 213–224 (2012)

[17] Zhang, X. List total coloring of pseudo-outerplanar graphs. *Discrete Math.*, 313: 2297–2306 (2013)

[18] Zhang, X., Liu, G., Wu, J.L. Edge covering pseudo-outerplanar graphs with forests. *Discrete Math.*, 312: 2788–2799 (2012)