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Strong Edge Coloring of Outerplane Graphs with Independent Crossings

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Abstract The strong chromatic index of a graph is the minimum number of colors needed in a proper edge coloring so that no edge is adjacent to two edges of the same color. An outerplane graph with independent crossings is a graph embedded in the plane in such a way that all vertices are on the outer face and two pairs of crossing edges share no common end vertex. It is proved that every outerplane graph with independent crossings and maximum degree Δ has strong chromatic index at most $4\Delta-6$ if $\Delta\geq 4$, and at most 8 if $\Delta\leq 3$. Both bounds are sharp.

Keywords outer-1-planar graph; IC-planar graph; strong edge coloring; crossing;
2020 MR Subject Classification 05C15; 05C10

1 Introduction

All graphs in this paper are simple and finite. For a graph G, we denote its vertex set, edge set, minimum degree, and maximum degree as $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$, respectively. By $N_G(v)$ we denote the set of the vertices adjacent to v in G. A vertex $v\in V(G)$ is a *k-vertex* (or *k^+-vertex*) if the *degree* $d_G(v):=|N_G(v)|$ of v in G is k (or at least k). For a vertex $v\in V(G)$, we denote by $E_G(v)$ the set of edges incident with v in G. For a set $S\subseteq V(G)$, let $E_G(S)=\bigcup\limits_{v\in S}E_G(v)$.

For an integer k, let $[k]=\{1,2,\cdots,k\}$. A *proper edge k-coloring* of a graph G is a mapping $\varphi: E(G)\to[k]$ so that $\varphi(e_1)\neq\varphi(e_2)$ if e_1 is adjacent to e_2 in G. A *strong edge k-coloring* is a proper edge k-coloring so that no edge is adjacent to two edges of the same color. The minimum integer k so that G has a strong edge k-coloring is the *strong chromatic index* of G, denoted by $\chi'_s(G)$.

The notion of strong edge coloring was firstly introduced by Fouquet and Jolivet[7] in 1983. At the end of 1985, Erdős and Nešetřil raised the following conjecture at a seminar in Prague:

Conjecture 1.1. *If G is a graph with maximum degree Δ, then*

$$\chi'_s(G)=\begin{cases}\dfrac{5}{4}\Delta^2 & \text{if } n \text{ is even},\\ \dfrac{1}{4}(5\Delta^2-2\Delta+1) & \text{if } n \text{ is odd}.\end{cases}$$

Since for every edge $e\in E(G)$, there are at most $2(\Delta(G)-1)\Delta(G)$ edges that are at distance at most 2 from e, we can greedily color the edges of G to obtain a strong edge coloring using

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$2(\Delta(G)-1)\Delta(G)+1$ colors. Therefore we trivially have $\chi'_s(G) \le 2\Delta(G)^2-2\Delta(G)+1 < 2\Delta(G)^2$. An interesting question relative to Conjecture 1.1 is to find a constant $\gamma < 2$ independent of G so that $\chi'_s(G) \le \gamma\Delta(G)^2$.

Surprisingly, the first novel result concerning this problem appeared in 1997, when Molloy and Reed[14] proved using probabilistic method that $\chi'_s(G) \le 1.998\Delta(G)^2$ provided $\Delta(G)$ is sufficiently large. The next improvement is due to Bruhn and Joos[4], who pulled the coefficient before $\Delta(G)^2$ down to 1.93 in 2015. Three years later, Bonamy, Perrett, and Postle[2] improved it to 1.835 (the journal version of this result was published in 2022[3]). The best known result until now is due to Hurley, de Verclos, and Kang [11], who proved in 2021 that $\chi'_s(G) \le 1.772\Delta(G)^2$ provided $\Delta(G)$ is sufficiently large. For P_5-free graphs G, Xu and Zhang[18] showed that $\chi'_s(G) \le 1.25\Delta(G)^2$. In particular, their result confirmed Conjecture 1.1 for P_5-free graphs with even maximum degree.

For planar graphs G, Faudree *et al.* [6] proved $\chi'_s(G) \le 4\Delta(G) + 4$. More generally, they showed if G is a graph in a minor-closed class $\mathcal{G}$ then $\chi'_s(G) \le \chi(\mathcal{G})\chi'(\mathcal{G})$, where $\chi(\mathcal{G})$ and $\chi'(\mathcal{G})$ denote the chromatic number and the chromatic index of the class $\mathcal{G}$ respectively. For planar graphs G with girth g, Guo, Zhang, and Zhang[8] showed that $\chi'_s(G) \le 3\Delta(G) - 2$ if $g \ge 8$, and $\chi'_s(G) \le 3\Delta(G) - 3$ if $g \ge 10$.

There are various famous subclasses of planar graphs in the literature.

A graph is *outerplanar* if it has a plane embedding so that all vertices lie on the outer face of the drawing. Hocquard, Ochem and Valicov[9] proved that every outerplanar graph G with $\Delta(G) \ge 3$ has $\chi'_s(G) \le 3\Delta(G) - 3$ and this bound is sharp.

A graph is *series-parallel* if it contains no subgraph isomorphic to a subdivision of a complete graph K_4 of four vertices. Wang, Wang and Wang [17] proved $\chi'_s(G) \le 3\Delta(G) - 2$ (being sharp) for every series-parallel graph G with $\Delta(G) \ge 3$. Note that a graph is outerplanar if and only if it contains no subgraph isomorphic to a subdivision of K_4 or a subdivision of a complete bipartite graph $K_{2,3}$. So a graph is outerplanar only if it is series-parallel.

A graph is *outer 1-planar* if it can be drawn in the plane so that all vertices are on the outer face and each edge is crossed at most once. This notion was firstly introduced in 1986 by Eggleton[5] who called them *outerplanar graphs with edge crossing number one* and were also investigated under the notion of *pseudo-outerplanar graphs*[23]. The outer-1-planarity generalizes the outerplanarity, and is also a combination of the outerplanarity and the 1-planarity. Formally, a graph is *1-planar* if it has a plane embedding so that each edge is crossed at most once. It is necessary to point out that every outer-1-planar graph is planar[1] and the class of outer-1-planar graphs is not minor-closed[10].

It may be interesting to say something by combining the outerplanarity with some other beyond-planarity. For example, a graph is *NIC-planar* if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing edges share at most one common end vertex[20]. By combining the outerplanarity with the NIC-planarity, Zhang[21] introduced a new graph class say *outerplane graphs with near-independent crossings* or *outer-NIC-planar graphs* and then investigated the total coloring problems on such a class.

As a subclass of NIC-planar graphs, the class of IC-planar graphs is well investigated in the literature (see e.g. [12, 13, 15, 16, 19, 22]). A graph is *IC-planar* if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing edges share no common end vertex. Combining the outerplanarity with IC-planarity, we obtain a new graph class say *outerplane graphs with independent crossings* or *outer-IC-planar graphs*. Formally, a graph is *outer-IC-planar* if it can be drawn in the plane so that all vertices are on the outer face and two pairs of crossing edges share no common end vertex, and such a drawing is called an *outerplane graphs with independent crossings*.

We investigate the strong edge coloring of outer-IC-planar graphs by proving the following.

Theorem 1.2. *If G is an outer-IC-planar graph with $\Delta(G) \geq 3$, then*

$$\chi'_s(G) \leq \begin{cases} 8, & \text{if } \Delta(G) = 3 \\ 4\Delta(G) - 6, & \text{if } \Delta(G) \geq 4. \end{cases}$$

It is easy to see that every pair of edges in each configuration are at distance at most 2, and the left (resp. right) configuration has exactly 8 (resp. $4\Delta - 6$) edges. Hence if an outer-IC-planar graph G contains the left (resp. right) configuration as a subgraph, then its strong chromatic index is exactly 8 (resp. $4\Delta - 6$) by Theorem 1.2.

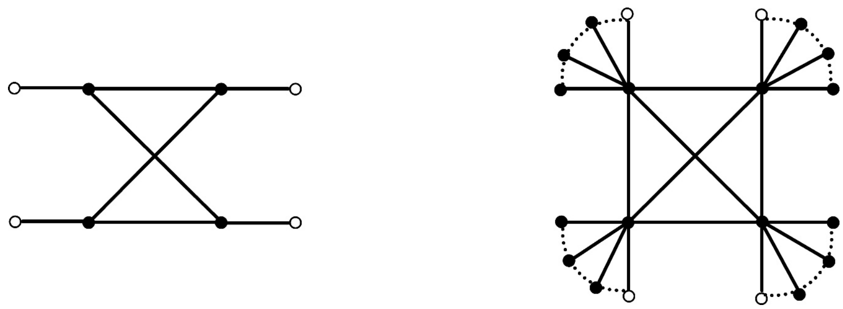

Figure 1.1. The sharpness of Theorem 1.2.

2 Structure of the Minimal Counterexample

A *pendant edge* of a graph G is an edge uv with $d_G(u) = 1$. We define the *partial order* $\prec$ on graphs such that $G_1 \prec G_2$ if and only if.

- $|E(G_1)| < |E(G_2)|$ or
- $|E(G_1)| = |E(G_2)|$ and G_1 contains more pendant edges than G_2.

Let $\mathcal{G}$ be a class of graphs. A graph G is k-minimal in $\mathcal{G}$ if

- $\chi'_s\ (G) > k$, and
- $\chi'_s\ (H) \leq k$ for every graph $H \in \mathcal{G}$ with $H \prec G$.

Let Δ be an integer and let $\mathcal{O}_\Delta$ be the class of outer-IC-planar graphs with $\Delta(G) \leq \Delta$. Instead of proving Theorem 1.2, we prove a slightly stronger result as follows.

Theorem 2.1. *If $G \in \mathcal{O}_\Delta$, then*

$$\chi'_s(G) \leq f(\Delta) = \begin{cases} 8, & \text{if } \Delta = 3, \\ 4\Delta - 6, & \text{if } \Delta \geq 4. \end{cases}$$

Suppose for a contradiction that Theorem 2.1 is false. Then we would find many counterexamples to that result, from which we choose G to be a minimum counterexample in terms of $\prec$. In other words, G is $f(\Delta)$-minimal in $\mathcal{O}_\Delta$. Moreover, we assume that G is already an outerplane graph with independent crossings.

In this section, we explore the structural properties of G and then use them to prove Theorem 2.1 in the next section. Now we need some additional notations that will be used afterwards.

If the vertices of G are labelled by $v_1, v_2, \cdots, v_n$, which lie in a clockwise ordering on the outer face, then we say that v_{i+1} or v_{i-1} is a *right-vertex* or a *left-vertex* of v_i, denoted by $v_{i+1}\mathbf{R}v_i$ or $v_{i-1}\mathbf{L}v_i$, respectively. For two distinct vertices v_i and v_j, we let

$$\mathcal{V}[v_i, v_j] = \begin{cases} \{v_i, v_{i+1}, \cdots, v_{j-1}, v_j\}, & \text{if } i < j, \\ \{v_i, v_{i+1}, \cdots, v_n, v_1, \cdots, v_{j-1}, v_j\}, & \text{if } i > j. \end{cases}$$

and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j] \setminus \{v_i, v_j\}$.

For two vertices $u, v \in V(G)$, the set $\mathcal{V}(u, v)$ is a *non-edge* if $u\mathbf{L}v$ and $uv \notin E(G)$, and is a *path* if either $u\mathbf{L}v$ and $uv \in E(G)$, or for every vertex $z \in \mathcal{V}(u, v)$ there exists vertices $x, y \in \mathcal{V}[u, v]$ such that $x\mathbf{L}z\mathbf{L}y$ and $xz, yz \in E(G)$.

An edge $uv \in E(G)$ is a *boundary edge* if either $u\mathbf{R}v$ or $u\mathbf{L}v$, and is a *chord* otherwise.

The subgraph obtained from G by removing all 1-vertices is denoted by G^*.

Lemma 2.2. *G^* is 2-connected and thus $\delta(G^*) \geq 2$.*

Proof. If not, then there would be a cut vertex v in G^*. Let $G_1, G_2, \cdots, G_p$ ($p \geq 2$) be the connected components of $G^* - v$. Let $S = E_{G^*}(v)$. The graph induced by $E(G_i) \cup S$ is denoted by G_i' for each $1 \leq i \leq p$. Since $G_i' \prec G$ and $G_i' \in \mathcal{O}_\Delta$, G_i' has a strong edge $f(\Delta)$-coloring φ_i. Now every two edges in S receive distinct colors under each φ_i. This allows us to permute the colors of each φ_i so that $\varphi_1(e) = \cdots = \varphi_p(e)$ for every $e \in S$. Combining $\varphi_1, \cdots, \varphi_p$ together, we obtain a strong edge $f(\Delta)$-coloring of G, a contradiction. □

Lemma 2.3. *Every chord of G is crossed.*

Proof. Suppose, for a contradiction G has a non-crossed chord xy. Let $S = E_G(x) \cup E_G(y)$ and let $G_1, G_2, \cdots, G_p$ ($p \geq 2$) be the connected components of $G - S$. The graph induced by $E(G_i) \cup S$ is denoted by G_i'. Clearly, $G_i' \in \mathcal{O}_\Delta$ for each $1 \leq i \leq p$.

If $G_i' \prec G$ for each $1 \leq i \leq p$, then G_i' has a strong edge $f(\Delta)$-coloring φ_i. Since every two edges of S is at distance at most two in each G_i', they receive distinct colors under each φ_i. Hence we permute the colors of each φ_i so that $\varphi_1(e) = \cdots = \varphi_p(e)$ for every $e \in S$, and then obtain a strong edge $f(\Delta)$-coloring of G by combining $\varphi_1, \cdots, \varphi_p$ together, a contradiction.

If $G_j' \not\prec G$ for some $1 \leq j \leq p$, then $G_j' \cong G$. It follows that there would be a vertex z such that $xz, yz \in E(G^*)$ and z is not incident with any pendant edges. We split z into two new vertices z_1 and z_2 so that xz_1, yz_2 are edges and z_1z_2 is a non-edge, and denote the resulting graph by H. Since $|E(H)| = |E(G)|$ and H has two more pendant edges than G, $H \prec G$. It follows that H has a strong edge $f(\Delta)$-coloring ϕ such that $\phi(xz_1) \neq \phi(yz_2)$. Now we restore a strong edge $f(\Delta)$-coloring φ of G from ϕ by letting $\varphi(xz) = \phi(xz_1)$, $\varphi(yz) = \phi(yz_2)$, and $\varphi(e) = \phi(e)$ for each $e \in E(G) \setminus \{xz, yz\}$, a contradiction. □

Lemma 2.4. *Every vertex of G is incident with at most one chord, so $d_{G^*}(u) \leq 3$ for every $u \in V(G^*)$.*

Proof. If not, then some vertex of G is incident with at least two crossed chords by Lemma 2.3. This is impossible as every pair of crossings of G are independent. □

Lemma 2.5. *If u is a vertex of G with at least one pendant edge, then*

(1) $\Delta \geq 4$;
(2) $d_{G^*}(u) = 3$;
(3) $d_{G^*}(v) = 3$ for every $v \in N_{G^*}(u)$ provided $\Delta \geq 5$.

Proof.

$$\{\varphi(e) \mid e \in E_{G-uw}(u) \cup E_{G-u}(N_G(u))\}.$$

If $\Delta \leq 3$, then $|F^\varphi(uw)| \leq 2 + 2 \times 2 = 6 < 8 = f(3)$, and thus φ can be extended to G by coloring uw with a color not in $F^\varphi(uw)$. This contradiction implies $\Delta \geq 4$ and thus proves (1).

If $|N_{G^*}(u)| \leq 2$, then $|F^\varphi(uw)| \leq (\Delta - 1) + 2(\Delta - 1) = 3\Delta - 3 < 4\Delta - 6 = f(\Delta)$ for $\Delta \geq 4$, and thus φ can also be extended to G, a contradiction implying $|N_{G^*}(u)| \geq 3$. Since $|N_{G^*}(u)| \leq 3$ by Lemma 2.4, $|N_{G^*}(u)| = 3$. This proves (2).

If $\Delta \geq 5$, $uv \in E(G^*)$ and $d_{G^*}(v) \neq 3$, then $d_{G^*}(v) = 2$ by Lemmas 2.2 and 2.4. We further have $d_G(v) = 2$ by (2). So $|F^\varphi(uw)| \leq (\Delta - 1) + 1 + 2(\Delta - 1) = 3\Delta - 2 < 4\Delta - 6 = f(\Delta)$, and thus φ can be extended again. This proves (3). □

Lemma 2.6. *Any two 2-verteices of G^* are not adjacent.*

Proof. Suppose u and v are two adjacent 2-vertices of G^*. By Lemma 2.5(2), $d_G(u) = d_G(v) = 2$. Let $x \in N_G(u) \setminus \{v\}$ and $y \in N_G(v) \setminus \{u\}$. Since $G - uv \prec G$ and $G - uv \in \mathcal{O}_\Delta$, $G - uv$ has a strong edge $f(\Delta)$-coloring φ. Now the set $F^\varphi(uv)$ of forbidden colors for uw is

$$\{\varphi(e) \mid e \in E_G(x) \cup E_G(y)\}.$$

Since $\Delta \geq 3$, $|F^\varphi(uv)| \leq 2\Delta < f(\Delta)$. Hence we can extend φ to G by coloring uv with a color not in $F^\varphi(uv)$. □

Lemma 2.7. *If $\Delta \geq 5$, then any 3-vertex of G^* is adjacent to at most one 2-vertex in G^*.*

Proof. Suppose that u has exactly three neighbors x, y, z in G^* such that $d_{G^*}(x) = d_{G^*}(y) = 2$. By Lemmas 2.5(2) and 2.5(3), u, x, or y is not incident with any pendant edge and thus $d_G(u) = 3$ and $d_G(x) = d_G(y) = 2$. Let $x' \in N_G(x) \setminus \{u\}$.

Since $G - ux \prec G$ and $G - ux \in \mathcal{O}_\Delta$, $G - ux$ has a strong edge $(4\Delta - 6)$-coloring φ. Now the set $F^\varphi(ux)$ of forbidden colors for ux is

$$\{\varphi(e) \mid e \in E_G(x') \cup E_G(y) \cup E_G(z)\},$$

which has size at most $\Delta + 2 + \Delta = 2\Delta + 2 < 4\Delta - 6$. Hence we can extend φ to G by coloring ux with a color not in $F^\varphi(ux)$. □

Lemma 2.8. *If a 3-vertex of G is adjacent to two 2-vertices x, y in G^*, then $N_G(x) \neq N_G(y)$.*

Proof. By Lemma 2.5(2), $d_G(x) = d_G(y) = 2$. Suppose for a contradiction that $N_G(x) = N_G(y) = \{u, v\}$ and $N_G(u) = \{x, y, z\}$. By the minimality of G, $G - ux$ has a strong edge $f(\Delta)$-coloring φ. Now the set $F^\varphi(ux)$ of forbidden colors for ux is

$$\{\varphi(e) \mid e \in E_G(y) \cup E_G(v) \cup E_G(z)\},$$

having size at most $2 + \Delta + \Delta - 1 = 2\Delta + 1 < f(\Delta)$. Hence φ can be extended to G, a contradiction. □

Lemma 2.9. *If u is a vertex of G incident with at least one pendant edge, then u is adjacent to at most one 2-vertex, and furthermore, if u is adjacent to exactly one 2-vertex, then other neighbors of u in G^* are 4^+-vertices.*

Proof. By Lemma 2.5(1), $\Delta \geq 4$. By Lemma 2.5(2), incident with a pendant edge uw, u has three neighbors, say u_1, u_2 and u_3, in G^*.

If $d_G(u_1) = 2$ and $d_G(u_2) \leq 3$, the set $F^\varphi(uw)$ of forbidden colors for uw is

$$\{\varphi(e) \mid e \in E_G(u_1) \cup E_G(u_2) \cup E_G(u_3) \cup E_{G-uw}(u)\}.$$

Since $|F^\varphi(uv)| \leq 2 + 3 + \Delta + \Delta - 1 - 3 = 2\Delta + 1 < 4\Delta - 6$, we can extend φ to G by coloring uw with a color not in $F^\varphi(uw)$, a contradiction. This proves the required result. □

Lemma 2.10. Assume that ux and vy are two mutually crossed chords of G so that u, v, x, y are located in a clockwise ordering on the outer face of G and the size of $\mathcal{V}[v, u]$ is minimum.

(1) Each of $\mathcal{V}(v, x)$, $\mathcal{V}(x, y)$ and $\mathcal{V}(y, u)$ forms a non-edge or a path.
(2) If $uv \notin E(G)$, then either u or v is not incident with pendant edges.
(3) $d_G(z) = 2$ for every vertex $z \in \mathcal{V}(v, u) \setminus \{x, y\}$ and each of $\mathcal{V}(v, x)$, $\mathcal{V}(x, y)$, and $\mathcal{V}(y, u)$ has size at most one.
(4) If $\Delta \geq 4$, then $\mathcal{V}(v, x)$ and $\mathcal{V}(u, y)$ are not non-edges.
(5) If $\Delta \geq 4$, then $uv \notin E(G)$ and $|E_G(u) \cup E_G(v)| \leq \Delta + 3$.
(6) If $\Delta \geq 5$, then vx and uy are boundary edges.

Proof. (1). If $\mathcal{V}(v, x)$ does not form a non-edge, then there is a vertex $z \in \mathcal{V}(v, x) \neq \emptyset$. If further $\mathcal{V}(v, x)$ does not form a path, then the vertex v or x would separate z from y, contradicting Lemma 2.2. So $\mathcal{V}(v, x)$ forms a non-edge or a path, and so do $\mathcal{V}(x, y)$ and $\mathcal{V}(y, u)$.

(2). Suppose, for a contradiction, that there are two pendant edges uu' and vv'. Let G' be the graph derived from G by removing edges uu' and vv' and adding new edge uv. Since G' has less edges than G and $\Delta(G') = \Delta(G)$, $G' \prec G$ and $G' \in \mathcal{O}_\Delta$. So G' has a strong edge $f(\Delta)$-coloring φ. Now we construct a strong edge $f(\Delta)$-coloring ϕ of G by setting $\phi(uu') = \phi(vv') = \varphi(uv)$ and $\phi(e) = \varphi(e)$ for each $e \in E(G) \cap E(G')$.

(3). If $z \in \mathcal{V}(v, u) \setminus \{x, y\}$, then by the choice of the chords ux and vy and by Lemma 2.3, z is not incident with any chord. This implies $d_{G^*}(z) = 2$ as $\delta(G^*) \geq 2$ by Lemma 2.2. By Lemma 2.5(2), z is not incident with any pendant edge, and thus $d_G(z) = d_{G^*}(z) = 2$. We then claim $|\mathcal{V}(v, x)| \leq 1$. If not, then there would be two adjacent 2-vertices $z_1, z_2 \in \mathcal{V}(v, x)$ by (1), contradicting Lemma 2.6. Similarly, we have $|\mathcal{V}(x, y)| \leq 1$ and $|\mathcal{V}(y, u)| \leq 1$.

(4). Suppose for a contradiction that $\mathcal{V}(v, x)$ is a non-edge. If $\mathcal{V}(y, u)$ is a non-edge, then we can easily redraw G to avoid the crossing produced by ux crossing vy. If $\mathcal{V}(x, y)$ is a non-edge, then G^* has a cut-vertex u, contradicting Lemma 2.2. Hence both $\mathcal{V}(x, y)$ and $\mathcal{V}(y, u)$ are paths by (1). Since $d_{G^*}(x) = 2$ by Lemma 2.4, $d_G(x) = 2$ by Lemma 2.5(2) and xy is a boundary edge by Lemma 2.6 and by (3).

Let $z \in N_{G^*}(y) \setminus \{v, x\}$. Note that it may be possible that $z = u$.

If there is a pendant edge yw, then $G - yw$ has a strong edge $(4\Delta - 6)$-coloring φ as $G - yw \prec G$ and $G - yw \in \mathcal{O}_\Delta$. Now the set $F^\varphi(yw)$ of forbidden colors for yw is

$$\{\varphi(e) \mid e \in E_G(x) \cup E_G(z) \cup E_G(v) \cup E_{G-yw}(y)\}.$$

If $z = u$, then $|E_G(z) \cup E_G(v)| \leq \max\{\Delta + 3, 2\Delta - 1\} = 2\Delta - 1$ by Lemma 2.4 and by (2), which follows $|E_G(x) \cup E_G(z) \cup E_G(v)| \leq 2 + (2\Delta - 1) - 1 = 2\Delta$. If $z \neq u$, then by (3), $d_G(z) = 2$ and thus $|E_G(x) \cup E_G(z) \cup E_G(v)| \leq 2 + 2 + \Delta = \Delta + 4 \leq 2\Delta$. Hence $|F^\varphi(yw)| \leq 2\Delta + (\Delta - 1) - 2 = 3\Delta - 3 < 4\Delta - 6$ and φ can be extended to G.

If y is not incident with any pendant edges, then $G - xy$ has a strong edge $(4\Delta - 6)$-coloring φ as $G - xy \prec G$ and $G - xy \in \mathcal{O}_\Delta$. Now the set $F^\varphi(xy)$ of forbidden colors for xy is

$$\{\varphi(e) \mid e \in E_G(x) \cup E_G(z) \cup E_G(v)\},$$

which has size at most $2\Delta < 4\Delta - 6$ by above arguments. Hence φ can also be extended to G.

Therefore, $\mathcal{V}(v, x)$ is not a non-edge, and by symmetry, $\mathcal{V}(y, u)$ is not a non-edge either.

(5). Assume $uv \in E(G)$ for a contradiction. It follows that uv is a boundary edge by Lemma 2.3. By (1) and (4), $\mathcal{V}(v, x)$ and $\mathcal{V}(u, y)$ are paths.

If $\mathcal{V}(x,y)$ is a non-edge, then $d_G(x) = d_G(y) = 2$ by Lemma 2.5(2), and furthermore, vx and uy are boundary edges by (3) and Lemma 2.6. Now G has at most $2\Delta - 1$ edges and is trivially strongly edge $(4\Delta - 6)$-colorable.

If $\mathcal{V}(x,y)$ is not a non-edge, then it is a path by (1). By (3), G is isomorphic to one of graphs in Figure 2.1, each of which is clearly strongly $(4\Delta - 6)$-colorable.

Hence $uv \notin E(G)$. Now by Lemma 2.4 and by (2), $|E_G(u) \cup E_G(v)| \le \Delta + 3$.

(6). If $|\mathcal{V}(v,x)| = 1$, then let $z \in \mathcal{V}(v,x)$. By (3), $d_G(z) = 2$. This implies $vz, xz \in E(G)$ by (1). If v is incident with at least one pendant edge, then $\Delta \le 4$ by Lemma 2.5(3), a contradiction. Hence v is not incident with any pendant edge and thus $d_G(v) \le 3$ by Lemma 2.4. By Lemma 2.6, $d_G(v) = 3$. Similarly, $d_G(x) = 3$.

Let $G' = G - vz$. Since $G' \prec G$ and $G' \in \mathcal{O}_\Delta$, G' has a strong edge $(4\Delta - 6)$-coloring φ. Now the set $F^\varphi(vz)$ of forbidden colors for vz is

$$\big\{\varphi(e) \bigm| e \in E_G(x) \cup E_G(y) \cup E_G(w)\big\},$$

where $w \in N_G(v) \setminus \{y,z\}$. Since $|F^\varphi(vz)| \le 3 + 2\Delta < 4\Delta - 6$, we can extend φ to G by coloring vz with a color not in $F^\varphi(vz)$. Therefore, $\mathcal{V}(v,x) = \emptyset$ by (3).

If vx is not a boundary edge, then $\mathcal{V}(v,x)$ is a non-edge and thus $d_{G^*}(v) = d_{G^*}(x) = 2$. It follows $d_{G^*}(y) = 3$ by Lemma 2.6, and thus $\mathcal{V}(x,y)$ is a path by (1). If $\mathcal{V}(x,y) \ne \emptyset$, then there is a vertex z such that yz is a boundary edge. By (3), $d_{G^*}(z) = d_G(z) = 2$. If $\mathcal{V}(x,y) = \emptyset$, then xy is a boundary edge. In each case y is adjacent to two 2-vertices in G^*, contradicting Lemma 2.7. Therefore, vx is a boundary edge, and so does uy by the symmetry. This proves (6). □

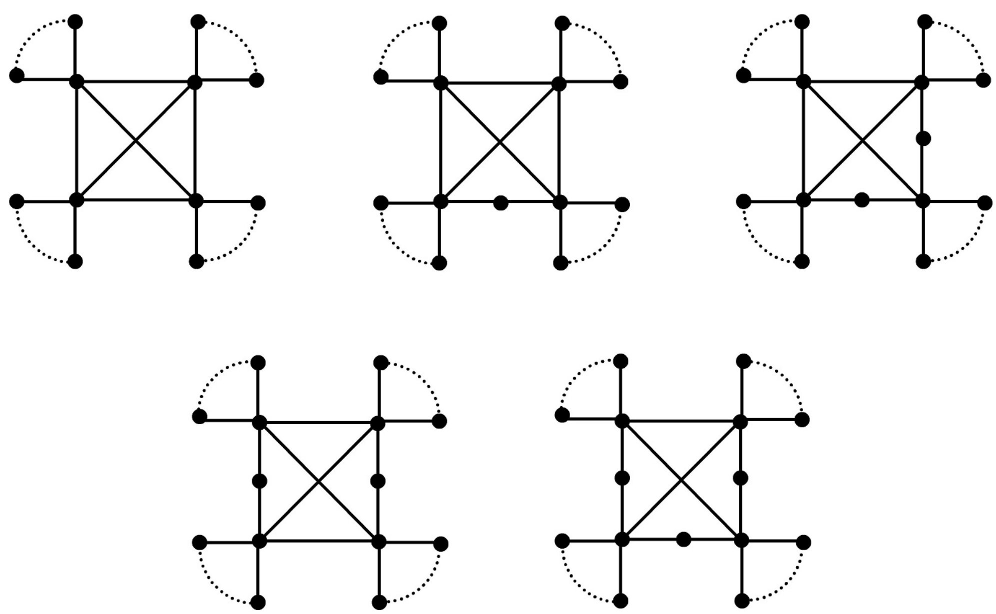

Figure 2.1. Special graphs in the proof of Lemma 2.10(5).

3 Completing the Proof of Theorem 2.1

Let G be a minimal counterexample to the result with respect to the partial order $\prec$. If there is no crossing in G, then G is an outer-plane graph and thus $\chi'_s(G) \le 3\Delta - 3 \le 4\Delta - 6$ (see [17, Theorem 5]). So we assume that there is at least one pair of mutually crossed chords in G.

Choose ux and vy to be two mutually crossed chords of G so that u, v, x, y are located in a clockwise ordering on the outer face of G and the size of $\mathcal{V}[v, u]$ is minimum.

Case 1. $\Delta \geq 5$.

By Lemma 2.10(6), both vx and uy are boundary edges. If $\mathcal{V}(x, y)$ is a non-edge, then $d_{G^*}(x) = d_{G^*}(y) = 2$ by Lemma 2.2. By Lemma 2.5(3), v is not incident with any pendant edges. This implies $d_G(v) = d_{G^*}(v) \leq 3$ by Lemma 2.4. However, this is impossible by Lemmas 2.6 and 2.7. Hence $\mathcal{V}(x, y)$ is a path by Lemma 2.10(1). By Lemma 2.10(3), $|\mathcal{V}(x, y)| \leq 1$.

If $|\mathcal{V}(x, y)| = 1$, then there is a vertex $z \in \mathcal{V}(x, y)$ such that $zx, zy \in E(G)$ and $d_G(z) = 2$ by Lemma 2.10(3). By Lemma 2.5(3), x or y is not incident with any pandent edges and thus $d_G(x) = d_G(y) = 3$. By the minimality of G, $G - yz$ has a strong edge $(4\Delta - 6)$-coloring φ. Now the set $F^\varphi(yz)$ of forbidden colors for yz is

$$\{\varphi(e) \mid e \in E_G(x) \cup E_G(u) \cup E_G(v)\},$$

By Lemma 2.10(5). $F^\varphi(yz) \leq 3 + (\Delta + 3) - 2 = \Delta + 4 < 4\Delta - 6$, and thus φ can be extended to G.

If $|\mathcal{V}(x, y)| = 0$, then xy is a boundary edge. By the minimality of G, $G - xy$ has a strong edge $(4\Delta - 6)$-coloring φ. Now the set $F^\varphi(xy)$ of forbidden colors for xy is

$$\{\varphi(e) \mid e \in E_G(x) \cup E_G(y) \cup E_G(u) \cup E_G(v)\},$$

which has size at most $2(\Delta - 1) + (\Delta + 3) - 4 = 3\Delta - 3 < 4\Delta - 6$ by Lemma 2.10(5). Hence φ can also be extended to G.

Case 2. $\Delta = 4$.

By Lemma 2.10(5), $uv \notin E(G)$. By Lemmas 2.10(1) and 2.10(4), $\mathcal{V}(v, x)$ and $\mathcal{V}(y, u)$ are paths. By Lemma 2.10(2), we assume, without loss of generality, that u is not incident with any pendant edges, and thus $d_G(u) \leq 3$ by Lemma 2.4.

If $\mathcal{V}(x, y)$ is a non-edge, then $d_{G^*}(x) = d_{G^*}(y) = 2$ by Lemma 2.2. Now $\mathcal{V}(v, x)$ and $\mathcal{V}(y, u)$ are not non-edges, and thus are paths by Lemma 2.10(1). If vx is not a boundary edge, then there is a vertex z such that $xz \in E(G)$ and $d_G(z) = 2$ by Lemma 2.10(3), contradiction Lemma 2.6. Hence vx is a boundary edge and so does uy. It follows $N_G(x) = N_G(y)$. However, this is impossible by Lemmas 2.6 or 2.8 as $d_G(u) \leq 3$. Hence $\mathcal{V}(x, y)$ is a path by Lemma 2.10(1). By Lemma 2.10(3), $|\mathcal{V}(x, y)| \leq 1$.

If $|\mathcal{V}(x, y)| = 1$, then there is a vertex $z \in \mathcal{V}(x, y)$ such that $zx, zy \in E(G)$ and $d_G(z) = 2$ by Lemma 2.10(3). By Lemmas 2.9, there are no pendant edges incident with x and y, so $d_G(x) = d_G(y) = 3$. If v is incident with some pendant edge, then vx is a boundary edge by Lemmas 2.9 and 2.10(3), and thus $|E_G(v) \cup E_G(x)| \leq 6$. If v is not incident with any pendant edges, then we also have $|E_G(v) \cup E_G(x)| \leq 6$ by Lemma 2.4.

By the minimality of G, $G - yz$ has a strong edge 10-coloring φ. Let $w \in N_G(y) \setminus \{z, v\}$. Note that w may be u and thus $d_G(w) \leq 3$ by Lemma 2.10(3). Now the set $F^\varphi(yz)$ of forbidden colors for yz is

$$\{\varphi(e) \mid e \in E_G(w) \cup E_G(v) \cup E_G(x)\}.$$

Since $|F^\varphi(yz)| \leq 3 + 6 < 10$, φ can be extended to G.

If $|\mathcal{V}(x, y)| = 0$, i.e., xy is a boundary edge, then we claim that x and y are not incident with any pendant edges.

If y is incident with a pendant edge yy', then by the minimality of G, $G - yy'$ has a strong edge 10-coloring φ. Let $w \in N_G(y) \setminus \{v, x\}$. Note that w may be u and $|E_G(w) \setminus \{ux\}| \leq 2$ by Lemma 2.10(3). If x is incident with some pendant edge, then vx is a boundary edge by Lemmas 2.9 and 2.10(3) as $d_G(u) \leq 3$, and thus $|E_G(v) \cup E_G(x)| \leq 7$. If x is not incident with any pendant edges, then $d_G(x) \le 3$ by Lemma 2.4 and we also have $|E_G(v) \cup E_G(x)| \le 7$. Note that the set

$$\{\varphi(e) \mid e \in (E_G(w) \setminus \{ux\}) \cup E_G(v) \cup E_G(x)\}$$

of forbidden colors for yy' has size at most $2 + 7 = 9 < 10$, and thus φ can be extended to G.

If x is incident with a pendant edge xx', then by the minimality of G, $G - xx'$ has a strong edge 10-coloring φ. By Lemmas 2.9 and 2.10(3), vx is a boundary edge. Since y is not incident with any pendant edges now, the set

$$\{\varphi(e) \mid e \in E_{G-xx'}(x) \cup E_G(y) \cup E_G(u) \cup E_G(v)\}$$

of $F^{\varphi}(xx')$ of forbidden colors for xx' has size at most $3+3+3+4-4 = 9 < 10$. Hence φ can be extended to G, a contradiction.

Now, there are no pendant edges incident with x or y. Then any strong edge 10-coloring φ of $G - xy$ can be extended to G, as the set of forbidden colors for xy is

$$\{\varphi(e) \mid e \in E_{G-xy}(x) \cup E_{G-xy}(y) \cup E_G(u) \cup E_G(v)\},$$

which has size at most $2 + 2 + 3 + 4 - 2 = 9 < 10$.

Case 3. $\Delta = 3$

By Lemma 2.5(1), there are no pendant edges in G.

If $\mathcal{V}(x, y)$ is a non-edge, then $d_G(x) = d_G(y) = 2$ and vx and uy are boundary edges by Lemmas 2.2, 2.6, 2.10(1), and 2.10(3), contradicting Lemma 2.8. So $\mathcal{V}(x, y)$ is a path with $|\mathcal{V}(x, y)| \le 1$ by Lemmas 2.10(1) and 2.10(3).

If $uv \in E(G)$, then uv is a boundary edge by Lemma 2.3. Now G is isomorphic to one of graphs in Figure 3.1 by Lemmas 2.8 and 2.10(3), each of which is strongly edge 8-colorable. Hence $uv \notin E(G)$.

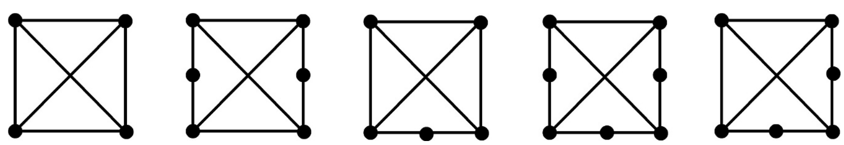

Figure 3.1. Special graphs in the proof of Case 3.

Case 3.1. $|\mathcal{V}(x, y)| = 1$.

Let $z_1 \in \mathcal{V}(x, y)$ such that xz_1, yz_1 are boundary edges and $d_G(z_1) = 2$. By Lemma 2.6, $\mathcal{V}(v, x)$ cannot be a non-edge, and thus it is a path by Lemma 2.10(1). By symmetry, $\mathcal{V}(y, u)$ is a path too. By Lemma 2.10(3), $\max\{|\mathcal{V}(v, x)|, |\mathcal{V}(u, y)|\} \le 1$.

If $|\mathcal{V}(v, x)| = |\mathcal{V}(u, y)| = 0$, then any strong edge 8-coloring φ of $G - yz_1$ can be extended to G as the set of forbidden colors for yz_1 is

$$\{\varphi(e) \mid e \in E_G(u) \cup E_G(v) \cup E_G(x)\},$$

having size at most $3 + 3 + 3 - 2 = 7$.

If $|\mathcal{V}(v, x)| = 1$ and $|\mathcal{V}(u, y)| = 0$, then let z_2 be a 2-vertex such that vz_2x is a boundary path. Then any strong edge 8-coloring φ of $G - xz_1$ can be extended to G as the set

$$\{\varphi(e) \mid e \in E_G(y) \cup E_G(u) \cup E_G(z_2)\}$$

of forbidden colors for xz_1 has size at most $3 + 3 + 2 - 1 = 7$.

If $|\mathcal{V}(v, x)| = |\mathcal{V}(u, y)| = 1$, let z_2 and z_3 be 2-vertices such that vz_2x and uz_3y are boundary paths. By Lemma 2.6, $d_G(u) = d_G(v) = 3$. Let $u' \in N_G(u) \setminus \{x, z_3\}$ and $v' \in N_G(v) \setminus \{y, z_2\}$.

By the minimality of G, then $G' = G - yz_1$ has a strong edge 8-coloring φ. Now the set $F^\varphi(yz_1)$ of forbidden colors for yz_1 is

$$\{\varphi(vv'), \varphi(vz_2), \varphi(z_2x), \varphi(vy), \varphi(ux), \varphi(xz_1), \varphi(uz_3), \varphi(yz_3)\}.$$

If $|F^\varphi(yz_1)| \leq 7$, then φ can be easily extended to G. Hence we assume, without loss of generality, that $\varphi(vv') = 1$, $\varphi(vz_2) = 2$, $\varphi(z_2x) = 3$, $\varphi(vy) = 4$, $\varphi(ux) = 5$, $\varphi(xz_1) = 6$, $\varphi(uz_3) = 7$ and $\varphi(yz_3) = 8$. It follows $\varphi(uu') \neq \varphi(yz_3) = 8$. This makes it possible to complete a strong edge 8-coloring of G by recoloring z_2x with 8 and then coloring yz_1 with 3.

Case 3.2. $|\mathcal{V}(x,y)| = 0$

By the minimality of G, $G - xy$ has a strong edge 8-coloring φ. The set $F^\varphi(xy)$ of forbidden color for xy is

$$\{\varphi(e) \mid e \in E_{G-xy}(x) \cup E_{G-xy}(y) \cup E_G(u) \cup E_G(v)\}.$$

Clearly, $|F^\varphi(xy)| \leq 2 + 2 + 3 + 3 - 2 = 8$.

If $|F^\varphi(xy)| \leq 7$, we can easily extend φ to G. If $|F^\varphi(xy)| = 8$, then there are 2-vertices z_2 and z_3 such that vz_2x and uz_3y are boundary paths. By Lemma 2.6, $d_G(u) = d_G(v) = 3$. Let $u' \in N_G(u) \setminus \{x, z_3\}$ and $v' \in N_G(v) \setminus \{y, z_2\}$. Since $|F^\varphi(xy)| = 8$, we assume, without loss of generality, that $\varphi(uu') = 1$, $\varphi(uz_3) = 2$, $\varphi(ux) = 3$, $\varphi(vy) = 4$, $\varphi(vv') = 5$, $\varphi(vz_2) = 6$, $\varphi(z_3y) = 7$ and $\varphi(z_2x) = 8$.

If we are able to recolor uz_3 (resp. vz_2) with 5 or 6 (resp. 1 or 2) so that the resulting coloring is still a strong edge coloring, then φ can be finally extended to G by coloring xy with 2 (resp. 6). Hence $\varphi(E_G(u')) = \{1, 5, 6\}$ and $\varphi(E_G(v')) = \{1, 2, 5\}$. In such a situation we can complete a strong edge 8-coloring of G by recoloring uz_3 and vz_2 with 8, z_2x with 2, and then coloring xy with 6.

Conflict of Interest

The authors declare no conflict of interest.

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