



Equitable partition of planar graphs

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ABSTRACT

An equitable k -partition of a graph G is a collection of induced subgraphs $(G[V_1], G[V_2], \dots, G[V_k])$ of G such that (V_1, V_2, \dots, V_k) is a partition of $V(G)$ and $-1 \leq |V_i| - |V_j| \leq 1$ for all $1 \leq i < j \leq k$. We prove that every planar graph admits an equitable 2-partition into 3-degenerate graphs, an equitable 3-partition into 2-degenerate graphs, and an equitable 3-partition into two forests and one graph.

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1. Introduction

All graphs in this paper are simple and finite. A k -partition of a graph G is a collection of induced subgraphs $(G[V_1], G[V_2], \dots, G[V_k])$ such that (V_1, V_2, \dots, V_k) is a partition of $V(G)$. Such a k -partition is equitable if

$$||V_i| - |V_j|| \leq 1$$

for all $i, j \in \{1, 2, \dots, k\}$. If there is no confusion, then we use (V_1, V_2, \dots, V_k) to denote a k -partition $(G[V_1], G[V_2], \dots, G[V_k])$ of G . We write $\Delta(G)$ to denote the maximum degree of a graph G .

In 1970, Hajnal and Szemerédi [9] proved a conjecture of Erdős, stating that every graph admits an equitable k -partition into empty subgraphs, if $k > \Delta(G)$. In 2008, Kierstead and Kostochka [10] found a short proof. In 2010, Kierstead, Kostochka, Mydlarz, and Szemerédi [12] designed a fast algorithm to find such an equitable k -partition. The bound on k in the Hajnal–Szemerédi Theorem is sharp because of complete graphs for instance. Thus, there have been many results in this field trying to obtain better lower bounds on the number k of parts for special graph classes. Motivated by Brooks' theorem, Chen, Lih, and Wu [5] conjectured that a connected graph G admits an equitable $\Delta(G)$ -partition into empty graphs if and only if it is not $K_{\Delta(G)+1}$, an odd cycle, or $K_{\Delta(G), \Delta(G)}$ (for odd $\Delta(G)$). They proved this conjecture for $\Delta(G) \leq 3$ and Kierstead and Kostochka [11] proved the conjecture for $\Delta(G) = 4$. For planar graphs, Zhang and Yap [20] proved this conjecture for $\Delta(G) \geq 13$, and Nakprasit [15] proved it for $\Delta(G) \geq 9$; in other words, he proved that every planar graph G has an equitable k -partition into empty subgraphs if $k \geq \max(\Delta(G), 9)$.

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If we relax the condition on each part, then it is possible to reduce the number of parts significantly. For instance, Williams, Vandenbussche, and Yu [17] proved that for all $k \geq 3$, every planar graph of minimum degree at least 2 and girth at least 10 has an equitable k -partition into graphs of maximum degree at most 1.

We will mostly focus on the degeneracy of graphs. A graph is d -degenerate if every non-null subgraph has a vertex of degree at most d . Note that a graph is 0-degenerate if it has no edges, and 1-degenerate if it is a forest. Kostochka, Nakprasit, and Pemmaraju [13] studied the existence of an equitable k -partition of a d -degenerate graph into $(d - 1)$ -degenerate graphs.

Theorem 1.1 (Kostochka, Nakprasit, and Pemmaraju [13]). *For $k \geq 3$ and $d \geq 2$, every d -degenerate graph has an equitable k -partition into $(d - 1)$ -degenerate subgraphs.*

This implies that every 5-degenerate graph admits an equitable 3-partition into 4-degenerate subgraphs, an equitable 9-partition into 3-degenerate subgraphs, an equitable 27-partition into 2-degenerate subgraphs, and an equitable 81-partition into forests.

Now we restrict our attention to planar graphs. As planar graphs are 5-degenerate, every planar graph admits an equitable 81-partition into forests. How far can we reduce 81? Esperet, Lemoine, and Maffray [7] proved that 81 can be improved to 4.

Theorem 1.2 (Esperet, Lemoine, and Maffray [7]). *For all $k \geq 4$, every planar graph admits an equitable k -partition into forests.*

However it is not known whether 4 is tight. Indeed, Esperet, Lemoine, and Maffray [7] proposed the following problem:

Problem 1.3 (Esperet, Lemoine, and Maffray [7]). *Does every planar graph G admit an equitable 3-partition into forests?*

This problem still remains open and is known to have affirmative answers in the following cases:

- G is 2-degenerate, by Theorem 1.1 (even if G is non-planar),
- the girth of G is at least 5, due to Wu, Zhang, and Li [18],
- no two cycles of length at most 4 share vertices in G , due to Zhang [19],
- G has no triangles, and no two cycles of length 4 are adjacent, due to Zhang [19],
- G has an acyclic 4-coloring, due to Esperet, Lemoine, and Maffray [7].

By relaxing the condition further, we may ask the following question.

Problem 1.4. *For each i , what is the minimum integer k_i such that for all integers $k \geq k_i$, every planar graph admits an equitable k -partition into i -degenerate subgraphs?*

It is easy to see that $k_0 = \infty$ by considering $K_{1,n}$ for large n , see Meyer [14]. Since every planar graph is 5-degenerate, $k_i = 1$ for all $i \geq 5$. Theorem 1.2 implies that $k_1 \leq 4$. Not every planar graph admits a (not necessarily equitable) 2-partition into forests, shown by Chartrand and Kronk [4]. Thus, $k_1 \geq 3$.

Our first and second theorems prove that $k_3 = k_4 = 2$ and $k_2 \in \{2, 3\}$.

Theorem 2.1. *Every planar graph admits an equitable 2-partition into 3-degenerate graphs.*

Theorem 2.2. *Every planar graph admits an equitable 3-partition into 2-degenerate graphs.*

Our third theorem shows a weaker variant of Problem 1.3.

Theorem 3.1. *Every planar graph admits an equitable 3-partition into two forests and one graph.*

The rest of this paper is organized as follows. In Section 2, we prove Theorems 2.1 and 2.2, and moreover, show that every triangle-free planar graph admits an equitable 2-partition into 2-degenerate graphs. In Section 3, we prove Theorem 3.1 and illustrate some discussions towards Problem 1.3 and its relative problems.

2. Equitable partition into degenerate graphs

For a graph G and disjoint sets U, V of vertices of G , we denote by $e_G(U, V)$ the number of edges between U and V . If $U = \{u\}$ or $V = \{v\}$, then we simply write $e_G(u, V)$ or $e_G(U, v)$ for $e_G(U, V)$. For a vertex set $S \subseteq V(G)$ and vertices $v \in S$ and $u \notin V(G) - S$, let us write $S - v$ for the set $S - \{v\}$ and $S + u$ for the set $S \cup \{u\}$.

Our first theorem shows that $k_3 = k_4 = 2$.

Theorem 2.1. *Every planar graph admits an equitable 2-partition into 3-degenerate graphs.*

Proof. Let G be an n -vertex planar graph. We proceed by induction on $|E(G)|$. We may assume that G has at least one edge and $n \geq 4$.

As G is planar, it has a vertex v such that $0 < \deg(v) \leq 5$. Let v_1 be a neighbor of v . By the induction hypothesis, there is an equitable 2-partition (V_1, V_2) of $G - vv_1$ into 3-degenerate graphs. We may assume, without loss of generality, that $v \in V_1$. If $e_G(v, V_1 - v) \leq 3$, then (V_1, V_2) is an equitable 2-partition of G into 3-degenerate graphs. So we may assume that $e_G(v, V_1 - v) \geq 4$, and so $e_G(v, V_2) \leq 1$. Therefore, $V_2 + v$ induces a 3-degenerate subgraph of G .

If there is a vertex $w \in V_2$ so that $e_G(w, V_1 - v) \leq 3$, then $(V_1 - v + w, V_2 - w + v)$ is an equitable 2-partition of G into 3-degenerate graphs. Hence we assume that $e_G(w, V_1 - v) \geq 4$ for every $w \in V_2$, which implies that

$$e_G(V_2, V_1 - v) \geq 4|V_2| \geq 4\lfloor n/2 \rfloor \geq 2n - 2.$$

On the other hand, the graph induced by the edges between $V_1 - v$ and V_2 is a bipartite planar graph on $n - 1$ vertices, and therefore $e_G(V_2, V_1 - v) \leq 2(n - 1) - 4 = 2n - 6$, contradicting the other inequality. \square

Now we show that $2 \leq k_2 \leq 3$.

Theorem 2.2. *Every planar graph admits an equitable 3-partition into 2-degenerate graphs.*

Proof. Let G be an n -vertex planar graph. We proceed by induction on $|E(G)|$. We may assume that G has at least one edge and at least 4 vertices.

Since G is planar, there is a vertex v such that $1 \leq \deg(v) \leq 5$. Let v_1 be a neighbor of v . By applying the induction hypothesis to the graph $G - vv_1$, we obtain an equitable 3-partition (V_1, V_2, V_3) of $G - vv_1$ into 2-degenerate graphs. We may assume, without loss of generality, that $v \in V_1$. If $e_G(v, V_1 - v) \leq 2$, then (V_1, V_2, V_3) is also an equitable 3-partition of G into 2-degenerate graphs. So we may assume that $e_G(v, V_1 - v) \geq 3$, which implies that $e_G(v, V_2) \leq 2$ and $e_G(v, V_3) \leq 2$. Therefore, both $V_2 + v$ and $V_3 + v$ induce 2-degenerate subgraphs of G .

If there is a vertex $w \in V_2$ so that $e_G(w, V_1 - v) \leq 2$, then $V_1 - v + w$ induces a 2-degenerate subgraph of G . Hence, $(V_1 - v + w, V_2 - w + v, V_3)$ is an equitable 3-partition of G into 2-degenerate graphs. Now we assume that $e_G(w, V_1 - v) \geq 3$ for every $w \in V_2$, and by symmetry, we assume further that $e_G(w, V_1 - v) \geq 3$ for every $w \in V_3$. This implies that

$$e_G(V_2 \cup V_3, V_1 - v) \geq 3|V_2 \cup V_3| \geq 3 \cdot 2\lfloor n/3 \rfloor \geq 2(n - 2).$$

On the other hand, the graph induced by the edges between $V_2 \cup V_3$ and $V_1 - v$ is a bipartite planar graph on $n - 1$ vertices, and therefore

$$e_G(V_2 \cup V_3, V_1 - v) \leq 2(n - 1) - 4,$$

contradicting the previous inequality. \square

We do not know whether $k_2 = 2$.

Problem 2.3. Does every planar graph admit an equitable 2-partition into 2-degenerate graphs?

We remark that Thomassen [16] proved that every planar graph admits a 2-partition into a 2-degenerate graph and a forest.

The following theorem shows that a possible counterexample to Problem 2.3 shall contain triangles.

Theorem 2.4. *Every triangle-free planar graph admits an equitable 2-partition into 2-degenerate graphs.*

Proof. We first prove by using the discharging method that every triangle-free planar graph contains a vertex of degree at most 2 or an edge with one end having degree 3 and the other having degree at most 6.

Suppose there exists a triangle-free planar graph G not satisfying the condition above. We assign initial charge $\mu(v) = \deg(v)$ to each vertex v of G , and let each vertex of degree at least 7 send $1/3$ to each of its neighbors of degree 3. After this discharging, the final charge $\mu'(v)$ of each vertex v of degree 3 is exactly $3 + 3 \times 1/3 = 4$ since vertices of degree 3 are only adjacent to vertices of degree at least 7. If v is a vertex of degree in $\{4, 5, 6\}$, then $\mu'(v) = \mu(v) \geq 4$. Lastly, each vertex v of degree at least 7 has final charge $\mu'(v) \geq \deg(v) - \deg(v)/3 \geq 14/3 > 4$. Since G is triangle-free and has minimum degree at least 3,

$$4n > 2e(G) = \sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} \mu'(v) \geq 4n,$$

a contradiction. Therefore, the claim holds.

Now we prove the theorem. Let G be an n -vertex triangle-free planar graph. We proceed by induction on n . If there is a vertex v of degree at most two, then by applying the induction hypothesis to the graph $G - v$, we obtain an equitable 2-partition (V_1, V_2) of $G - v$ into 2-degenerate graphs with $|V_1| \leq |V_2|$. Then, $(V_1 + v, V_2)$ is an equitable 2-partition of G into 2-degenerate graphs.

So we assume that every vertex of G has degree at least 3. Then, by the claim, there is an edge uv with $\deg(u) = 3$ and $\deg(v) \leq 6$. Applying the induction hypothesis to the graph $G - \{u, v\}$, we obtain an equitable 2-partition (V_1, V_2) of $G - \{u, v\}$ into two 2-degenerate graphs. Since v has at most five neighbors in $G - \{u, v\}$, we may assume that V_1 contains at most two neighbors of v . Then $(V_1 + v, V_2 + u)$ is an equitable 2-partition of G into 2-degenerate graphs. This completes the proof. \square

3. Equitable partition into 2 forests and 1 graph

In this section, we aim to prove the following theorem.

Theorem 3.1. *Every planar graph admits an equitable 3-partition into two forests and one graph.*

An *acyclic k -coloring* of a graph is a proper k -coloring such that there is no cycle consisting of two colors. In other words, if a graph has an acyclic k -coloring, then its vertex set can be partitioned into k independent sets A_1, A_2, \dots, A_k such that $A_i \cup A_j$ induces a forest for all $i, j \in \{1, 2, \dots, k\}$. Borodin proved the following theorem, initially conjectured by Grünbaum [8].

Theorem 3.2 (Borodin [1]). *Every planar graph has an acyclic 5-coloring.*

To prove Theorem 1.2, Esperet, Lemoine, and Maffray used Theorem 3.2 and try to combine two color classes in an acyclic 5-coloring of planar graphs to produce large induced forests. We extend their idea.

3.1. Key proposition

To prove Theorem 3.1, we prove the following stronger statement.

Proposition 3.3. *Let $k > \ell \geq 1$ be integers. Let A_1, A_2, \dots, A_k be sets such that $|\bigcup_{i=1}^k A_i| = n$. Let*

$$q = \left\lfloor \frac{2}{k + \ell - 1} \left(n + \frac{k - \ell}{2} \right) \right\rfloor.$$

Then there exists a partition $(B_0, B_1, \dots, B_\ell)$ of $\bigcup_{i=1}^k A_i$ into $\ell + 1$ sets, possibly empty, such that

- (i) *for each $1 \leq i \leq \ell$, B_i is a subset of the union of two members of $\{A_1, A_2, \dots, A_k\}$,*
- (ii) *$|B_i| \geq q + 1$ if $1 \leq i \leq \lceil n - \frac{k + \ell - 1}{2} q \rceil$,*
- (iii) *$|B_i| \geq q$ if $\lceil n - \frac{k + \ell - 1}{2} q \rceil < i \leq \ell$,*
- (iv) *there exists $I \subseteq \{1, 2, \dots, k\}$ with $|I| = k - \ell - 1$ such that $B_0 \subseteq \bigcup_{i \in I} A_i$.*

Let us first see why Proposition 3.3 together with Theorem 3.2 implies Theorem 3.1

Proof of Theorem 3.1. Let G be a planar graph with n vertices. Then G has an acyclic 5-coloring by Theorem 3.2 and so there exist sets A_1, A_2, \dots, A_5 such that $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = V(G)$ and $A_i \cup A_j$ induces a forest for all $1 \leq i < j \leq 5$. By applying Proposition 3.3 with $k := 5$ and $\ell := 2$, we have a partition (B_0, B_1, B_2) of $V(G)$ such that $|B_1|, |B_2| \geq q$ and both $G[B_1]$ and $G[B_2]$ are forests, where $q = \lfloor \frac{2}{6}(n + \frac{3}{2}) \rfloor = \lfloor \frac{n+1}{3} \rfloor$. If $n \not\equiv 2 \pmod{3}$, then we take $B'_1 \subseteq B_1$ and $B'_2 \subseteq B_2$ such that $|B'_1| = |B'_2| = \lfloor n/3 \rfloor$. If $n \equiv 2 \pmod{3}$, then we take $B'_1 \subseteq B_1$ and $B'_2 \subseteq B_2$ such that $|B'_1| = |B'_2| = \lfloor n/3 \rfloor + 1$. Let $B'_0 = V(G) \setminus (B'_1 \cup B'_2)$. Then (B'_0, B'_1, B'_2) is a desired equitable 3-partition. \square

Here is a key lemma to prove Proposition 3.3 inductively.

Lemma 3.4. *Let $2 \leq \ell < k$ and n be positive integers and let $q = \lfloor \frac{2n+k-\ell}{k+\ell-1} \rfloor$ and $n' = n - q$. Then*

$$\left\lfloor \frac{2n' + k - \ell}{k + \ell - 3} \right\rfloor = \begin{cases} q + 1 & \text{if } \lceil n - \frac{k + \ell - 1}{2} q \rceil \geq \ell - 1, \\ q & \text{otherwise,} \end{cases}$$

and

$$n - \frac{k + \ell - 1}{2} q = n' - \frac{k + \ell - 3}{2} q.$$

Proof. Let $m := k + \ell - 1$, and let r be the integer such that $2n + k - \ell = mq + r$ and $0 \leq r < m$. Then

$$\left\lfloor \frac{2n' + k - \ell}{k + \ell - 3} \right\rfloor = \left\lfloor \frac{2(n - q) + k - \ell}{m - 2} \right\rfloor = \left\lfloor \frac{(m - 2)q + r}{m - 2} \right\rfloor = q + \left\lfloor \frac{r}{m - 2} \right\rfloor.$$

Note that $\lceil n - mq/2 \rceil \geq \ell - 1$ if and only if $r \geq k + \ell - 3 = m - 2$. Thus the first equation holds. The second equation is trivial. This completes the proof. \square

Proof of Proposition 3.3. We first observe that $\lceil n - \frac{k+\ell-1}{2}q \rceil \leq \ell - 1$, since

$$\frac{2n - 2\ell + 2}{k + \ell - 1} \leq \left\lfloor \frac{(2n - 2\ell + 2) + (k + \ell - 2)}{k + \ell - 1} \right\rfloor = \left\lfloor \frac{2n + k - \ell}{k + \ell - 1} \right\rfloor = q.$$

We may assume that $A_i \cap A_j = \emptyset$ for all $i \neq j$. We proceed by induction on ℓ . Let $a_i = |A_i|$ for all $1 \leq i \leq k$.

If $\ell = 1$, then by the pigeonhole principle, there exist $1 \leq i < j \leq k$ such that $a_i + a_j \geq \lceil \frac{2n}{k} \rceil$. Observe that

$$q = \left\lfloor \frac{2}{k} \left(n + \frac{k-1}{2} \right) \right\rfloor = \left\lfloor \frac{2n}{k} + \frac{k-1}{k} \right\rfloor = \left\lfloor \frac{2n}{k} \right\rfloor$$

and therefore we take a set $B_1 = A_i \cup A_j$. Then $|B_1| \geq q$ and B_0 is the union of the $k-2$ members of $\{A_1, A_2, \dots, A_k\} \setminus \{A_i, A_j\}$. Now we may assume that $\ell > 1$.

Suppose that there exist $i \neq j$ such that $a_i \leq q \leq a_i + a_j$. Without loss of generality, let us assume $i = 1$ and $j = 2$. Then there exists B_ℓ such that $A_1 \subseteq B_\ell \subseteq A_1 \cup A_2$ and $|B_\ell| = q$. Let

$$n' = n - q \quad \text{and} \quad q' = \left\lfloor \frac{2}{k + \ell - 3} \left(n' + \frac{k - \ell}{2} \right) \right\rfloor.$$

By applying the induction hypothesis to the $k-1$ sets $A_2 - B_\ell, A_3, A_4, \dots, A_k$, we obtain a partition $(B_0, B_1, B_2, \dots, B_{\ell-1})$ of $\bigcup_{i=2}^k A_i - B_\ell$, such that for $1 \leq i \leq \ell - 1$, the set B_i is a subset of the union of two members of $\{A_2, A_3, \dots, A_k\}$ and B_0 is a subset of the union of $k - \ell - 1$ members of $\{A_2, A_3, \dots, A_k\}$. If $\lceil n - \frac{k+\ell-1}{2}q \rceil = \ell - 1$, then by Lemma 3.4, $q' = q + 1$ and therefore the induction hypothesis provides that $|B_i| \geq q + 1$ for all $1 \leq i \leq \ell - 1$. If $\lceil n - \frac{k+\ell-1}{2}q \rceil < \ell - 1$, then again by Lemma 3.4, $q' = q$ and $n - \frac{k+\ell-1}{2}q = n' - \frac{k+\ell-3}{2}q'$ and therefore we deduce (ii) and (iii) by the induction hypothesis.

Thus, we may assume that for all $i \neq j$, $a_i > q$ or $a_i + a_j < q$. Note that if $a_i > q$, then $a_i + a_j > q$ and therefore $a_j > q$. Thus we deduce that either

- (I) $a_i > q$ for all $1 \leq i \leq k$, or
- (II) $a_i + a_j < q$ for all $i \neq j$.

By the pigeonhole principle there exist $1 \leq i < j \leq k$ such that $a_i + a_j \geq \lceil \frac{2n}{k} \rceil$. Since

$$\left\lfloor \frac{2n}{k} \right\rfloor = \left\lfloor \frac{2}{k} \left(n + \frac{k-1}{2} \right) \right\rfloor \geq \left\lfloor \frac{2}{k + \ell - 1} \left(n + \frac{k - \ell}{2} \right) \right\rfloor = q,$$

(II) does not hold and so (I) holds. Then we take $B_i = A_i$ for $i = 1, 2, \dots, \ell - 1$, $B_\ell = A_\ell \cup A_{\ell+1}$, and $B_0 = \bigcup_{i=\ell+2}^k A_i$. \square

Proposition 3.3 is best possible in the sense that we cannot increase q ; consider the case that A_1, A_2, \dots, A_k are disjoint sets with $|A_1| = \ell a$, $|A_2| = |A_3| = \dots = |A_k| = a$ for some positive integer a . There are no ℓ disjoint sets of size at least $2a + 1$, each contained in the union of two members of $\{A_1, \dots, A_k\}$, so we cannot increase q from $2a$ to $2a + 1$.

When $\ell = k - 1$, then we obtain the following from Proposition 3.3, which is due to Esperet, Lemoine, and Maffray [7].

Corollary 3.5 (Esperet, Lemoine, and Maffray [7]). *Let $k > 1$ be an integer. Let A_1, A_2, \dots, A_k be sets such that $|\bigcup_{i=1}^k A_i| = n$. Then there exists a partition (B_1, \dots, B_{k-1}) of $\bigcup_{i=1}^k A_i$ into $k - 1$ sets such that for each $1 \leq i \leq k - 1$, B_i is a subset of the union of two members of $\{A_1, A_2, \dots, A_k\}$, and $|B_i| = \lceil \frac{n}{k-1} \rceil$ or $|B_i| = \lfloor \frac{n}{k-1} \rfloor$*

Proof. We apply Proposition 3.3 with $\ell = k - 1$ to obtain $(B_0, B_1, \dots, B_{k-1})$. Then $q = \lfloor \frac{1}{k-1}(n + \frac{1}{2}) \rfloor = \lfloor \frac{n}{k-1} \rfloor$. As B_0 is a subset of the union of 0 sets, $B_0 = \emptyset$. And, $\lceil n - \frac{k+\ell-1}{2}q \rceil = n - (k-1)q$ is exactly the remainder when dividing n by $k-1$ and therefore $|B_i| = q + 1$ for all $1 \leq i \leq \lceil n - \frac{k+\ell-1}{2}q \rceil$ and $|B_i| = q$ for all $\lceil n - \frac{k+\ell-1}{2}q \rceil < i \leq k - 1$. Thus $(B_1, B_2, \dots, B_{k-1})$ is a desired partition. \square

3.2. Discussions

Borodin and Ivanova [2] and Chen and Raspaud [6] independently showed that every planar graph without cycles of length 4 or 5 has an acyclic 4-coloring. By Corollary 3.5, if a planar graph has no cycle of length 4 or 5, then it admits an equitable 3-partition into forests. By Proposition 3.3, we have the following variation of Problem 1.3.

Corollary 3.6. *If a planar graph has no cycle of length 4 or 5, then it admits a partition of its vertex set into three sets A_1, A_2, A_3 such that*

- (i) both A_1 and A_2 induce forests and $|A_1|, |A_2| \geq \lfloor \frac{2}{5}(n + 1) \rfloor$,
- (ii) A_3 is independent.

By the four color theorem, Corollary 3.5 implies that every planar graph admits an equitable 3-partition into bipartite graphs. We also deduce the following variant.

Corollary 3.7. Every n -vertex planar graph admits a partition of its vertex set into three sets A_1, A_2, A_3 such that

- (i) both A_1 and A_2 induce bipartite subgraphs and $|A_1|, |A_2| \geq \lfloor \frac{2}{5}(n+1) \rfloor$,
- (ii) A_3 is independent.

A linear forest is a forest of maximum degree at most 2. Cai, Xie, and Yang [3] showed that for every planar graph G , its vertices can be colored by $\Delta(G) + 7$ colors such that the union of any two color classes induces a linear forest. Combined with Proposition 3.3 and Corollary 3.5, we deduce that every planar graph G admits an equitable $(\Delta(G) + 6)$ -partition into linear forests.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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