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List Edge Coloring of Outer-1-planar Graphs

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Abstract A graph is outer-1-planar if it can be drawn in the plane so that all vertices are on the outer face and each edge is crossed at most once. It is known that the list edge chromatic number $\chi'_l(G)$ of any outer-1-planar graph G with maximum degree $\Delta(G) \geq 5$ is exactly its maximum degree. In this paper, we prove $\chi'_l(G) = \Delta(G)$ for outer-1-planar graphs G with $\Delta(G) = 4$ and with the crossing distance being at least 3.

Keywords outerplanar graph; outer-1-planar graph; crossing distance; list edge coloring
2000 MR Subject Classification 05C15; 05C10

1 Introduction

In this paper, all graphs are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. The *order* $|G|$ of a graph G is $|V(G)|$ and the *size* of G is $|E(G)|$. The *distance* $d_G(u, w)$ between two vertices u and w of a connected graph G is the minimum length of the path (i.e., the number of edges on the path) connecting them.

The problem of coloring a graph arises in many practical areas such as pattern matching, sports scheduling, designing seating plans, exam timetabling, the scheduling of taxis, and solving Sudoku puzzles[9]. There are many types of colorings of graphs, and in this paper we mainly focus on the edge coloring. Precisely, an *edge coloring* of a graph G is an assignment of colors to the edges of G such that every pair of adjacent edges receive different colors. An *edge k-coloring* of a graph G is an edge coloring of G from a set of k colors. The minimum positive integer k for which G has an edge k-coloring, denoted by $\chi'(G)$, is the *edge chromatic number* of G. The well-known Vizing's Theorem states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for any simple graph G.

Suppose that a set $L(e)$ of colors, called a *list* of e, is assigned to each edge $e \in E(G)$. *L-coloring* an edge e means coloring e with a color in $L(e)$. An *edge L-coloring* of G is an edge coloring c so that $c(e) \in L(e)$ for every $e \in E(G)$. We say that G is *edge k-choosable* if G has an edge L-coloring whenever $|L(e)| = k$ for every $e \in E(G)$. The minimum integer k for which G is edge k-choosable is the *list edge chromatic number* of G, denoted by $\chi'_l(G)$.

The most famous open problem concerning list edge coloring is probably the *list edge coloring conjecture* (LECC for short):

$$\chi'_l(G) = \chi'(G)$$

for any graph G. This conjecture has a fuzzy origin. Jensen and Toft overview its history in their book [6].

LECC is regarded to be very difficult, and is still widely open. Some partial results were however obtained in the special case of planar graphs. For example, LECC is true for planar graphs with maximum degree at least 12[3], series-parallel graphs[7], outerplanar graphs[11],

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near-outerplanar graphs[5], and pseudo-outerplanar (outer-1-planar) graphs with maximum degree at least 5[10].

A graph is *outer-1-planar* if it can be drawn in the plane so that all vertices are on the outer face and each edge is crossed at most once. For example, $K_{2,3}$ and K_4 are outer-1-planar graphs. Outer-1-planar graphs were first introduced by Eggleton[4] who called them *outerplanar graphs with edge crossing number one*, and were also investigated under the notion of *pseudo-outerplanar graphs* by Zhang, Liu and Wu[13, 14].

In this paper, we use elementary notions and notations from [15]. A drawing of an outer-1-planar graph in the plane preserving its outer-1-planarity is an *outer-1-plane graph*, and we call it *good* if the number of its crossings is as small as possible. Note that every crossing in an outer-1-plane graph G is generated by two mutually crossed edges, thus every crossing c corresponds to a vertex set $M_G(c)$ of size four, where $M_G(c)$ consists of the end-vertices of the two edges that generate c. For crossings c_1 and c_2 in an outer-1-plane graph G, define $d_G(c_1, c_2) = \min\{d_G(v_1, v_2) \mid v_1 \in M_G(c_1) \text{ and } v_2 \in M_G(c_2)\}$ to be the *distance* in the drawing G between c_1 and c_2. By

$$\vartheta(G) = \min\{d_{G'}(c_1, c_2) \mid G' \textit{ is a good drawing of } G, \textit{ and } c_1, c_2 \textit{ are distinct crossings of } G'\}$$

we denote the *crossing distance* of an outer-1-planar graph G. Note that we will set $\vartheta(G) = \infty$ if G has a good drawing with at most one crossing.

For every outer-1-planar graph G, it was proved in [14, Theorem 5.3] that $\chi'(G) = \Delta(G)$ if $\Delta(G) \geq 4$, and in [10, Theorem 2.5] that $\chi'_l(G) = \chi'(G) = \Delta(G)$ if $\Delta(G) \geq 5$. In this paper we will prove the following results.

Theorem 1.1. *Let G be an outer-1-planar graph. If $\Delta(G) = 4$ and $\vartheta(G) \geq 3$, then $\chi'_l(G) = \chi'(G) = 4$.*

Theorem 1.2. *If G is an outer-1-planar graph with $\Delta(G) = 3$, then $\chi'_l(G) \leq 4$.*

The following corollary from Theorem 1.2 is immediate.

Corollary 1.1. *If G is an outer-1-planar graph with $\Delta(G) = 3$ and $\chi'(G) = 4$, then $\chi'_l(G) = \chi'(G)$.*

Note that there exist infinitely many outer-1-planar graph G with $\Delta(G) = 3$ and $\chi'(G) = 4$, see [16, Theorem 3.2].

2 Preliminaries

Throughout this section, G will be a 2-connected good outer-1-plane graph, and by $v_1, \cdots, v_{|G|}$ we denote the vertices of G with clockwise ordering on the boundary.

Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \cdots, v_j\}$ and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j] \backslash \{v_i, v_j\}$, where the subscripts are taken modulo $|G|$. Set $\mathcal{V}[v_i, v_i] = V(G)$ and $\mathcal{V}(v_i, v_i) = V(G) \setminus \{v_i\}$. By $G[v_i, v_j]$ and $G(v_i, v_j)$, we denote the subgraph of G induced by $\mathcal{V}[v_i, v_j]$ and $\mathcal{V}(v_i, v_j)$, respectively. If there is no edge between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then $\hat{G}_{i,j}$ denotes the graph obtained from $G[v_i, v_j]$ by adding edge v_iv_j if it does not exist in $G[v_i, v_j]$ (otherwise $\hat{G}_{i,j}$ is $G[v_i, v_j]$ itself). Clearly, $\hat{G}_{i,j}$ is a 2-connected good outer-1-plane graph if G is such a graph.

A vertex set $\mathcal{V}[v_i, v_j]$ with $i \neq j$ is a *non-edge* if $j = i + 1$ and $v_iv_j \notin E(G)$, and is a *path* if $v_kv_{k+1} \in E(G)$ for all $i \leq k < j$. An edge v_iv_j in G is a *chord* if $|j - i| \neq 1$ or $|G| - 1$. By $\mathcal{C}[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$.

In any figure of this paper, the degree of a solid (or hollow) vertex is exactly (or at least) the number of edges that are incident with it, respectively, and a solid vertex is distinct to every another vertex but two hollow vertices may be identified unless stated otherwise.

We now collect some useful results that will be applied in the next sections.

Lemma 2.1. [14, Claim 1] *Let v_a and v_b be vertices of G. If there are no crossed chords in $\mathcal{C}[v_a, v_b]$ and no edges between $\mathcal{V}(v_a, v_b)$ and $\mathcal{V}(v_b, v_a)$, then $\mathcal{V}[v_a, v_b]$ is either non-edge or path.*

In what follows, when mentioning the configuration G_i with $1 \le i \le 14$ or S_i with $1 \le i \le 3$ we always refer to the corresponding picture in Figures 2.1 or 3.1.

Saying that G *contains* G_i or S_i, we mean that G contains a subgraph isomorphic to G_i or S_i such that the degree in G of any solid (resp. hollow) vertex in that picture is exactly (resp. at least) the number of edges that are incident with it there.

For two distinct vertices v_a and v_b on the outer boundary of G, saying $G[v_a, v_b]$ *properly contains* G_i or S_i, we mean that $G[v_a, v_b]$ contains G_i or S_i so that neither v_a nor v_b corresponds to a solid vertex or a hollow vertex with a degree restriction in the picture of G_i or S_i (e.g., the vertices y in G_1, G_9 and G_{10}, and the vertex z in S_1).

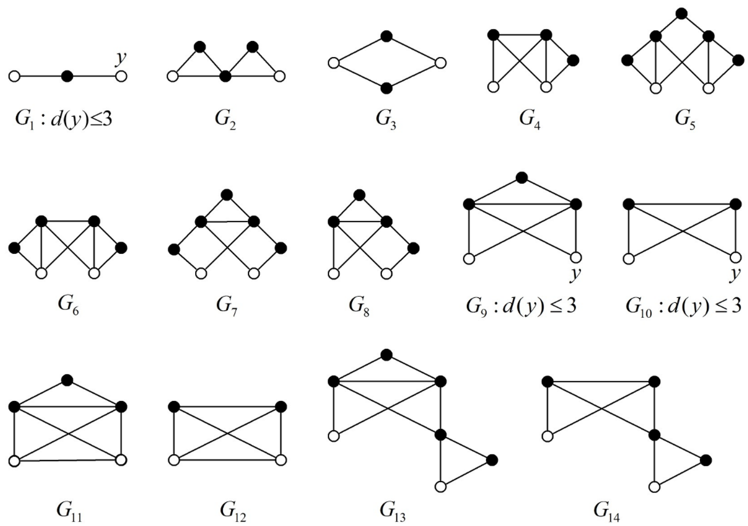

Figure 2.1. Local structures in outer-1-plane graph with $\vartheta(G) \ge 1$ and $\Delta(G) \le 4$

Lemma 2.2. *Let $\mathcal{V}[v_a, v_b]$ with $b - a \ge 3$ be a path in G. If $\Delta(G) \le 4$ and there are no crossed chords in $\mathcal{C}[v_a, v_b]$ and no edges between $\mathcal{V}(v_a, v_b)$ and $\mathcal{V}(v_b, v_a)$, then $G[v_a, v_b]$ properly contains G_1 or G_2.*

Proof. If $\mathcal{C}[v_a, v_b] \setminus \{v_a v_b\} = \emptyset$ (note that the chord $v_a v_b$ may not really exist), then $d(v_{a+1}) = d(v_{a+2}) = 2$ and G_1 is properly contained. If there is at least one chord in $\mathcal{C}[v_a, v_b] \setminus \{v_a v_b\}$, then choose one, say $v_{a'} v_{b'}$ with $a \le a' < b' \le b$, so that there is no other chord in $\mathcal{C}[v_{a'}, v_{b'}]$. If $b' - a' \ge 3$, then $d(v_{a'+1}) = d(v_{a'+2}) = 2$ and G_1 is properly contained. If $b' - a' = 2$, then $d(v_{a'+1}) = 2$. Choose $t \in \{a', b'\}$ such that $v_t \ne v_a, v_b$. If $d(v_t) \le 3$, then G_1 is properly contained. If $d(v_t) = 4$, then there is another one chord $v_t v_{c'}$ with $a \le c' \le b$ and $c' \ne a', b'$. If $|c' - t| = 2$, then $d(v_{t-1}) = d(v_{t+1}) = 2$, and thus G_2 is properly contained. If $|c' - t| \ge 3$, then let $a := \min\{c', t\}$, $b := \max\{c', t\}$ and come back to the first line of this proof. Since $t \ne a, b$, $|c' - t| < |b - a|$, which implies that this iterative process will terminate. □

Lemma 2.3. *Let $v_i v_j$ cross $v_k v_l$ in G with $i < k < j < l$ so that there are no other crossed chords besides $v_i v_j$ and $v_k v_l$ in $\mathcal{C}[v_i, v_l]$. If $\Delta(G) \le 4$ and $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \ge 4$, then $G[v_i, v_l]$ properly contains G_1 or G_2.*

Proof. Without loss of generality, assume that $\mathcal{V}[v_i, v_k] \ge 4$. This implies that $k - i \ge 3$. Note that there are no edges between $\mathcal{V}(v_i, v_k)$ and $\mathcal{V}(v_k, v_i)$, since G is outer-1-planar. By Lemma

2.1, $\mathcal{V}[v_i, v_k]$ is a path. By Lemma 2.2, $G[v_i, v_k]$ properly contains G_1 or G_2. Since for any vertex in $\mathcal{V}(v_i, v_k)$, its degree in $G[v_i, v_k]$ is the same as that in $G[v_i, v_l]$. Hence $G[v_i, v_l]$ properly contained G_1 or G_2. □

3 Local Structures

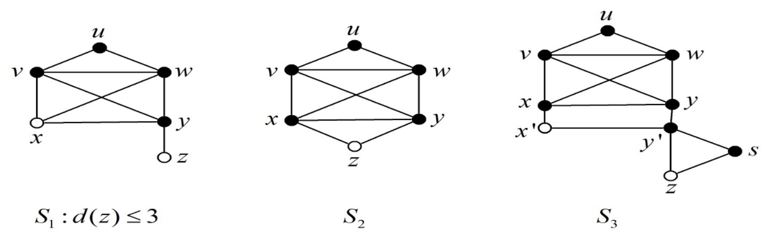

Figure 3.1. Three special configurations

Lemma 3.1. *Let G be a good 2-connected outer-1-plane graph with vertices $v_1, v_2, \cdots, v_n$ lying clockwise on its outer boundary, where $n = |G|$. If $\Delta(G) \leq 4$ and $\vartheta(G) \geq 1$, then $G[v_1, v_n]$ properly contains one of the configurations (see Figures 2.1 and 3.1)*

(1) G_1, G_3 if $n = 4$, unless $\mathcal{V}[v_1, v_4]$ is a path and $v_1v_3, v_2v_4 \in E(G)$;

(2) $G_1, \cdots, G_4, G_{10}$, G_{12} if $n = 5$, unless $\mathcal{V}[v_1, v_5]$ is a path and $v_1v_4, v_2v_4, v_2v_5 \in E(G)$;

(3) $G_1, \cdots, G_4, G_6, G_8, \cdots, G_{12}, G_{14}$ if $n = 6$;

(4) $G_1, \cdots, G_{14}$ if $n \geq 7$;

(5) $G_1, \cdots, G_{10}, G_{12}, G_{13}, G_{14}$, S_1, S_3 if $n \geq 8$ and $\vartheta(G) \geq 3$.

Proof. If there is no crossing in G, then $v_1v_2 \cdots v_n$ forms a path since G is 2-connected. Under this condition, one can easily show, by Lemma 2.2, that $G[v_1, v_n]$ properly contains G_1 if $n = 4$, and G_1 or G_2 if $n \geq 5$. Hence in the following we always assume that there is at least one crossing in G.

Case 3.1.1. $n = 4$**.**

Suppose that v_1v_3 crosses v_2v_4. Since G is good and 2-connected, at least two of v_1v_2, v_2v_3 and v_3v_4 belong to $E(G)$. If $v_2v_3 \notin E(G)$, then $v_1v_2, v_3v_4 \in E(G)$ and G_3 is properly contained. If $v_2v_3 \in E(G)$ and $\{v_1v_2, v_3v_4\} \nsubseteq E(G)$, then G_1 is properly contained. If $v_1v_2, v_2v_3, v_3v_4 \in E(G)$, then $\mathcal{V}[v_1, v_4]$ is a path.

Case 3.1.2. $n = 5$**.**

By symmetry, we consider three cases. First, if v_1v_3 crosses v_2v_4, then at least two of v_1v_2, v_2v_3 and v_3v_4 belong to $E(G)$, since G is good and 2-connected. If $v_2v_3 \notin E(G)$, then $v_1v_2, v_3v_4 \in E(G)$ and G_3 is properly contained. If $v_2v_3 \in E(G)$ and $\{v_1v_2, v_3v_4\} \nsubseteq E(G)$, then G_1 is properly contained If $v_1v_2, v_2v_3, v_3v_4 \in E(G)$, then G_{12} is properly contained if $v_1v_4 \in E(G)$, and G_{10} is contained if $v_1v_4 \notin E(G)$.

Second, if v_1v_3 crosses v_2v_5, then $v_3v_4, v_4v_5 \in E(G)$ and $d(v_4) = 2$, since G is 2-connected. If $\{v_2v_3, v_3v_5\} \nsubseteq E(G)$, then $d(v_3) \leq 3$ and G_1 is properly contained. If $\{v_2v_3, v_3v_5\} \subseteq E(G)$, then $G[v_1, v_n]$ properly contains G_3 if $v_1v_2 \notin E(G)$, or G_4 if $v_1v_2 \in E(G)$.

Third, if v_1v_4 crosses v_2v_5, then $v_2v_3, v_3v_4 \in E(G)$ and $d(v_3) = 2$, since G is 2-connected. If $\{v_1v_2, v_2v_4, v_4v_5\} \nsubseteq E(G)$, then $\min\{d(v_2), d(v_4)\} \leq 3$ and G_1 is properly contained. If $\{v_1v_2, v_2v_4, v_4v_5\} \subseteq E(G)$, then the excluded case occurs.

Case 3.1.3. $n = 6$**.**

Suppose that v_iv_j crosses v_kv_l with $1 \leq i < k < j < l \leq 6$. In what follows, we consider three major cases. Note that there is no edge between $\mathcal{V}(v_i, v_l)$ and $\mathcal{V}(v_l, v_i)$, since G is an outer-1-plane graph. So, the graph $\hat{G}_{i,l}$ is a 2-connected good outer-1-plane graph, and thus the results (1) and (2) can be applied to $\hat{G}_{i,l}$.

Subcase 3.1.3.1. $|\mathcal{V}[v_i, v_l]| = 4$.

By (1), $\mathcal{V}[v_i, v_l]$ is a path, because otherwise $\hat{G}_{i,l}[v_i, v_l]$ properly contains G_1 or G_3 , and so does $G[v_1, v_6]$.

If $v_iv_l \in E(G)$, then G_{12} is properly contained. Therefore, we consider the case that $v_iv_l \notin E(G)$. By symmetry, we discuss the following two subcases.

If $i = 1$, then $v_4v_5, v_4v_6 \in E(G)$, because otherwise $d(v_4) \leq 3$ and G_{10} is properly contained. This implies that $v_5v_6 \in E(G)$ since G is 2-connected. Therefore, G_{14} is properly contained.

If $i = 2$, then $d(v_2) \leq 3$ or $d(v_5) \leq 3$, because otherwise v_2v_6 crosses v_1v_5, contradicting the fact that $\vartheta(G) \geq 1$. Therefore, G_{10} is properly contained.

Subcase 3.1.3.2. $|\mathcal{V}[v_i, v_l]| = 5$.

Assume, without loss of generality, that $i = 1$. By (2), $\mathcal{V}[v_1, v_5]$ is a path and $v_1v_4, v_2v_4, v_2v_5 \in E(G)$, because otherwise $\hat{G}_{1,5}[v_1, v_5]$ (and thus $G[v_1, v_5]$) properly contains one configurations from the list in (2). If $v_1v_5 \in E(G)$, then G_{11} is properly contained. If $v_1v_5 \notin E(G)$, then $d(v_5) \leq 3$ and G_9 is properly contained.

Subcase 3.1.3.3. $|\mathcal{V}[v_i, v_l]| = 6$.

If $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \geq 4$, then by the fact that $\vartheta(G) \geq 1$ and by Lemma 2.3, $G[v_i, v_l]$ properly contains G_1 or G_2, and so does $G[v_1, v_n]$.

Hence we assume that $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \leq 3$. By symmetry we consider two subcases.

First, if v_1v_4 crosses v_2v_6, then by the 2-connectedness of G, we have $v_2v_3, v_3v_4, v_4v_5, v_5v_6 \in E(G)$. This implies that $d(v_3) = 2$. If $\{v_1v_2, v_2v_4\} \not\subseteq E(G)$, then $d(v_2) \leq 3$ and G_1 is properly contained. If $\{v_1v_2, v_2v_4\} \subseteq E(G)$, then $v_4v_6 \notin E(G)$ since $\Delta(G) \leq 4$, which implies that G_8 is properly contained.

Second, if v_1v_4 crosses v_3v_6, then by the the 2-connectedness of G, we have $v_1v_2, v_2v_3, v_4v_5, v_5v_6 \in E(G)$ and $d(v_2) = d(v_5) = 2$. If $\{v_1v_3, v_3v_4, v_4v_6\} \not\subseteq E(G)$, then v_3 or v_4 has degree at most 3 and G_1 is properly contained. If $\{v_1v_3, v_3v_4, v_4v_6\} \subseteq E(G)$, then G_6 is properly contained.

Case 3.1.4. $n \geq 7$.

We prove (4) by induction on n. First, we prove it for $n = 7$ in Case 3.1.4.1, and then assume that the result holds for good 2-connected outer-1-plane graphs G with order n', where $7 \leq n' < n$. In Case 3.1.4.2, we prove (4) for $n \geq 8$, where the above induction hypothesis will be frequently applied.

Case 3.1.4.1. $n = 7$.

Suppose that v_iv_j crosses v_kv_l with $1 \leq i < k < j < l \leq 7$. Three major cases are considered as follows. Again, note that there is no edge between $\mathcal{V}(v_i, v_l)$ and $\mathcal{V}(v_l, v_i)$, since G is an outer-1-plane graph. So, the graph $\hat{G}_{i,l}$ is a 2-connected good outer-1-plane graph, and thus the results (1), (2) and (3) can be applied to $\hat{G}_{i,l}$.

Subcase 3.1.4.1.1. $|\mathcal{V}[v_i, v_l]| = 4$.

By (1), we assume that $\mathcal{V}[v_i, v_l]$ is a path and $v_iv_j, v_kv_l \in E(G)$ (the reason for this is the same as the one we stated in Subcase 3.1.3.1. Here and below, we do not repeatedly explain why and how the previous results can be applied). If $v_iv_l \in E(G)$, then G_{12} is properly contained. Therefore, we consider the case that $v_iv_l \notin E(G)$. By symmetry, the following two subcases are considered.

If $i = 1$, then $d(v_4) = 4$ (otherwise G_{10} is properly contained), which implies that $v_4v_6 \in E(G)$ or $v_4v_7 \in E(G)$. If $v_4v_6 \in E(G)$, then by the 2-connectedness of G, $v_4v_5, v_5v_6 \in E(G)$ and $d(v_5) = 2$, which implies that G_{14} is properly contained. If $v_4v_6 \notin E(G)$ and $v_4v_7 \in E(G)$, then $v_4v_5, v_5v_6, v_6v_7 \in E(G)$ since G is 2-connected, which implies that $d(v_6) = 2$ and $d(v_5) \leq 3$. Hence G_1 is properly contained in $G[v_1, v_7]$.

If $i = 2$, then $d(v_2) = d(v_5) = 4$, because otherwise G_{10} is properly contained. This case appears only if $v_1v_2, v_2v_7, v_5v_6, v_6v_7, v_5v_7 \in E(G)$ since $\vartheta(G) \geq 1$, and thus G_{14} is properly contained.

Subcase 3.1.4.1.2. $|\mathcal{V}[v_i, v_l]| = 5$**.**

By symmetry, we consider two subcases.

First, if $i = 1$, then by (2), we assume that $\mathcal{V}[v_1, v_5]$ is a path and $v_1v_4, v_2v_4, v_2v_5 \in E(G)$. If $v_1v_5 \in E(G)$, then G_{11} is properly contained. Therefore we assume that $v_1v_5 \notin E(G)$. If $d(v_5) \leq 3$, then G_9 is properly contained. If $d(v_5) = 4$, then $v_5v_6, v_5v_7 \in E(G)$, and by the 2-connectedness of G, we also have $v_6v_7 \in E(G)$ and $d(v_6) = 2$. This implies that G_{13} is properly contained.

Second, if $i = 2$, then by (2), we assume that $\mathcal{V}[v_2, v_6]$ is a path and $v_2v_5, v_3v_5, v_3v_6 \in E(G)$. If $v_2v_6 \in E(G)$, then G_{11} is properly contained. If $v_2v_6 \notin E(G)$, then $d(v_2) \leq 3$ or $d(v_7) \leq 3$, because otherwise v_1v_6 crosses v_2v_7, contradicting the fact that $\vartheta(G) \geq 1$. Therefore, G_9 is properly contained.

Subcase 3.1.4.1.3. $|\mathcal{V}[v_i, v_l]| = 6$**.**

By (3), $G[v_i, v_l]$ properly contains $G_1, \cdots, G_4, G_6, G_8, \cdots, G_{12}, G_{14}$, and so does $G[v_1, v_7]$.

Subcase 3.1.4.1.4. $|\mathcal{V}[v_i, v_l]| = 7$**.**

If $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \geq 4$, then by Lemma 2.3, $G[v_i, v_l]$ properly contains G_1 or G_2, and so does $G[v_1, v_n]$. Here, note that there is no other crossed chords besides v_iv_j and v_kv_l in $\mathcal{C}[v_i, v_l]$, since $\vartheta(G) \geq 1$.

Hence we leave an unique case, that is, the case when v_1v_5 crosses v_3v_7. Since G is 2-connected, $\mathcal{V}[v_1, v_7]$ is a path and $d(v_2) = d(v_4) = d(v_6) = 2$. If $\{v_1v_3, v_5v_7\} \nsubseteq E(G)$, then $G[v_1, v_7]$ properly contains G_1 if $v_3v_5 \notin E(G)$, and G_7 otherwise. If $\{v_1v_3, v_5v_7\} \subseteq E(G)$, then $v_3v_5 \notin E(G)$ since $\Delta(G) \leq 4$, and thus G_5 is properly contained.

Case 3.1.4.2. $n \geq 8$.

Choose two mutually crossed chords v_iv_j and v_kv_l with $1 \leq i < k < j < l \leq n$ so that $l - i$ is as minimum as possible. Clearly, there is no other crossed chord besides v_iv_j and v_kv_l in $\mathcal{C}[v_i, v_l]$ by this choice.

If $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \geq 4$, then by Lemma 2.3, $G[v_i, v_l]$ properly contains G_1 or G_2, and so does $G[v_1, v_n]$. Hence we assume that $\max\{|\mathcal{V}[v_i, v_k]|, |\mathcal{V}[v_k, v_j]|, |\mathcal{V}[v_j, v_l]|\} \leq 3$ (i.e., $\max\{k - i, j - k, l - j\} \leq 2$). This implies that $|\mathcal{V}[v_i, v_l]| \leq 7$. If $|\mathcal{V}[v_i, v_l]| = 7$ (resp. $|\mathcal{V}[v_i, v_l]| = 6$), then by Case 3.1.4.1 (resp. by (3)), $\hat{G}_{i,l}[v_i, v_l]$ properly contains one of the required configurations, and so does $G[v_1, v_n]$.

If $|\mathcal{V}[v_i, v_l]| = 5$ (resp. $|\mathcal{V}[v_i, v_l]| = 4$), then by (2) (resp. by (1)), we only consider the case that $\mathcal{V}[v_i, v_l]$ is a path with $j - k = 2$ and $v_kv_j \in E(G)$ (resp. with $j - k = 1$). If $v_iv_l \in E(G)$, then G_{11} (resp. G_{12}) is properly contained. Therefore, we assume that $v_iv_l \notin E(G)$.

Since $i \neq 1$ or $l \neq n$, we assume, by symmetry, that $l \neq n$. If $d(v_l) \leq 3$, then G_9 (resp. G_{10}) is properly contained. If $d(v_l) = 4$, then by the fact that $\vartheta(G) \geq 1$, there is a non-crossed chord v_lv_s with $1 \leq s \leq n$ and $s \neq i, k$.

Subcase 3.1.4.2.1. $i \neq 1$.

It is easy to see that $l < s \leq n$ or $1 \leq s < i$. If $1 \leq s < i$, then $d(v_i) = 4$, because otherwise G_9 (resp. G_{10}) is properly contained. This implies that there is a non-crossed chord v_iv_t with $s \leq t < i$, since $\vartheta(G) \geq 1$. Therefore, we shall consider two major cases: (a) there is a non-crossed chord v_lv_s with $l < s \leq n$, or (b) there is non-crossed chord v_iv_t with $s \leq t < i$. Clearly, this two cases are symmetry. Hence we just need consider one, say (a).

Choose one s from those satisfying the condition (a) so that $s - l$ is as large as possible. At this moment, there is no chord in the form v_lv_t with $1 \leq t < i$, because otherwise v_s would be a cut vertex separating $\mathcal{V}(v_t, v_s)$ from $\mathcal{V}(v_s, v_t)$. Hence by the choice of s and by the fact that v_lv_s is non-crossed, there is no edge between $\mathcal{V}(v_i, v_s)$ and $\mathcal{V}(v_s, v_i)$.

If $s - l = 2$, then by the 2-connectedness of G, $v_lv_{l+1}, v_{l+1}v_s \in E(G)$ and $d(v_{l+1}) = 2$, which implies that G_{13} (resp. G_{14}) is properly contained. If $s-l \geq 3$, then $\hat{G}_{i,s}$ is a 2-connected good outer-1-plane graph with order n', where $7 \leq n' = s-i+1 < s \leq n$ (note that $l-i \geq 3$). By the induction hypothesis, $\hat{G}_{i,s}[v_i, v_s]$ properly contains one of the configurations among $G_1, \cdots, G_{14}$. Since there is no edge between $\mathcal{V}(v_i, v_s)$ and $\mathcal{V}(v_s, v_i)$, any configuration properly contained in $\hat{G}_{i,s}[v_i, v_s]$ is properly contained in $G[v_i, v_s]$, and then in $G[v_1, v_n]$.

Subcase 3.1.4.2.2. $i = 1$.

In this case we have $l < s \leq n$. Choose such an s so that $s-l$ is as large as possible. If $s-l = 2$, then by the 2-connectedness of G, $v_l v_{l+1}, v_{l+1} v_s \in E(G)$ and $d(v_{l+1}) = 2$, which implies that G_{13} (resp. G_{14}) is properly contained. If $s-l \geq 3$ and $s \neq n$, then applying the induction hypothesis to the graph $\hat{G}_{i,s}$ and we can obtain the required result as we have done in Subcase 3.1.4.2.1. Note that there is no edge between $\mathcal{V}(v_i, v_s)$ and $\mathcal{V}(v_s, v_i)$ by the choice of s. At last, we are left the case that $s-l \geq 3$ and $s = n$.

If there is a pair of crossed chords $v_{i'} v_{j'}$ and $v_{k'} v_{l'}$ with $l \leq i' < k' < j' < l' \leq s$, then before augmenting, we can properly choose such a pair in advance so that $l' - i'$ is as minimum as possible. Now, we come back to the first line of Case 3.1.4.2 by setting $i := i', j := j', k := k'$ and $l := l'$. Note that $i' \neq 1$, and thus the new i is not 1. Therefore, this second round of arguments will not involve the current subcase and thus one of the required configurations can be properly contained in $G[v_1, v_n]$.

Hence we assume that there are no crossed chords in $\mathcal{C}[v_l, v_s]$. Since $v_l v_s$ is not crossed, there are no edges between $\mathcal{V}(v_l, v_s)$ and $\mathcal{V}(v_s, v_l)$. Since $s-l \geq 3$, $\mathcal{V}[v_i, v_l]$ is a path by Lemma 2.1. So, by Lemma 2.2, $G[v_l, v_s]$ properly contains G_1 or G_2, and so does $G[v_1, v_n]$.

Case 3.1.5. $n \geq 8$ **and** $\vartheta(G) \geq 3$**.**

We prove (5) for by induction on n. First, we prove it for $n = 8$ in Case 3.1.5.1, and then assume that the result holds for good 2-connected outer-1-plane graphs G with order n', where $8 \leq n' < n$. In Case 3.1.5.2, we prove (5) for $n \geq 9$, where the above induction hypothesis will be frequently applied.

Case 3.1.5.1. $n = 8$.

We assume that $G[v_1, v_n]$ properly contains G_{11}, as otherwise the result follows from (4). Let $G[v_i, v_l] \cong G_{11}$, where $i \leq i < l \leq n$. Since G is 2-connected, v_i, v_l both have degree 4 (otherwise one of them is a cut-vertex of G), and there exist chords $v_l v_s$ and $v_t v_i$ such that $s \neq i$ and $t \neq l$. If $s < i$, then v_s becomes a cut-vertex of G unless $s = 1$ and $l = n$, since $v_l v_s$ is non-crossed by the face that $\vartheta(G) \geq 1$. Hence $l < s \leq n$ if $l \neq n$. Similar, $1 \leq t < i$ if $i \neq 1$. Since $l - i = 4$, either $i \neq 1$ or $l \neq n$. Without loss of generality, we assume the latter, and thus $l < s \leq n$. If $s - l \geq 2$, then v_s is a cut-vertex separating v_l from v_{l+1}, contradicting the 2-connectedness of G. Hence $s = l + 1$.

If $v_i v_s \in E(G)$, then v_s is a cut-vertex separating $\mathcal{V}[v_i, v_l]$ from $V(G) \backslash \mathcal{V}[v_i, v_s]$, contradicting the 2-connectedness of G. Hence $v_i v_s \notin E(G)$ and $d(v_s) \leq 3$, since $n = 8$. This implies that $G[v_1, v_8]$ properly contains S_1.

Case 3.1.5.2. $n \geq 9$.

As in Case 3.1.5.1, we may assume that $G[v_i, v_l] \cong G_{11}$, where $l \leq i < l \leq n$. Since $l - i = 4$, either $i \neq 1$ or $l \neq n$. Without loss of generality, assume that $l \neq n$. By the same argument as in Case 3.1.5.1, we can prove that $v_l v_{l+1} \in E(G)$, $v_i v_{i-1} \in E(G)$ if $i \geq 2$, and $v_1 v_n \in E(G)$ if $i = 1$.

If we meet the case that $i = 1$ and $v_1 v_n \in E(G)$, then $l + 1 = 6 < n$. By relabelling the vertices of G from $v_1, v_2, \cdots, v_{n-1}, v_n$ to $v_2, v_3, \cdots, v_n, v_1$, we translate this case to the one that $i \geq 2$, $l \neq n$ and $v_l v_{l+1}, v_i v_{i-1} \in E(G)$.

Therefore, we assume $i \geq 2$ and $v_i v_{i-1} \in E(G)$ in the following.

Since $n \geq 9$ and $|\mathcal{V}[v_{i-1}, v_{l+1}]| = 7$, either $i - 1 \neq 1$ or $l + 1 \neq n$. Without loss of generality, assume the latter. If $d(v_{l+1}) \leq 3$, then S_1 is properly contained. If $d(v_{l+1}) = 4$, then there is a chord $v_{l+1} v_s$ with $1 \leq s \leq n$ so that $s > l + 1$ or $s < i - 1$. Since $\vartheta(G) \geq 3$, $v_{l+1} v_s$ is non-crossed.

If $s < i-1$, then $i-1 \neq 1$ and thus $d(v_{i-1}) = 4$ as otherwise S_1 is properly contained. This implies that there is a chord $v_{i-1}v_t$ with $s \le t < i-1$. Therefore, there is either (a) a chord $v_{l+1}v_s$ with $l+1 < s \le n$, or (b) a chord $v_{i-1}v_t$ with $1 \le t < i-1$. By symmetry, we assume that (a) exists. Among those s satisfying (a), choose one s so that $s-(l+1)$ is as large as possible.

If there is a chord v_tv_{l+1} so that $1 \le t < i-1$, then v_tv_{l+1} is non-crossed since $\vartheta(G) \ge 3$, which implies that there is no edge between $\mathcal{V}(v_t, v_{l+1})$ and $\mathcal{V}(v_{l+1}, v_t)$. Therefore, $\hat{G}_{t,l+1}$ is a 2-connected good outer-1-planar graph with order n', where $8 \le n' = (l+1)-t+1 < n$ (note that $t < i-1$ and $l-i = 4$), and then by the induction hypothesis, $\hat{G}_{t,l+1}[v_t, v_{l+1}]$ properly contains one of the configurations from the list in (5). Since no edge exists between $\mathcal{V}(v_t, v_{l+1})$ and $\mathcal{V}(v_{l+1}, v_t)$, any configuration properly contained in $\hat{G}_{t,l+1}[v_t, v_{l+1}]$ is properly contained in $G[v_t, v_{l+1}]$, and then in $G[v_1, v_n]$.

Hence we assume that there is no chord v_tv_{l+1} with $1 \le t < i-1$. At this stage, there is no edge between $\mathcal{V}(v_{i-1}, v_s)$ and $\mathcal{V}(v_s, v_{i-1})$ by the choice of s and the fact that $v_{l+1}v_s$ in non-crossed.

If $s = l+3$, then $v_{l+1}v_{l+2}, v_{l+2}v_s \in E(G)$ and $d(v_{l+2}) = 2$ by the 2-connectedness of G. Since v_{l+1} has degree 4, by the choice of s, we shall have $v_{i-1}v_{l+1} \in E(G)$, which implies that S_3 is properly contained in $G[v_{i-1}, v_s]$, and thus in $G[v_1, v_n]$.

If $s = l+4$, then by (1), $G[v_{l+1}, v_s]$ (and so does $G[v_1, v_n]$) properly contains G_1 or G_3, unless $v_{l+1}v_{l+3}$ crosses $v_{l+2}v_s$, which case contradicts the fact that $\vartheta(G) \ge 3$.

If $s = l+5$, then by (2), $G[v_{l+1}, v_s]$ (and so does $G[v_1, v_n]$) properly contains one of the configurations among $G_1, \cdots, G_4, G_{10}, G_{12}$, unless $v_{l+1}v_{l+4}$ crosses $v_{l+2}v_5$, which case contradicts the fact that $\vartheta(G) \ge 3$.

If $s = l+6$, then by (3), $G[v_{l+1}, v_s]$ (and so does $G[v_1, v_n]$) properly contains one of the configurations among $G_1, \cdots, G_4, G_6, G_8, G_9, G_{10}, G_{12}, G_{14}$, unless $G[v_{l+2}, v_s] \cong G_{11}$, which case contradicts the fact that $\vartheta(G) \ge 3$..

If $s = l+7$, then by (4), $G[v_{l+1}, v_s]$ (and so does $G[v_1, v_n]$) properly contains one of the configurations among $G_1, \cdots, G_{10}, G_{12}, G_{13}, G_{14}$, unless $G[v_{l+2}, v_{l+6}] \cong G_{11}$ and $v_{l+1}v_{l+2}, v_{l+6}v_s \in E(G)$. This contradicts the fact that $\vartheta(G) \ge 3$.

If $s \ge l+8$, then applying the induction hypothesis to the graph $G[v_{l+1}, v_s]$, which has order $8 \le s-l < n$, we then conclude that $G[v_{l+1}, v_s]$ properly contains at least one configuration from the list in (5), and so does $G[v_1, v_n]$. □

Theorem 3.1. *Every outer-1-planar graph G with crossing distance at least 3 contains one of the configurations $G_1, \cdots, G_{10}, G_{12}, G_{13}, G_{14}$, S_1, S_2 or S_3 as long as $\Delta(G) \le 4$ and $\delta(G) \ge 2$.*

Proof. If G is 2-connected, then let $H = G$, otherwise let H be an end-block of G (i.e., a 2-connected subgraph having only one cut-vertex of G). Clearly $|H| \ge 3$ since $\delta(G) \ge 2$. Let $u_1, u_2, \cdots, u_{|H|}$ be the vertices of H with clockwise ordering on the boundary, where u_1 is the cut-vertex of G if H is an end-block of G.

If $|H| \ge 8$, then by Lemma 3.1(5), $H[u_1, u_{u_{|H|}}]$ properly contains one of those configurations, which is also contained in G.

If $6 \le |H| \le 7$, then by Lemmas 3.1(3) and 3.1(4), $H[u_1, u_{u_{|H|}}]$ properly contains the configurations $G_1, \cdots, G_{10}, G_{12}, G_{13}$ or G_{14} (and then G contains one of them) if it does not properly contain G_{11}. Suppose that $H[u_1, u_{u_{|H|}}]$ properly contains G_{11}. Let $H[u_i, u_l] \cong G_{11}$ and assume, without loss of generality, that $i < l$. Suppose there is an edge $v_lv_s \in E(H)$ with $s \neq i, j, k$, because otherwise v_l has degree 3 in G and thus G contains G_9. If $s \neq 1$, then u_s has degree at most 3 in G since $|H| \le 7$, and thus G contains S_1. If $s = 1$ and $|H| = 7$, then $i = 3$ and $l = 7$, because otherwise u_1 would be a cut-vertex of H, contradicts the 2-connectedness of H. This implies $u_1u_2, u_2u_3 \in E(H)$ and thus $H[u_1, u_{u_{|H|}}]$ properly contains S_1, which is also contained in G. If $s = 1$ and $|H| = 6$, then by the 2-connectedness of H, $i = 2, l = 6$ and $u_1u_2 \in E(H)$, which implies that H is isomorphic to the graph S_2 with u_1 corresponding to the vertex z in that picture of S_2 in Figure 3.1, and then G contains S_2.

If $|H| = 5$, then by Lemma 3.1(2), $H[u_1, u_5]$ properly contains $G_1, \cdots, G_4$ or G_{12}, and then G contains one of them, or $\mathcal{V}[u_1, u_5]$ is a path and $u_1u_4, u_2u_4, u_2u_5 \in E(G)$. In the latter case, we have $d(u_5) \leq 3$ and then G contains G_9.

If $|H| = 4$, then by Lemma 3.1(1), $H[u_1, u_4]$ properly contains G_1 or G_3, and then G contains one of them, or $\mathcal{V}[u_1, u_4]$ is a path and $u_1u_3, u_2u_4 \in E(G)$. In this case, we have $d(u_4) \leq 3$ and then G contains G_{10}.

If $|H| = 3$, then u_2 and u_3 are two adjacent vertices of degree 2 in G. Hence G contains G_1. □

4 List coloring results

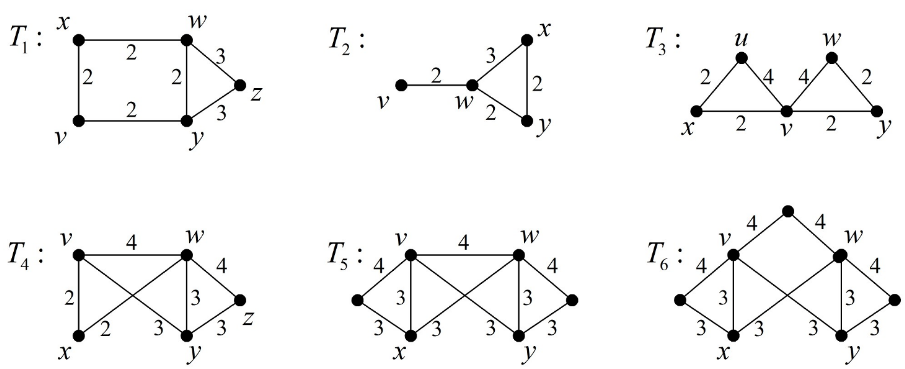

Figure 4.1. Six edge-choosable graphs

Each of the graphs T_i in Figure 1 is assumed to have an associated function L_i that assigns a list of colors to each edge, of the size indicated by that edge.

Theorem 4.1. *(a) If $L_1(wx) \cap L_1(vy) = \emptyset$, then T_1 can be L_1-colored.*
(b) For each integer $2 \leq i \leq 6$, the graph T_i is L_i-colorable.

Proof. We consider each graph separately.

Case 4.1.1. T_1. If the edges wx, wy, vy are L_1-colored with colors a, b, c, say, and this coloring cannot be extended to wz and yz, then there is a color d such that $L_1(wz) = \{a, b, d\}$ and $L_1(yz) = \{b, c, d\}$. These lists then determine a and c uniquely, as $\{a\} = L_1(wz) \setminus L_1(yz)$ and $\{c\} = L_1(yz) \setminus L_1(wz)$.

There are four different ways in which the edges wx and vy can be L_1-colored. Since $L_1(wx) \cap L_1(vy) = \emptyset$, at most one of these four colorings uses both colors in $L_1(vx)$, at most one uses both colors in $L_1(wy)$, and (as we have just seen) at most one can be extended to wy and vx but cannot then be extended to wz and yz. Thus at least one of the four L_1-colorings of wx and vy can be extended to an L_1-coloring of T_1.

Case 4.1.2. T_2. If we can give vw and xy the same color, or alternatively give wy a color not in $L_2(vw)$, then the remaining edges are easily colored. So we may assume $L_2(vw) \cap L_2(xy) = \emptyset$ and $L_2(vw) = L_2(wy)$. But then $L_2(wy) \cap L_2(xy) = \emptyset$, and any L_2-coloring of the three edges at w can be extended to xy.

Case 4.1.3. T_3. Let P denote the path $uxvyw$. L_3-color the edges in order along P. Then there is at least one color available for each of uv and vw, and the only problem is if there is exactly one and it is the same color, say d, in each case. If this happens, then the colors along P take the form a, b, c, a' and $L_3(uv) = \{a, b, c, d\}$ and $L_3(vw) = \{a', b, c, d\}$, where possibly $a' = a$ but otherwise the colors are distinct. We may assume that $L_3(wy) = \{a', c\}$, as otherwise we could change the color of wy and color uv and vw with d and a'; and we may assume that $L_3(vy) = \{b, c\}$, as otherwise we could change the color of vy and color uv and wy with c and vw with an available color. In a similar way, we may assume that $L_3(ux) = \{a, b\}$ and $L_3(vx) = \{b, c\}$. But then we can recolor the edges of P with b, c, b, c and uv, vw with a, d.

Case 4.1.4. T_4. If possible, L_4-color vy and wx with the same color, and then color vx; the remaining edges can then be colored since T_2 is L_2-colorable. So we may assume that $L_4(vy) \cap L_4(wx) = \emptyset$.

If possible, L_4-color vx and wy with the same color, and then color wx; the remaining edges can then be colored since they form a 4-cycle with at least two colors available for each edge, and a 4-cycle is edge-2-choosable. So we may assume that $L_4(vx) \cap L_4(wy) = \emptyset$.

If possible, L_4-color vw with a color not in $L_4(vx) \cup L_4(wx)$; the remaining edges can then be colored since T_1 is L_1-colorable. So we may assume that $L_4(vw) = L_4(vx) \cup L_4(wx)$, say $L_4(vx) = \{a, b\}$, $L_4(wx) = \{c, d\}$, and $L_4(vw) = \{a, b, c, d\}$.

If possible, L_4-color vy and wy with distinct colors not in $L_4(vw)$; then vx and wx are easily colored (they can be treated as non-adjacent, as their lists are disjoint), and the remaining edges can be colored in the order yz, wz, vw. So we may assume that there is a color e such that $L_4(vy) = \{a, b, e\}$ and $L_4(wy) = \{c, d, e\}$.

Now choose a color $p \in \{a, b, c, d, e\} \setminus L_4(wz)$. It is easy to see that the edges not incident with z can be L_4-colored so that p is used on an edge at w. (If $p \in \{a, b, c, d\}$ then we can use p on vw, and if we use c or d on vw then e is used on wy.) Then the remaining edges yz, wz can be colored in this order.

Case 4.1.5. T_5. If possible, L_5-color vy and wx with the same color, leaving at least three possible colors for vw. Since T_2 is L_2-colorable, at most one of these colors for vw cannot be extended to the triangle on the right, and at most one cannot be extended to the triangle on the left, and so at least one can be extended to both triangles, giving an L_5-coloring of T_5.

So we may assume that $L_5(vy) \cap L_5(wx) = \emptyset$. There are then nine different ways in which the two edges vy and wx can be L_5-colored. At most four of these ways use two colors from $L_5(vw)$, the worst case being when $L_5(vw)$ contains two colors from $L_5(vy)$ and two from $L_5(wx)$. Similarly, at most two of these ways use two colors from $L_5(vx)$ and at most two use two colors from $L_5(wy)$. So there is at least one way of L_5-coloring vy and wx with two colors that are not both in $L_5(vw)$, not both in $L_5(vx)$, and not both in $L_5(wy)$. The remaining edges can then be colored in the same way as before.

Case 4.1.6. T_6. As in Case 4.1.5, L_6-color vy and wx either with the same color, or with two different colors that are not both in $L_6(vx)$ and not both in $L_6(wy)$. The remaining edges can be colored by applying Case 4.1.2 twice. ☐

Theorem 4.2. *(a) R_0 is L-colorable unless each color in $L(vw)$ is in the lists of exactly two adjacent edges of F;*
(b) Each of the graphs R_1, R_2, R_3 is L-colorable.

Each of the graphs in Figure 4.2 is assumed to have an associated function, which for convenience we denote by the same letter L in each case, that assigns a list of colors to each edge, of the size indicated by that edge. In each graph, let F denote the 4-cycle $vxwy$, with edges vx, wx, wy, vy in that order. Note that F will be frequently used in the next proofs.

Proof. We consider each graph separately.

Case 4.2.1. R_0. Since a 4-cycle is edge-2-choosable, the edges of F can be L-colored, with colors a, b, c, d, say. Assume that vw cannot now be colored. Then a, b, c, d are all different and $L(vw) = \{a, b, c, d\}$. If the list of some edge of F has a color not in $\{a, b, c, d\}$ then we can recolor that edge with such a color and use the freed color for vw; this contradiction shows that all lists are subsets of $\{a, b, c, d\}$. If the lists of two opposite (i.e., non-adjacent) edges of F have a color in common, then we can color those two edges with that color and the coloring is easily completed; this contradiction shows that each color occurs in the lists of two adjacent edges of F, which proves (a).

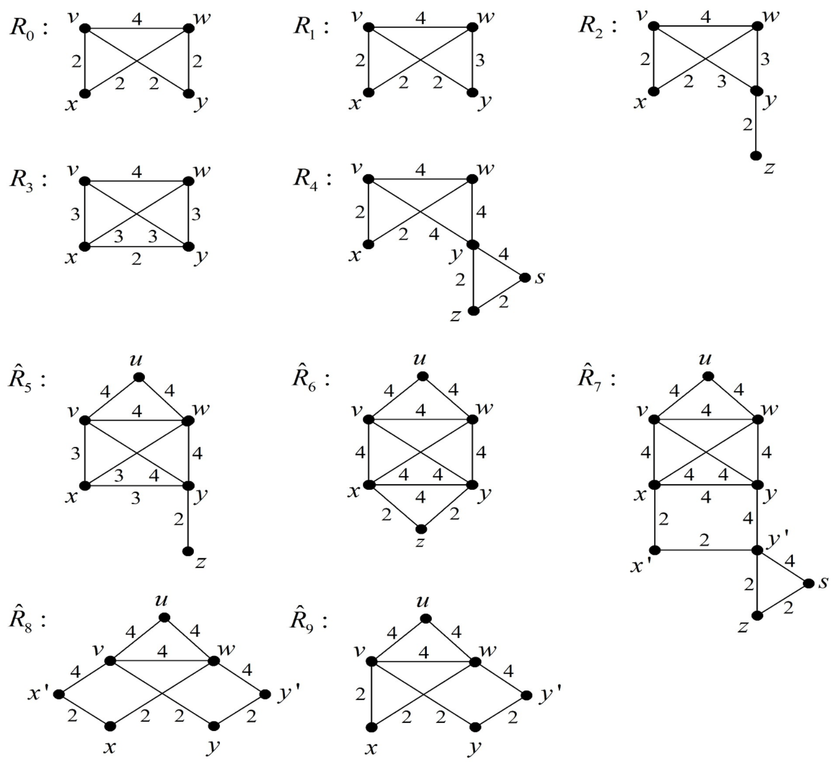

Figure 4.2. More graphs for list-edge-coloring

Case 4.2.2. R_1. Delete a color from $L(wy)$ so that the list of every edge of F now has two colors. If these lists are of the form described in (a), delete a different color from $L(wy)$ instead; the coloring can now be completed.

Case 4.2.3. R_2. We will color yz first; for $e \in \{vy, wy\}$, let $L_0(e)$ be obtained from $L(e)$ by deleting the color of yz; for every other edge of $R_2 - yz$, let $L_0(e) = L(e)$. We must color yz so that $R_2 - yz$ has an L_0-coloring. We may assume that $L(yz) \subseteq L(vy) \cap L(wy)$, as otherwise we can color yz so that at least one of $L_0(vy)$ and $L_0(wy)$ contains three colors, and the result will follow by Case 4.2.2 (possibly with v and w interchanged).

Let $L(yz) = \{p, q\}$. Color yz with p. If the coloring cannot be completed, then the lists L_0 have the form described in (a), with $L(vw) = L_0(vw) = \{a, b, c, d\}$ and each of these colors being in the lists of two adjacent edges of F. Note that q is in both $L_0(vy)$ and $L_0(wy)$, and p is in neither of these lists, so that either (i) $p \in L_0(vx) \cap L_0(wx)$ or (ii) $p \in \{a, b, c, d\}$. If we recolor yz with q then we can complete the coloring, either by using p on two nonadjacent edges of F, in case (i), or by using it on one edge, in case (ii).

Case 4.2.4. R_3. Let $L(vw) = \{a, b, c, d\}$. If possible, use the same color on xy and vw; the edges of the 4-cycle $vxwy$ can then be colored from the remaining lists of at least two colors. So we may assume that $L(xy) = \{p, q\}$, where $p, q \notin \{a, b, c, d\}$.

If the list of some edge of the 4-cycle F contains p, first color xy with q and then extend the coloring to F so that some edge has color p; now we can color vw. If no edge of F has p in its list, color xy with color p, and extend this coloring to the remaining edges by Case 4.2.2. $\square$

For each graph R_i in Figure 4.2, let $\widehat{R}_i$ denote the graph obtained from R_i by adding a new vertex u with neighbors v and w, and let L be extended to $\widehat{R}_i$ by giving lists of size 4 to the new edges uv and uw.

Theorem 4.3. *(a) $\widehat{R}_1$ is L-colorable;*
(b) $\widehat{R}_2$ is L-colorable if (i) some color is in the lists of two opposite (non-adjacent) edges of F, or (ii) $L(vy) = L(wy)$.

Proof. We know from Theorem 4.2 that R_1 and R_2 are L-colorable; so consider an L- coloring of one of them; we will try to extend it to $\widehat{R}_1$ or $\widehat{R}_2$ as appropriate. Let p be the color given to yz in R_2. For $e \in \{vy, wy\}$ in $\widehat{R}_2$, let $L_0(e) = L(e) \setminus \{p\}$; for every other edge of $\widehat{R}_2$, and every edge of $\widehat{R}_1$, let $L_0(e) = L(e)$. As in Theorem 4.2, we will assume in $\widehat{R}_2$ that $L(yz) \subseteq L(vy) \cap L(wy)$, as otherwise we can color yz so that at least one of $L_0(vy)$ and $L_0(wy)$ contains three colors, which is covered by the proof for $\widehat{R}_1$.

Let the edges vx, xw, wy, yv of F have colors a, b, a', b' in this order, and let vw have color c, where possibly $a = a'$ and/or $b = b'$ but the colors are otherwise distinct. There is at least one color available for each of uv and uw, and the only problem arises if it is the same color, say d, in each case, i.e., $L(uv) = \{a, b', c, d\}$ and $L(uw) = \{a', b, c, d\}$, where clearly $d \notin \{a, a', b, b?c\}$. Assume that this problem cannot be avoided by recoloring one or more of the edges of F and/or vw. Then the following hold.

- **(O1)** $L_0(e) \subseteq \{a, a', b, b', c\}$ for each $e \in E(F)$, and $L(vw) \subseteq \{a, a', b, b', c, d\}$; otherwise we can avoid the problem by recoloring a single edge.
- **(O2)** If $d \in L(vw)$ then $L_0(e) \subseteq \{a, a', b, b'\}$ for each $e \in E(F)$. To see this, recolor vw with d and apply (O1) with c and d interchanged.
- **(O3)** If $b \neq b'$ then $b \notin L_0(vy)$ and $b' \notin L_0(wx)$, as otherwise we can avoid the problem by recoloring vy with b (the same as wx) or wx with b' (the same as vy).
- **(O4)** If $b \neq b'$ and $b' \in L(vw)$ then $c \notin L_0(vy)$, as otherwise we can avoid the problem by swapping the colors of vy, vw from b', c to c, b'.
- **(O5)** In $\widehat{R}_2$, $L(yz) = \{p, q\}$ where, for each $e \in \{vy, wy\}$, $p \in L(e) \setminus L_0(e)$, $q \in L_0(e)$, and $L(e) = L_0(e) \cup \{p\}$.

We now consider four cases.

Case 4.3.1. $a = a'$ and $b = b'$. Then $L(uv) = L(uw) = L(vw) = \{a, b, c, d\}$, and $L_0(e) = \{a, b\}$ for every edge $e \in E(F)$ by (O1) and (O2). This is impossible in $\widehat{R}_1$ (where $|L_0(wy)| = 3$), and it can be avoided in $\widehat{R}_2$ by changing the color of yz. In the latter case, by (O5), $L(yz) = \{a, p\}$ or $\{b, p\}$, where $p \notin \{a, b\}$. If $L(yz) = \{a, p\}$ then we can swap the colors of wy, yz from a, p to p, a and (re)color uv, vw, wu with c, d, a if $p = c$ and with d, c, a otherwise; the case $L(yz) = \{b, p\}$ is similar.

Case 4.3.2. $a = a'$ and $b \neq b'$. Then $L(vw) \subset \{a, b, b', c, d\}$. Thus $c \notin L_0(vy)$ by (O2) and (O4), and $b \notin L_0(vy)$ by (O3), and so $L_0(vy) = \{a, b'\}$. Similarly, $L_0(wx) = \{a, b\}$. Now we can avoid the problem by recoloring vy and wx with a, and then recoloring vx and wy, unless $L_0(vx) = L_0(vy) = \{a, b'\}$ and $L_0(wx) = L_0(wy) = \{a, b\}$. This is impossible in $\widehat{R}_1$, and it can be avoided in $\widehat{R}_2$ by changing the color of yz. In the latter case, by (O5), $L(yz) = \{a, p\}$ for some color $p \notin \{b, b'\}$ (since $p \notin L_0(vy) \cup L_0(wy)$), and we can swap the colors of wy and yz and (re)color uv, vw, wu exactly as described at the end of Case 4.3.1.

Case 4.3.3. $a \neq a'$ and $b = b'$. This is equivalent to Case 4.3.2. (Interchange v and w.)
Case 4.3.4. $a \neq a'$ and $b \neq b'$. We may assume that no color occurs in the lists L_0 of two opposite edges of F, as otherwise we could use it on those two edges and get a new coloring that is covered by a previous case. Thus each of the colors a, a', b, b', c occurs in the list L_0 of either at most one, or two adjacent, edges of F.

The proof of (b) is now easily completed. If (i) holds, then we can choose the color of yz so that some color occurs in the lists L_0 of two opposite edges of F, and this coloring can be extended to R_0 by Theorem 4.2; as just remarked, this is covered by a previous case. If (ii) holds, then we can avoid the problem by swapping the colors of vy and wy, thereby recoloring vy with $a' \notin L(uv)$ and wy with $b' \notin L(uw)$. So from now on we assume the graph is $\widehat{R}_1$.

We claim that c does not occur in the list of any edge of F. Suppose it does, say $c \in L_0(vy) = L(vy)$. (The other cases are similar.) By (O2) and (O4), this implies that $\{b', d\} \cap L(vw) = \emptyset$ and so $L(vw) = \{a, a', b, c\}$; thus, by the analogs of (O4) for other edges of F, $c \notin L_0(vx)$ since $a \in L(vw)$, $c \notin L_0(wx)$ since $b \in L(vw)$, and $c \notin L_0(wy)$ since $a' \in L_0(vw)$. It follows that b' is in the list of another edge of F, as otherwise the lists of vx, wx and wy contain the colors a, a' and b twice each, and some color is in the lists of two opposite edges. So we can color vy and uw with c, some other edge of F with b', and then complete the coloring using two different colors from $\{a, a', b\}$ on F and the third color on vw; and now uv can be colored as it has two adjacent edges with the same color, c.

It follows that $L_0(e) \subset \{a, a', b, b'\}$ for every edge e of F in $\widehat{R}_1$, and so one of these colors must occur on at least three edges, which contradicts the first paragraph of Case 4.3.4. This completes the proof of Theorem 4.3. □

Lemma 4.1. *Let $T(syz)$ be a triangle with an associated function L that assigns lists of two colors to edges sz and yz and a list of four colors to sy. Then there are distinct colors a, b, c, d such that $L(yz) = \{a, c\}$, $\{b, d\} \in L(sy)$, and $T(syz)$ can be L-colored with the two edges at y receiving any of the three pairs of colors $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$.*

Proof. Let $L(yz) = \{a, c\}$ and let b, d be two colors in $L(sy) \setminus \{a, c\}$, labeled so that $L(sz) \neq \{a, b\}, \{b, c\}$ or $\{c, d\}$. (In other words, if $L(sz)$ comprises one of a, c and one of b, d, then swap the labels of the colors so that $L(sz) = \{a, d\}$.) Then the result clearly holds. □

Lemma 4.2. *Let p be an arbitrary but fixed color. For each of the following conditions, there is an L-coloring of the edges of $\widehat{R}_7$ below xy in Figure 4.2 such that the condition holds:*
(C1) xx' and yy' have different colors;
(C2) xx' does not have color p;
(C3) yy' does not have color p.

Proof. Choose colors a, b, c, d that satisfy the conclusion of Lemma 4.1 for triangle $T(sy'z)$ in $\widehat{R}_7$. The conclusion of Lemma 4.2 clearly holds for (C2), as we can color xx' differently from p and continue the coloring in order along the walk $xx'y'zsy'y$. The same holds for (C1) if b or $c \in L(xx')$, as then we can color xx' and sy' with b, or xx' and $y'z$ with c, and then color the remaining edges in order along the above walk. So in proving (C1) we may assume that $L(xx') \cap \{b, c\} = \emptyset$. We must prove (C1) and (C3). There are two cases.
Case 4.2.1. $L(x'y') \neq \{b, c\}$. Color $x'y'$ with $q \notin \{b, c\}$, then color xx', and let p denote the color of xx' if we are proving (C1). Since q cannot equal both a and d, assume w.l.o.g. $q \neq a$, and color $y'z, sy'$ with a, b if $L(yy') = \{b, c, p, q\}$ and with c, b otherwise; then the coloring can be completed with yy' not receiving color p.
Case 4.2.2. $L(x'y') = \{b, c\}$. In proving (C1), give xx' a color and call it p, noting that $p \notin \{b, c\}$ by the assumption made before Case 4.2.1. Color $x'y'$, $y'z$, sy' with b, c, d if $L(yy') = \{a, b, c, p\}$ and with c, a, b otherwise. Then the coloring can be completed with yy' not receiving color p. □

Lemma 4.3. *Let $T(vwx)$ be a triangle and vy be an edge incident with it. If there is an associated function L that assigns lists of two colors to edges vy, wx and vx and a list of four colors to vw so that $L(vy) \cap L(wx) = \emptyset$, then we can L-color vy, wx and vx so that there are at least two colors still available for L-coloring vw while extending this partial coloring if (i) $L(vw) \neq L(vy) \cup L(wx)$, or (ii) $L(vx) \not\subset L(vw)$.*

Proof. Suppose that $L(vy) = \{a, b\}$ and $L(wx) = \{c, d\}$.

(i) Since $L(vw) \neq \{a, b, c, d\}$, either $\{a, c\} \not\subset L(vw)$ or $\{b, d\} \not\subset L(vw)$. Assume, w.l.o.g., that $\{a, c\} \not\subset L(vw)$. If $L(vx) \neq \{a, c\}$, then color vy, wx with a, c, and color vx with a color in $L(vx) \setminus \{a, c\}$. If $L(vx) = \{a, c\}$, then color vy, wx, vx with b, c, a. In any of the above two cases, there are at least two colors still available for L-coloring vw.

(ii) L-color vx with a color not in $L(vw)$, and vy, wx can then be easily colored. Obviously, there are at least two colors still available for L-coloring vw at this stage. ☐

Theorem 4.4. *Each of the graphs R_4 and $\widehat{R}_4, \cdots, \widehat{R}_9$ is L-colorable.*

Proof. Obviously, if $\widehat{R}_4$ is L-colorable then so is R_4, and so we will consider just the last four graphs.

Case 4.4.1. $\widehat{R}_4$. By Lemma 4.1, there are distinct colors a, b, c, d such that the edges of the triangle syz can be L-colored with the two edges at y receiving any of the three pairs of colors $\{a, b\}, \{b, c\}$ and $\{c, d\}$. If $L(wy) \neq \{a, b, c, d\}$ then at least one of these colorings will leave wy with at least three available colors, and the result will hold since $\widehat{R}_1$ is L-colorable by Theorem 4.3(a). Thus we may assume that $L(wy) = \{a, b, c, d\}$. By symmetry, we may also assume that $L(vy) = \{a, b, c, d\}$. Now L-color yz with color c, so that there are at least two colors (b and d) available for sy that enable sz to be colored as well. The coloring can now be completed by Theorem 4.3(b)(ii) applied to $\widehat{R}_4 \setminus \{yz, sz\}$, which is essentially the same as $\widehat{R}_2$.

Case 4.4.2. $\widehat{R}_5$. If $L(xy)$ contains a color that is not in $L(e)$ for some $e \in E(F)$, color xy with such a color and then color yz; now e still has three available colors, and so the remaining edges can be colored since $\widehat{R}_1$ is L-colorable by Theorem 4.3(a). So we may assume that $L(xy) \subseteq L(e)$ for every $e \in E(F)$. If we now color xy with a color from $L(xy) \setminus L(yz)$, then the remaining two colors from $L(xy)$ are still available for every edge of F, and the coloring can be completed by Theorem 4.3(b)(i).

Case 4.4.3. $\widehat{R}_6$ and $\widehat{R}_7$. To enable these graphs to be discussed together, let x' and y' both denote the vertex z in $\widehat{R}_6$, and note that the conclusion of Lemma 4.2 easily holds for $\widehat{R}_6$.

Suppose first that, for some edge $e \in E(F)$, $L(xy)$ contains a color $p \notin L(e)$. If e is incident with y, L-color the edges below xy so that (C2) of Lemma 4.2 holds, and then color xy so that p is used at y (i.e., color xy with p if yy' is not already colored with p); if e is incident with x, do the same with x and y interchanged, using (C3). Then the edge e still has at least three available colors, and so the remaining edges can be colored since $\widehat{R}_1$ is L-colorable by Theorem 4.3. We deduce from this that $L(e) = L(xy) = \{a, b, c, d\}$, say, for every edge $e \in E(F)$.

Now color the edges below xy so that (C1) holds, say xx' and yy' have different colors a' and b' respectively, and relabel a, b, c, d if necessary so that $a' \notin \{b, c, d\}$ and $b' \notin \{a, c, d\}$. Then the coloring can be completed as follows. Color xy with d. If $a \notin L(uv)$ then color the edges vx, xw, wy, yv of F with c, b, c, a and color the remaining edges in the order vw, uw, uv; otherwise, if $a \in L(uv)$, color the edges of F with b, c, a, c, color uv with a, and then color vw and uw.

Case 4.4.4. $\widehat{R}_8$. If possible, L-color wy' with a color in $L(wy') \setminus L(uw)$, and then it is easy to color the remaining edges in this order $yy', vy, wx, xx', vw, vx', uv, uw$. So we assume that $L(uw) = L(wy') = \{a, b, c, d\}$. By symmetry, we also assume that $L(uv) = L(vx')$.

If possible, L-color wy' with a color in $L(wy') \setminus L(vw)$, and color in this order yy', vy, wx, xx'; the remaining edges can then be colored since T_2 is L_2-colorable as in Theorem 4.1(b). So we assume that $L(vw) = L(wy') = \{a, b, c, d\}$, and by symmetry, assume that $L(vx') = L(vw) = \{a, b, c, d\}$. This implies that $L(uv) = \{a, b, c, d\}$.

If possible, L-color vy with a color in $L(vy) \setminus L(vw)$, and continue the coloring in order along the walk $yy'wxx'$; the remaining edges can then be colored since T_2 is L_2-colorable. So we assume that $L(vy) \subset L(vw)$, and by symmetry, assume that $L(wx) \subset L(vw)$.

If possible, L-color vy and wx with a same color, and color in this order yy', wy', xx'; the remaining edges can then be colored since T_2 is L_2-colorable. So we assume, w.l.o.g., that $L(vy) = \{a, b\}$ and $L(wx) = \{c, d\}$.

Since either $L(yy') \neq \{a, d\}$ or $L(yy') \neq \{b, d\}$, we assume, w.l.o.g., that $L(yy') \neq \{a, d\}$. Now color $uv, uw, vw, vx', vy, wx, wy'$ with d, a, b, c, a, c, d, respectively, and then xx' and yy' can be easily colored.

Case 4.4.5. $\widehat{R}_9$. Remove the edge wy' and add a new edge wy so that $L(wy) = L(wy')$. It is easy to see that the resulting graph $\widehat{R}_9^*$ is L-colorable only if $\widehat{R}_9$ is L-colorable.

We now consider $\widehat{R}_9^*$. If possible, L-color yy' with a color in $L(yy') \setminus L(vy)$; the remaining edges can then be colored since $\widehat{R}_1$ is L-colorable by Theorem 4.3(a). So we assume that $L(yy') = L(vy) = \{a, b\}$.

In the following we consider $\widehat{R}_9$. If possible, L-color vy and wx with a same color, and color in this order yy', vx; the remaining edges can then be colored since T_2 is L_2-colorable. So we assume that $L(vy) = \{c, d\}$.

If $L(vw) \neq L(vy) \cup L(wx)$ or $L(vx) \not\subset L(vw)$, then by Lemma 4.3 we can L-color vy, wx, vx so that there are at least two colors still available for L-coloring vw. Now it is easy to color yy' and the remaining edges can be L-colored since T_2 is L_2-colorable. So we assume that $L(vx) \subset L(vw) = L(vy) \cup L(wx) = \{a, b, c, d\}$.

Since either $L(vx) \neq \{a, c\}$ or $L(vx) \neq \{b, c\}$, we assume, w.l.o.g., that $L(vx) \neq \{a, c\}$. Color vy, yy', wx with a, b, c, and L-color vx, vw, wy' with $p \notin \{a, c\}$, $q \notin \{a, c, p\}$, $r \notin \{b, c, q\}$, respectively. Note that $p \in L(vx) \subset \{a, b, c, d\}$, $q \in L(vw) = \{a, b, c, d\}$ and $q \neq p$. Thus $\{p, q\} = \{b, d\}$.

We now finish the L-coloring of $\widehat{R}_9$ by coloring uv and uw properly. The only problem is if there is a color s so that $L(uv) = \{a, p, q, s\}$ and $L(uw) = \{c, q, r, s\}$.

If $|\{b, c, q\}| = 2$, then erase the color on wy' and L-color wy' with a color $r' \notin \{b, c, q, r\}$. Hence we can finish the L-coloring of $\widehat{R}_9$ by coloring uv, uw with s, r. So we assume that $|\{b, c, q\}| = 3$, which implies that $q \neq b$, and then $q = d, p = b$.

Under this condition, we exchange the colors on vw and wx, and then color uv, uw with d, s. This completes the proof since we get an L-coloring of $\widehat{R}_9$. □

5 Proof of the Main Results

Theorem 5.1. *Let H be a graph and let $L(e)$ be a list of four colors, for each $e \in E(H)$. If H is not edge L-colorable but every proper subgraph of H has an edge L-coloring, then H does not contain any of the configurations $G_1, \cdots, G_{14}$, S_1, S_2 or S_3*

Proof. We consider each configuration C separately. We let H' be the subgraph of H obtained by removing all the solid vertices in C. We choose an edge L-coloring of H', which exists by hypothesis, and for each edge $e \in E(H) \setminus E(H')$ we denote by $A(e)$ the set of available colors for e, comprising those colors in $L(e)$ that are not used on any colored edge adjacent to e. If we can prove that $H - E(H')$ is edge A-colorable, then it will follow that H is edge L-colorable, and this contradiction will show that H cannot contain the configuration C. In each case, it suffices to describe how to construct an edge A-coloring of $H - E(H')$.

In every case except for G_1 and G_3, $H - E(H')$ is isomorphic to a graph that has already been proved to be A-colorable for a suitable list assignment A, as shown in this table.

Configuration	G_2	G_4	G_5	G_6	G_7	G_8	G_9
Equivalent graph	T_3	T_4	T_6	T_5	$\widehat{R}_8$	$\widehat{R}_9$	$\widehat{R}_1$
Theorem	4.1	4.1	4.1	4.1	4.4	4.4	4.3

Configuration	G_{10}	G_{12}	G_{13}	G_{14}	S_1	S_2	S_3
Equivalent graph	R_1	R_3	$\widehat{R}_4$	R_4	$\widehat{R}_5$	$\widehat{R}_6$	$\widehat{R}_7$
Theorem	4.2	4.2	4.4	4.4	4.4	4.4	4.4

It remains to consider the graphs G_1 and G_3. In G_1, let u denote the solid vertex and let x denote the left vertex in that picture. It is easy to see that $|A(ux)| \geq 1$ and $|A(uy)| \geq 2$, and so we can color the edges in the order ux, uy. In G_3, the result holds since every edge of the 4-cycle has at least two available colors and it is well-known that a 4-cycle is edge 2-choosable. This proves Theorem 5.1. □

Proofs of Theorems 1.1 and 1.2. Let H be the minimum counterexample to the theorem and let $L(e)$ be a list of four colors for each $e \in E(H)$. First of all, it is easy to see that $\delta(H) \geq 2$. Since every proper subgraph of H has an edge L-coloring, H does not contain any of the configurations listed in Theorem 5.1. On the other hand, if $\Delta(H) = 4$ and $\vartheta(H) \geq 3$, then H contains one of the configurations $G_1, \cdots, G_{14}$, S_1, S_2 or S_3 by Theorem 3.1, contradicting Theorem 5.1. If $\Delta(G) = 3$, then H contains the configuration G_1 or G_3 or G_{10}, which is an immediate corollary from the result of Zhang, Liu and Wu [14, Theorem 4.2]. However, this also contradicts Theorem 5.1, which implies that H does not contain G_1 or G_3 or G_{10}. □

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