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Upper bound on the sum of powers of the degrees of graphs with few crossings per edge

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ABSTRACT

A graph is q-planar if it can be drawn in the plane so that each edge is crossed by at most q other edges. For fixed integers $q \geq 1$ and $k \geq 2$, it is proven that $2(n-1)^k + o(n)$ is an asymptotically tight upper bound on the sum of the k-th powers of the degrees of any simple q-planar graph with order n. As a result, an open problem listed at the end of the paper *J. Czap, J. Harant, D. Hudák, Discrete Appl. Math. 165 (2014) 146–151* is solved.

1. Introduction

In this paper, all graphs are simple and undirected. For a vertex v of a graph G, we denote by $N_G(v)$ the set of vertices that are adjacent to v in G. The *degree* of v in G, denoted by $d_G(v)$, is exactly the value of $|N_G(v)|$. By $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, we denote the vertex set, the edge set, the maximum degree and the minimum degree of a graph G, respectively. Sometimes we use Δ and δ instead of $\Delta(G)$ and $\delta(G)$ for convenience, respectively. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. For two disjoint subset A, $B \subseteq V(G)$, $e(A, B)$ is the number of edges that have one end-vertex in A and the other in B. If G_1 and G_2 are two disjoint graphs, then $G_1 + G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For other undefined notation, we refer the readers to [4].

It is well-known that the sum of the degrees of the vertices of a graph G with order n and size m is twice the number of edges. Formally, $\sum_{v \in V(G)} d_G(v) = 2m \leq n(n-1)$. Actually, we are also interested in upper bounds on the expression $\sum_k(G) := \sum_{v \in V(G)} d_G^k(v)$ for all integers $k \geq 2$. Note that $\Sigma_k(G)$ is well-known as the first general Zagreb index [17], or the general zeroth-order Randić index [18,29], which is an important molecular descriptor and has been closely correlated with many chemical properties [11]. It attracts more and more attention from chemists and mathematicians including [1,2,7,14,15,19,21–24,30].

In [6], De Caen proved that $\sum_2(G) \leq m(\frac{2m}{n-1} + n - 2)$, which is tight for complete graphs. This bound was generalized to hypergraphs by Bey [3] and improved to $m(\frac{2m}{n-1} + \frac{n-2}{n-1}\Delta + (\Delta - \delta)(1 - \frac{\Delta}{n-1}))$ by Das [10]. De Caen's inequality was used by Li and Pan [16] to provide an upper bound on the largest eigenvalue of the Laplacian of a graph. In [8], Cioabă generalized Das's bound to $\sum_{k+1}(G) \leq \frac{2m}{n}(\sum_k(G) + (n-1)(\Delta^k - \delta^k)) - ((\Delta^k - \delta^k)/n)\sum_2(G)$ for a positive integer. In [5], Brandt, Harant and Naumann gave an upper bound on $\Sigma_k(G)$ with $k \geq 2$ if G is a triangle-free k-chromatic graph.

We now focus on graphs with few crossings per edge. A graph is *q-planar* if it can be drawn in the plane so that each edge is crossed by at most q other edges. Obviously, a 0-planar graph is just a planar graph. Harant, Jendrol' and Madaras

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[12] proved that $\sum_k(G) \le 2(n-1)^k + 4^k(n-4) + 2\cdot 3^k - 2((\delta+1)^k - \delta^k)(3n-6-m)$ if G is a planar graph with order n and size m. This implies

$$\sum_k(G) \le 2(n-1)^k + o(n) \tag{1.1}$$

for a planar graph G. The upper bound on the sum of the k-th powers of the degrees of 1-planar graphs was investigated by Xu et al. [27], and by Czap, Harant and Hudák [9]. In particular, Czap, Harant and Hudák [9] showed that (1.1) holds for 1-planar graphs, and guessed that (1.1) holds for q-planar graphs with any integer $q \ge 2$. In this paper, we solve this open problem. In other words, we prove

Theorem 1.1. *If G is a q-planar graph with $q \ge 1$, then $\sum_k(G) \le 2(n-1)^k + o(n)$, and this bound is asymptotically tight.*

2. Preliminaries

The *crossing number* $cr(G)$ is the minimum number of edge crossings of a plane drawing of the graph G. The study of crossing numbers originated in Turán's brick factory problem, in which Turán [26] asked for a factory plan that minimized the number of crossings between tracks connecting brick kilns to storage sites. Mathematically, this problem can be formalized as asking for the crossing number of a complete bipartite graph. Concerning this problem, Zarankiewicz [28] gave a drawing of $K_{m,\,n}$ which demonstrates that

$$cr(K_{m,n}) \le Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

and conjectured that $cr(K_{m,n}) = Z(m,n)$. Kleitman [13] showed that

$$cr(K_{m,n}) = Z(m,n), \ \ m \le 6 \tag{2.1}$$

and gave a lower bound on $cr(K_{m,\,n})$ as follows:

$$cr(K_{m,n}) \ge \frac{1}{5} m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \ \ m \ge 5, \tag{2.2}$$

Let $\mathcal{G}_r$ be the class of graphs G with the property that
$cr(H) \le r \cdot e(H)$ for every subgraph H of G,
where r is a nonnegative real number. Suppose that $K_{m,n} \in \mathcal{G}_r$. By the definition of $\mathcal{G}_r$ and by (2.1) and (2.2), if $m \ge 5$, then $rmn \ge \frac{1}{5} m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \ge \frac{1}{5} m(m-1) \frac{n(n-2)}{4}$, which implies that

$$n \le \frac{20r}{m-1} + 2, \tag{2.3}$$

if $m = 4$, then $4rn \ge 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \ge \frac{n(n-2)}{2}$, which implies that

$$n \le 8r + 2, \tag{2.4}$$

and if $m = 3$, then $3rn \ge \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \ge \frac{n(n-2)}{4}$, which implies that

$$n \le 12r + 2, \tag{2.5}$$

Let s_m denote the maximum integer such that $K_{m,s_m} \in \mathcal{G}_r$. It is now easy to conclude from (2.3), (2.4) and (2.5) that

$$s_m - 1 \le \frac{20r}{m-1} + 1, \ \ m \ge 5, \tag{2.6}$$

$$s_4 - 1 \le 8r + 1, \tag{2.7}$$

$$s_3 - 1 \le 12r - 1. \tag{2.8}$$

Lemma 2.1. *[20] If G is a q-planar graph with $q \ge 1$, and with order n and size m, then*

$$m \le \sqrt{16.875q} \cdot n.$$

By Lemma 2.1, the average degree of any q-planar is $\frac{2m}{n} \le \sqrt{67.5q}$. Hence we conclude

Lemma 2.2. *If G is a q-planar graph with $q \ge 1$, then $\delta(G) \le \sqrt{67.5q}$.* □

For the crossing number of q-planar graph with $q \ge 1$, it is not hard to deduce the following

Lemma 2.3. *[9] If G is a q-planar graph with $q \ge 1$ and with size m, then $cr(G) \le 0.5qm$.*

Hence, if G is a q-planar graph, then $G \in \mathcal{G}_{0.5q}$.

Let $\mathcal{P}(M, a, b, c, k)$ be the following optimization problem on variables $x_1, \ldots, x_c$.

$$max \sum_{i=1}^{c} \left(x_i^k - (x_i - 1)^k\right)$$
$$s.t.\ \ a \le x_i \le b, i \in \{1, 2, \ldots, c\} \tag{2.9}$$

$$\sum_{i=1}^{c} x_i \le M \tag{2.10}$$

$$ac \le M$$
$$M, a, b, c, k, x_1, \ldots, x_c \text{ are positive integers} \tag{2.11}$$

Lemma 2.4. *[9] If $(x_1, \ldots, x_c)$ is a feasible solution of $\mathcal{P}(M, a, b, c, k)$ such that $a < x_i < b$ for at most one value $i \in \{1, \ldots, c\}$ and $\sum_{i=1}^{c} x_i = M$, then it is an optimal solution of $\mathcal{P}(M, a, b, c, k)$.*

Lemma 2.5. *If k, ρ and δ are positive integers such that $\delta \ge 4$ and $\rho \ge \delta(\delta - 3) - 1$, then*

$$\left(\delta^k - (\delta - 1)^k\right)(\delta - 3) + \left(\rho + 2 - \delta(\delta - 3)\right)^k - \left(\rho + 1 - \delta(\delta - 3)\right)^k$$
$$\le (\delta - 3) + (\rho + 2)^k - (\rho + 1)^k.$$

Proof. Consider the optimization problem $\mathcal{P}(M, a, b, c, k)$ with $M = \rho + \delta - 1$, $a = 1$, $b = \rho + 2$ and $c = \delta - 2$. It is easy to see that $ac = \delta - 2 < \rho + \delta - 1 = M$ and then (2.11) holds.

First, choose $x_1 = \cdots = x_{\delta-3} = \delta$, and $x_{\delta-2} = \rho + 2 - \delta(\delta - 3)$. It is easy to check that $(x_1, \ldots, x_{\delta-2})$ is a feasible solution of $\mathcal{P}(M, a, b, c, k)$. Second, choose $\widetilde{x}_1 = \cdots = \widetilde{x}_{\delta-3} = 1$ and $\widetilde{x}_{\delta-2} = \rho + 2$. Clearly, $(\widetilde{x}_1, \ldots, \widetilde{x}_{\delta-2})$ is also a feasible solution of $\mathcal{P}(M, a, b, c, k)$, and moreover, it satisfies the condition in Lemma 2.4. Therefore, $(\widetilde{x}_1, \ldots, \widetilde{x}_{\delta-2})$ is an optimal solution. This implies

$$\sum_{i=1}^{\delta-2} \left(x_i^k - (x_i - 1)^k\right) \le \sum_{i=1}^{\delta-2} \left(\widetilde{x}_i^k - (\widetilde{x}_i - 1)^k\right),$$

as desired. □

3. Sum of Powers of the Degrees

Lemma 3.1. *Let G be a graph in $\mathcal{G}_r$ with order n. If v is a vertex of G with the minimum degree $\delta \ge 1$, then*

$$\sum_{x \in N_G(v)} d_G(x) \le 2n + (20r + \delta)2^\delta.$$

Proof. If $\delta \le 2$, then $\sum_{x \in N_G(v)} d_G(x) \le 2\Delta(G) \le 2n - 2$. Hence we assume that $\delta \ge 3$. For a vertex $u \in V(G)$ not belonging to $N_G(v) \cup \{v\}$, $t(u)$ denotes the value of $|N_G(u) \cap N_G(v)|$. Clearly, $0 \le t(u) \le \delta$.

Let $B_1, B_2, \ldots, B_p$ be all m-subsets of $N_G(v)$, where $p = \binom{\delta}{m}$, and let T_m be the set of vertices $u \in V(G) \setminus (N_G(v) \cup \{v\})$ such that $t(u) = m$, where $0 \le m \le \delta$. Set $\alpha_m = |T_m|$. It follows that

$$\alpha_0 + \alpha_1 + \cdots + \alpha_m + \delta + 1 = n. \tag{3.1}$$

Construct a bipartite graph $H = (\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} = T_m$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_p\}$. If a vertex $u \in \mathcal{A}$ has m neighbors in $N_G(v)$ that forms an m-set $S_i \in \mathcal{B}$, then add an edge between u and S_i. Clearly, in the graph H, every vertex in $\mathcal{A}$ has exactly one neighbor in $\mathcal{B}$, which implies

$$e(\mathcal{A}, \mathcal{B}) = |\mathcal{A}| = |T_m|. \tag{3.2}$$

On the other hand, in the graph H, every vertex in $\mathcal{B}$ has at most $s_m - 1$ neighbors in $\mathcal{A}$. Otherwise, there is a vertex in $\mathcal{B}$, say B_i, that has s_m neighbors $u_1, u_2, \ldots, u_{s_m}$ in $\mathcal{A}$. Therefore, the graph induced by $\{u_1, u_2, \ldots, u_{s_m}, v\}$ and B_i contains a K_{m,s_m+1} in G. However, $K_{m,s_m+1} \notin \mathcal{G}_r$ by the definition of s_m, a contradiction. Hence we conclude that

$$e(\mathcal{A}, \mathcal{B}) \le (s_m - 1)|\mathcal{B}| = (s_m - 1)p. \tag{3.3}$$

Combine (3.2) with (3.3), and then with (2.6), (2.7) and (2.8), we immediately have

$$\alpha_m = |T_m| \le \left(\frac{20r}{m - 1} + 1\right)\binom{\delta}{m}, \ \ m \ge 5$$

$$\alpha_4 = |T_4| \le (8r+1)\binom{\delta}{4}$$

$$\alpha_3 = |T_3| \le (12r+1)\binom{\delta}{3}.$$

For the convenience of the next computations, we write the above three inequalities into a common one (although it is weaker):

$$\alpha_m = |T_m| \le \left(\frac{20r}{m-2}+1\right)\binom{\delta}{m}, \ \ m \ge 3,$$

which implies

$$(m-2)\alpha_m \le (20r+m-2)\binom{\delta}{m} \le (20r+\delta-2)\binom{\delta}{m}. \tag{3.4}$$

By $e(v)$ we denote the number of edges in the subgraph induced by $N_G(v)\cup\{v\}$. Clearly,

$$e(v) \le \binom{\delta+1}{2} = \frac{1}{2}\delta(\delta+1) \tag{3.5}$$

Using (3.1), (3.4) and (3.5), we conclude

$$\begin{aligned}\sum_{x\in N_G(v)} d_G(x) &= 2e(v) - \delta + \sum_{u\in V(G)\setminus(N_G(v)\cup\{v\})} t(u)\\
&= 2e(v) - \delta + \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + \delta\alpha_\delta\\
&= 2(\alpha_0+\alpha_1+\cdots+\alpha_m+\delta+1) + 2e(v) - 3\delta - 2 - 2\alpha_0 - \alpha_1 + \sum_{m=3}^{\delta}(m-2)\alpha_m\\
&= 2n + 2e(v) - 3\delta - 2 - 2\alpha_0 - \alpha_1 + \sum_{m=3}^{\delta}(m-2)\alpha_m\\
&\le 2n + \delta(\delta-2) + (20r+\delta-2)\sum_{m=3}^{\delta}\binom{\delta}{m}\\
&= 2n + \delta(\delta-2) + (20r+\delta-2)\left(2^\delta - 1 - \delta - \frac{1}{2}\delta(\delta-1)\right)\\
&\le 2n + \delta(\delta-2) + (20r+\delta-2)2^\delta - (\delta-2)\left(1+\delta+\frac{1}{2}\delta(\delta-1)\right)\\
&= 2n + (20r+\delta-2)2^\delta - (\delta-2)\left(1+\frac{1}{2}\delta(\delta-1)\right)\\
&\le 2n + (20r+\delta)2^\delta,\end{aligned}$$

as desired. □

Lemma 3.2. *Let G be a graph in $\mathcal{G}_r$ with order n and minimum degree $\delta \ge 1$, and let $\rho := \rho(r,\delta) = \lfloor(20r+\delta)2^\delta\rfloor$. If v is a vertex of G with the minimum degree and $n \ge \rho+3$, then*

$$\sum_{x\in N_G(v)}\left(d_G(x)^k - (d_G(x)-1)^k\right) \le 2\left((n-1)^k-(n-2)^k\right) + (\delta-3) + (\rho+2)^k - (\rho+1)^k.$$

Proof. If $k=1$, then it is trivial. Therefore we assume that $k \ge 2$. If $\delta \le 2$, then

$$\sum_{x\in N_G(v)}\left(d_G(x)^k - (d_G(x)-1)^k\right) \le 2\left(\Delta(G)^k - (\Delta(G)-1)^k\right) \le 2\left((n-1)^k-(n-2)^k\right)$$

and the desired result holds. Note that $\delta-3+(\rho+2)^k-(\rho+1)^k \ge \delta-3+k(\rho+1)^{k-1} \ge \delta-3+2(\rho+1) \ge 0$. Hence we assume in the following that $\delta \ge 3$.

Consider the optimization problem $\mathcal{P}(M,a,b,c,k)$ with $a=c=\delta$, $b=n-1$ and $M=2n+\rho$. Since $M>\rho\ge\delta\cdot 2^\delta>\delta^2$, (2.11) holds.

Choose $x_1,\ldots,x_\delta$ to be the degrees of the δ neighbors of v in G, respectively, and then substitute them into $\mathcal{P}(M,a,b,c,k)$. Clearly, (2.9) is trivially satisfied. By Lemma 3.1, (2.10) is verified. Hence $(x_1,\ldots,x_\delta)$ is a feasible solution of $\mathcal{P}(2n+\rho,\delta, n-1,\delta,k)$.

We now construct another one feasible solution $(\widetilde{x}_1, \ldots, \widetilde{x}_\delta)$ of $\mathcal{P}(2n+\rho, \delta, n-1, \delta, k)$ as follows. If $\delta = 3$, then let $\widetilde{x}_1 = \rho + 2$, $\widetilde{x}_2 = \widetilde{x}_3 = n-1$, and if $\delta \geq 4$, then let $\widetilde{x}_1 = \ldots = \widetilde{x}_{\delta-3} = \delta$, $\widetilde{x}_{\delta-2} = \rho + 2 - \delta(\delta-3)$, and $\widetilde{x}_{\delta-1} = \widetilde{x}_\delta = n-1$. Since $n-1 \geq \rho+2 > \rho+2-\delta(\delta-3) > \delta \cdot 2^\delta - \delta(\delta-3) > \delta$, (2.9) holds and $\delta < \widetilde{x}_i < n-1$ for at most one value $i \in \{1, \ldots, c\}$. Since $\sum_{i=1}^{\delta} \widetilde{x}_i = 2n+\rho = M$, by Lemma 2.4, $(\widetilde{x}_1, \ldots, \widetilde{x}_\delta)$ is an optimal solution of $\mathcal{P}(2n+\rho, \delta, n-1, \delta, k)$.

Hence,

$$\sum_{x \in N_G(v)} \Big(d_G(x)^k - (d_G(x)-1)^k \Big) = \sum_{i=1}^{\delta} \Big(x_i^k - (x_i-1)^k \Big) \leq \sum_{i=1}^{\delta} \Big(\widetilde{x}_i^k - (\widetilde{x}_i - 1)^k \Big). \tag{3.6}$$

If $\delta = 3$, then

$$\sum_{i=1}^{\delta} \Big(\widetilde{x}_i^k - (\widetilde{x}_i - 1)^k \Big) = 2\Big((n-1)^k - (n-2)^k \Big) + (\rho+2)^k - (\rho+1)^k, \tag{3.7}$$

and if $\delta \geq 4$, then by Lemma 2.5 (note that $\rho \geq \delta 2^\delta \geq \delta(\delta-3) - 1$), we conclude

$$\begin{aligned} &\sum_{i=1}^{\delta} \Big(\widetilde{x}_i^k - (\widetilde{x}_i - 1)^k \Big) \\ &= 2\Big((n-1)^k - (n-2)^k \Big) + \Big(\delta^k - (\delta-1)^k \Big)(\delta-3) + \Big(\rho + 2 - \delta(\delta-3) \Big)^k - \Big(\rho + 1 - \delta(\delta-3) \Big)^k \\ &\leq 2\Big((n-1)^k - (n-2)^k \Big) + (\delta-3) + (\rho+2)^k - (\rho+1)^k. \end{aligned} \tag{3.8}$$

Combine (3.6) and (3.7) with (3.8), we prove the desired inequality. □

Lemma 3.3. *[12] If a, b and k are positive integers with $a < b$ and $k \geq 2$, then*

$$a^k - (a-1)^k < b^k - (b-1)^k.$$

Lemma 3.4. *Let G be a graph in $\mathcal{G}_r$ with order n and minimum degree $\delta_0 \leq \delta$, where δ is a fixed positive integer, and let $\rho := \rho(r, \delta) = \lfloor (20r+\delta)2^\delta \rfloor$. If v is a vertex of G with the minimum degree $\delta_0 \geq 1$ and $n \geq \rho + 3$, then*

$$\sum_{x \in N_G(v)} \Big(d_G(x)^k - (d_G(x)-1)^k \Big) \leq 2\Big((n-1)^k - (n-2)^k \Big) + (\delta-3) + (\rho+2)^k - (\rho+1)^k.$$

Proof. If $k = 1$, then there is nothing to prove. If $k \geq 2$, then by Lemmas 3.2 and 3.3,

$$\begin{aligned} \sum_{x \in N_G(v)} \Big(d_G(x)^k - (d_G(x)-1)^k \Big) &\leq 2\Big((n-1)^k - (n-2)^k \Big) + (\delta_0 - 3) + \Big(\rho(r, \delta_0) + 2 \Big)^k - \Big(\rho(r, \delta_0) + 1 \Big)^k \\ &\leq 2\Big((n-1)^k - (n-2)^k \Big) + (\delta-3) + \Big(\rho(r, \delta) + 2 \Big)^k - \Big(\rho(r, \delta) + 1 \Big)^k \\ &= 2\Big((n-1)^k - (n-2)^k \Big) + (\delta-3) + (\rho+2)^k - (\rho+1)^k, \end{aligned}$$

as desired. Note that $\rho(r, \delta_0) \leq \rho(r, \delta)$, since $\delta_0 \leq \delta$. □

Theorem 3.5. *Let G be a graph in $\mathcal{G}_r$ with order n and minimum degree $\delta_0 \leq \delta$, where δ is a fixed positive integer, and let $\rho := \rho(r, \delta) = \lfloor (20r+\delta)2^\delta \rfloor$, $\omega := \omega(r, \delta) = \delta - 3 + (\rho+2)^k - (\rho+1)^k$. If $n \geq \rho + 2$, then*

$$\sum_k (G) \leq 2(n-1)^k + (\omega + \delta^k)(n - \rho - 2) + \rho(\rho+1)^k. \tag{3.9}$$

Proof. Obviously, we can assume that there is no isolate vertex in G, since such a vertex contribute nothing to $\Sigma_k(G)$. Therefore, we assume $\delta \geq \delta_0 \geq 1$ in the following.

We prove (3.9) by induction on n. If $n = \rho + 2$, then $\sum_k(G) \leq n(n-1)^k = (\rho+2)(\rho+1)^k$, and (3.9) holds. Hence we assume that $n \geq \rho + 3$.

Let v be a vertex in G with minimum degree δ_0. Clearly, $H := G - v \in \mathcal{G}_r$ is a graph with order $n-1$. By the induction hypothesis and by Lemma 3.4,

$$\begin{aligned} \sum_k (G) &= \sum_k (H) + d_G(v)^k + \sum_{x \in N_G(v)} \Big(d_G(x)^k - (d_G(x)-1)^k \Big) \\ &\leq 2(n-2)^k + (\omega + \delta^k)(n - \rho - 3) + \rho(\rho+1)^k + \delta_0^k + 2\Big((n-1)^k - (n-2)^k \Big) + \omega \\ &\leq 2(n-1)^k + (\omega + \delta^k)(n - \rho - 2) + \rho(\rho+1)^k, \end{aligned}$$

as desired. □

4. Conclusions and remarks

Recall that $\rho := \rho(r, \delta) = \lfloor (20r + \delta)2^{\delta} \rfloor$ and $\omega := \omega(r, \delta) = \delta - 3 + (\rho + 2)^k - (\rho + 1)^k$. One can easily see that

$$(\rho + 2)^k - (\omega + \delta^k) = (\rho + 1)^k - (\delta^k + \delta + 3) \geq \Big((20r + \delta)2^{\delta}\Big)^k - (\delta^k + \delta + 3)$$
$$\geq \delta^k(2^{\delta k} - 1) - \delta - 3 \geq \delta^2(4^{\delta} - 1) - \delta - 3 > 0 \tag{4.1}$$

for $k \geq 2$ and $\delta \geq 2$.

If $n \geq \rho + 2$, $k \geq 2$ and $\delta \geq 2$, then by (4.1), $n((\rho + 2)^k - (\omega + \delta^k)) \geq (\rho + 2)((\rho + 2)^k - (\omega + \delta^k))$, which implies that

$$(\omega + \delta^k)(n - \rho - 2) + \rho(\rho + 1)^k \leq (\rho + 2)^k n + \rho(\rho + 1)^k - (\rho + 2)^{k+1} \tag{4.2}$$

Theorem 4.1. *Let $q \geq 1$ be an integer and let $\rho := \rho(0.5q, \sqrt{67.5q}) = \lfloor (10q + \sqrt{67.5q})2^{\sqrt{67.5q}} \rfloor := f(q)$. If G is a q-planar graph with order $n \geq \rho + 2$, then*

$$\sum_k (G) \leq 2(n - 1)^k + \Big(f(q) + 2\Big)^k n + f(q)\Big(f(q) + 1\Big)^k - \Big(f(q) + 2\Big)^{k+1} \tag{4.3}$$

$$= 2(n - 1)^k + o(n), \tag{4.4}$$

moreover, the upper bound in (4.4) is asymptotically tight.

Proof. By Lemma 2.3, $G \in \mathcal{G}_{0.5q}$, and by Lemma 2.2, the minimum degree of G is at most $\sqrt{67.5q}$. Substitute $r = 0.5q$ and $\delta = \sqrt{67.5q}$ into Theorem 3.5, and then use (3.9) and (4.2), we obtain the desired result.

For the tightness, consider the planar (so also q-planar with $q \geq 1$) graph $H := K_2 + P_{n-2}$. It has two vertices of degree $n - 1$, $n - 4$ vertices of degree 4, and two vertices of degree 3. Therefore,

$$\textstyle\sum_k (H) = 2(n - 1)^k + (n - 4)4^k + 2 \cdot 3^k = 2(n - 1)^k + 4^k \cdot n + 2 \cdot 3^k - 4^{k+1} = 2(n - 1)^k + o(n). \ \square$$

Theorem 4.1 implies Theorem 1.1, which says that $\sum_k (G) \leq 2(n - 1)^k + o(n)$ for any q-planar graph with $q \geq 1$. An interesting problem is now to determine what does the $o(n)$ part look like. Actually, (4.3) gives an expanded form of the $o(n)$ part, but the integer $f(q)$ there is a bit large. Therefore, we naturally propose the following problem.

Open Problem: *For a q-planar graph G, determine the smallest integer $z(q)$ such that*

$$\sum_k (G) \leq 2(n - 1)^k + \Big(z(q) + 2\Big)^k n + z(q)\Big(z(q) + 1\Big)^k - \Big(z(q) + 2\Big)^{k+1}. \tag{4.5}$$

Truszczyński [25] confirmed that $z(0) = 2$. In this paper we conclude for $q \geq 1$, from Theorem 4.1, that

$$z(q) \leq f(q) = \Big\lfloor \Big(10q + \sqrt{67.5q}\Big)2^{\sqrt{67.5q}} \Big\rfloor. \tag{4.6}$$

Actually, *fixing $z(q)$ or finding a better upper bound on $z(q)$ are both interesting.*

Specially, the main result of Czap et al. [9] showed that $\sum_k (G) \leq 2(n - 1)^k + 607^k \cdot n + 605 \cdot 606^k - 607^{k+1}$ if G is a 1-planar graph, which implies that $z(1) \leq 605$.

Let $\widehat{P_n}$ be the graph derived from the path P_n on n vertices by adding edges such that any two vertices with distance (in P_n) at most 2 or exactly $n - 1$ are adjacent. It is easy to see that $H := K_2 + \widehat{P_n}$ is a 1-planar graph (see Fig. 1 for two special examples) with two vertices of degree $n - 1$, $n - 6$ vertices of degree 6, and four vertices of degree 5, which results in

$$\textstyle\sum_k (H) = 2(n - 1)^k + (n - 6)6^k + 4 \cdot 5^k = 2(n - 1)^k + 6^k \cdot n + 4 \cdot 5^k - 6^{k+1}.$$

This implies $z(1) \geq 4$. Actually, *we conjecture that* $z(1) = 4$, that is, $\sum_k (G) \leq 2(n - 1)^k + (n - 6)6^k + 4 \cdot 5^k = 2(n - 1)^k + 6^k \cdot n + 4 \cdot 5^k - 6^{k+1}$ for any 1-planar graph G (being sharp).

For $q \geq 2$, if $z(q)$ has been fixed, then whether the upper bound in (4.5) is sharp? In order to confirm this guesswork positively, let us look first at the following computation:

$$2(n - 1)^k + \Big(z(q) + 2\Big)^k n + z(q)\Big(z(q) + 1\Big)^k - \Big(z(q) + 2\Big)^{k+1}$$
$$= 2(n - 1)^k + \Big(n - (z(q) + 2)\Big)\Big(z(q) + 2\Big)^k + z(q)\Big(z(q) + 1\Big)^k.$$

Clearly, what we should now do is to construct an n-vertex q-planar graph G_q with two vertices of degree $n - 1$, $z(q)$ vertices of degree $z(q) + 1$, and $n - \big(z(q) + 2\big)$ vertices of degree $z(q) + 2$. If n is sufficiently large (note that $z(q)$ is bounded by a constant by (4.6)), then G_q has two vertices of maximum degree $\Delta = n - 1$, $\delta - 1$ vertices of minimum degree δ, and $\Delta - \delta$ vertices of degree $\delta + 1$, which looks like the pictures in Fig. 1 that show the case for $q = 1$. Therefore, it is interesting to ask *whether (or how) such a graph G_q with $q \geq 2$ can be constructed.*

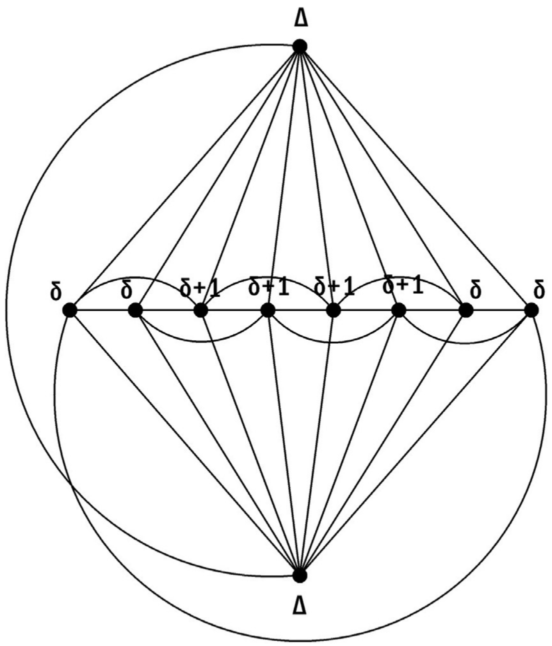

Fig. 1. $K_2 + \widehat{P}_8$ and $K_2 + \widehat{P}_7$.

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References

[1] A. Ali, I. Gutman, E. Milovanovic, I. Milovanovic, Sum of powers of the degrees of graphs: extremal results and bounds, MATCH Commun. Math. Comput. Chem. 80 (1) (2018) 5–84.
[2] L. Bedratyuk, O. Savenko, The star sequence and the general first Zagreb index, MATCH Commun. Math. Comput. Chem. 79 (2018) 407–414.
[3] C. Bey, An upper bound on the sum of the squares of the degrees in a hypergraph, Discrete Math. 269 (2003) 259–263.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, 2008. GTM 244.
[5] S. Brandt, J. Harant, S. Naumann, On degree sums of a triangle-free graph, Discrete Math. 337 (2014) 76–82.
[6] D. de Caen, An upper bound on the sum of squares of degrees in a graph, Discrete Math. 185 (1998) 245–248.
[7] Z. Chen, G. Su, L. Volkmann, Sufficient conditions on the zeroth-order general Randic index for maximally edge-connected graphs, Discrete Appl. Math. 218 (2017) 64–70.
[8] S.M. Cioabă, Sums of powers of the degrees of a graph, Discrete Math. 306 (2006) 1959–1964.
[9] J. Czap, J. Harant, D. Hudák, An upper bound on the sum of powers of the degrees of simple 1-planar graphs, Discrete Appl. Math. 165 (2014) 146–151.
[10] K.C. Das, Maximizing the sum of squares of the degrees of a graph, Discrete Math. 285 (2004) 57–66.
[11] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. total π-electron energy of alternant hydrocarbons, Chem. Phys. Lett. 17 (1972) 535–538.
[12] J. Harant, S. Jendrol', T. Madaras, Upper bounds on the sum of powers of the degrees of a simple planar graph, J. Graph Theory 67 (2011) 112–123.
[13] D.J. Kleitman, The crossing number of $k_{5,\,n}$, J. Combin. Theory 9 (1970) 315–323.
[14] Y. Lan, H. Liu, Z. Qin, Y. Shi, Degree powers in graphs with a forbidden forest, Discrete Math. 342 (3) (2019) 821–835.
[15] F. Li, M. Lu, On the zeroth-order general randic index of unicycle graphs with k pendant vertices, Ars Comb. 109 (2013) 229–237.
[16] J.S. Li, Y.L. Pan, de caen's inequality and bounds on the largest Laplacian eigenvalue of a graph, Linear Algebra Appl. 328 (2001) 153–160.
[17] X. Li, J. Zheng, A unified approach to the extremal trees for different indices, MATCH Commun. Math. Comput. Chem. 54 (1) (2005) 195–208.
[18] B. Liu, I. Gutman, Estimating the Zagreb and the general randć indices, MATCH Commun. Math. Comput. Chem. 57 (2007) 617–632.
[19] Y. Ma, S. Cao, Y. Shi, I. Gutman, M. Dehmer, B. Furtula, From the connectivity index to various Randić-type descriptors, MATCH Commun. Math. Comput. Chem. 80 (1) (2018) 85–106.
[20] J. Pach, G. Tóth, Graphs drawn with few crossings per edge, Combinatorica 17 (1997) 427–439.
[21] X.-F. Pan, H. Liu, M. Liu, Sharp bounds on the zeroth-order general Randic index of unicyclic graphs with given diameter, Appl. Math. Lett. 24 (2011) 687–691.
[22] Y. Shi, Note on two generalizations of the randić index, Appl. Math. Comput. 265 (2015) 1019–1025.
[23] G. Su, M. Meng, L. Cui, et al., The general zeroth-order Randic index of maximal outerplanar graphs and trees with k maximum degree vertices, Scienceasia 43 (2017) 387–393.
[24] G. Su, J. Tu, K.C. Das, Graphs with fixed number of pendent vertices and minimal zeroth-order general Randic index, Appl. Math. Comput. 270 (2015) 705–710.
[25] M. Truszczyński, Note on vertex degrees of planar graphs, J. Graph Theory 8 (1984) 171–176.
[26] P. Turán, A note of welcome, J. Graph Theory 1 (1977) 7–9.
[27] R. Xu, J.l. Wu, G. Wang, X. Zhang, On the sum of powers of the degrees of graphs, Bull. Aust. Math. Soc. 88 (3) (2013) 353–358.
[28] K. Zarankiewicz, On a problem of p. turán concerning graphs, Fund. Math. 41 (1954) 137–145.
[29] H. Zhang, S. Zhang, Unicyclic graphs with the first three smalllest and largest first general zagreb index, MATCH Commun. Math. Comput. Chem. 55 (2006) 427–438.
[30] M. Zhang, S. Li, On the zeroth-order general Randic index, J. Math. Chem. 49 (2011) 325–327.