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Equitable Coloring of Three Classes of 1-planar Graphs

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Abstract A graph is 1-planar if it can be drawn on a plane so that each edge is crossed by at most one other edge. A plane graph with near-independent crossings or independent crossings, say NIC-planar graph or IC-planar graph, is a 1-planar graph with the restriction that for any two crossings the four crossed edges are incident with at most one common vertex or no common vertices, respectively. In this paper, we prove that each 1-planar graph, NIC-planar graph or IC-planar graph with maximum degree Δ at least 15, 13 or 12 has an equitable Δ-coloring, respectively. This verifies the well-known Chen-Lih-Wu Conjecture for three classes of 1-planar graphs and improves some known results.

Keywords 1-planar graph; equitable coloring; independent crossing
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1 Introduction

A *k-coloring* of a graph G is a function f from $V(G)$ to the set $\{1, 2, \cdots, k\}$ such that $f(u) \neq f(v)$ if $uv \in E(G)$. We say a k-coloring of G *equitable* if the size of any two color classes differ by at most one. The smallest integer k such that G is equitably k-colorable is the *equitable chromatic number* of G, denoted by $\chi_{eq}(G)$. Note that a graph may have an equitable k-coloring but no equitable-$(k+1)$-colorings (check the balanced complete k-partite graph for example). Hence we need another parameter to fix the smallest integer k such that G is equitably k'-colorable for every $k' \geq k$. In this note, we use $\chi^*_{eq}(G)$ to denote this chromatic parameter and call it the *equitable chromatic threshold* of G. Clearly, $\chi_{eq}(G) \leq \chi^*_{eq}(G)$, but the gap between them can be any large. Take the complete bipartite graph $K_{2m+1,2m+1}$ for example, one can see that $\chi_{eq}(K_{2m+1,2m+1}) = 2$ but $\chi^*_{eq}(K_{2m+1,2m+1}) = 2m+2$.

An early result on equitable coloring of graphs due to Hajnal and Szemer[8] states that every graph G with $\Delta(G) \leq r$ has an equitable $(r+1)$-coloring, which answers a question of Erdős and implies $\chi^*_{eq}(G) \leq \Delta(G)+1$ for any graph G. This upper bound on $\chi^*_{eq}(G)$ is sharp, since the complete graph K_m admits no $(m-1)$-colorings, the odd cycles has no 2-colorings, and the complete bipartite graph $K_{2m+1,2m+1}$ has an equitable 2-coloring but no equitable $(2m+1)$-colorings. Actually, those classes of graphs are conjectured to be the only three classes with equitable chromatic threshold attaining this upper bound.

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Conjecture 1.1[4]**.** *For any connected graph G, except the complete graph, the odd cycle and the complete bipartite graph $K_{2m+1,2m+1}$, $\chi^*_{eq}(G) \leq \Delta(G)$.*

This conjecture is now confirmed for graphs with $\Delta \leq 3$ (see [4,6]), or $\Delta = 4$ (see [9]) or $\Delta \geq |G|/4$ (see [10]), bipartite graphs[15], interval graphs[5], outerplanar graphs[19], series-parallel graphs[24], pseudo-outerplanar graphs[18], planar graphs with $\Delta \geq 9$ (see [16,20]), 1-planar graphs with $\Delta \geq 17$ (see [22]), d-degenerate graphs with $d \leq (\Delta - 1)/14$ (see [12]) or with $d \leq \Delta/10$ and $\Delta \geq 46$ (see [11]), and graphs with $\Delta \geq 46$ and maximum average degree at most $\Delta/5$ (see [11]). One can refer to a nice survey by Lih[14] on equitable coloring of graphs for interesting reading.

A graph is *1-planar* if it can be drawn on a plane so that each edge is crossed by at most one other edge. The concept of the 1-planarity was introduced by Ringel[17] when he considered the vertex-face coloring of plane graphs, which can be translated to the vertex coloring of 1-planar graphs. In [17], Ringel gave the first result on the coloring of 1-planar graphs: every 1-planar graph is 7-colorable. Almost two decades later, Borodin[1,2] improved this bound to 6 and showed the sharpness of the new bound. A *plane graph with near-independent crossings* (*NIC-planar graph* for short), or *plane graph with independent crossings* (*IC-planar graph for short*) is a 1-planar graph with the restriction that for any two crossings the four crossed edges are incident with at most one common vertex, or with no common vertices, respectively. The NIC-planarity and IC-planarity was introduced by Zhang[21] in 2014 and by Král and Stacho[13] in 2010, respectively. By Borodin's result mentioned above, every NIC-planar graph is 6-colorable, but we do not know whether it can be improved. On the other hand, Král and Stacho[13] proved that every IC-planar graph is 5-colorable and this bound is sharp.

As reviewed above, Zhang[22] verified the Chen-Lih-Wu Conjecture for 1-planar graphs with maximum degree at least 17. In this note, we are to improve this lower bound to 15 and show that Chen-Lih-Wu Conjecture also holds for various subclasses of 1-planar graphs, especially for NIC-planar graphs with maximum degree at least 13 and IC-planar graphs with maximum degree at least 12.

2 Useful Lemmas

Lemma 2.1[7,21]**.** *Every 1-planar graph or NIC-planar graph contains a vertex of degree at most* 7 *or* 6*, respectively.*

Lemma 2.2. *Let $m \geq 1$ be a fixed integer. If any 1-planar graph (or NIC-planar graph, or IC-planar graph, respectively) of order mt is equitably m-colorable for any integer $t \geq 1$, then any 1-planar graph (or NIC-planar graph, or IC-planar graph, respectively) is equitably m-colorable.*

Proof. We just prove it for 1-planar graphs. If $|V(G)|$ is divisible by m, we success. If $|V(G)|$ is not divisible by m, then assume that $|V(G)| = mt - j$ with $0 < j < m$. We prove that either $j \leq 6$ or G has an equitably m-coloring.

If $m \leq 7$, then $0 < j \leq 6$ since $j < m \leq 7$. Suppose that $m \geq 8$. Let u be a vertex in G with $d(u) = \delta(G) \leq 7$ by Lemma 2.1. Using induction on $|V(G)|$, the graph $G - u$ admits an equitably m-coloring with color classes $V_1, \cdots, V_m$. Note that $|V_i| = t - 1$ or t for all $i \geq 1$. Assume that $N(u) \in \bigcup_{i=1}^{7} V_i$. If there exists a class V_i with $i \geq 8$ such that $|V_i| = t - 1$, then put u into V_i and get an equitably m-coloring of G. If $|V_i| = t$ for all $i \geq 8$, then $|V(G)| \geq (m - 7)t + 7(t - 1) + 1 = mt - 6$, which implies $j \leq 6$.

Since $G' = G \cup K_j$ with $j \leq 6$ is a 1-planar graph with order mt (note that K_6 is 1-planar), G' is equitably m-colorable by the assumption. Hence G has an equitable m-coloring by restricting the coloring of G' to G. □

Lemma 2.3. *If the set of the vertices of a 1-planar graph (or NIC-planar graph, respectively) contains an independent s-set I and there exists $A \subseteq V(G) \setminus I$ such that $|A| > \frac{s(\Delta(G)+5)}{2}$ (or $|A| > \frac{s(\Delta(G)+4)}{2}$, respectively) and $e(v, I) \geq 1$ for all $v \in A$, then A contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in I$.*

Proof. We only prove it for 1-planar graphs. Let A_1 be an r-subset of A so that $e(v, I) = 1$ for all $v \in A_1$. We have $r + 2(|A| - r) \leq s\Delta(G)$, which implies $e(I, A_1) = r > 5s$. Consequently, I contains a vertex γ which has at least six neighbors in A_1. Since K_7 is non-1-planar, there are two nonadjacent vertices α and β among the neighbors of γ in G. □

A graph is *edge-minimal* in terms of equitable coloring if G has no equitable m-colorings but any subgraph of G has an equitable m-coloring. Delete one edge xy with $d(x) = \delta(G) := \delta$ from an edge-minimal graph G and partition the set of vertices of $G' := G - xy$ equitably into m subsets $V'_1, \cdots, V'_m$ so that each of them is an independent set. Obviously, x and y belong to a same subset for otherwise G is equitably m-colorable. Without loss of generality, assume that $x, y \in V'_1$ and $N(x) \subseteq \bigcup_{i=1}^{\delta} V'_i$. Denote $V_1 = V'_1 \setminus \{x\}$ and $V_i = V'_i$ for each $2 \leq i \leq m$.

We define $\mathcal{R}$ recursively. Let $V_1 \in \mathcal{R}$ and $V_j \in \mathcal{R}$ if there exists a vertex in V_j which has no neighbors in some $V_i \in \mathcal{R}$. Let $r = |\mathcal{R}|$, $A := \bigcup_{V_i \in \mathcal{R}} V_i$, $B := V(G) \setminus A$, $A' := A \cup \{x\}$ and $B' := B \setminus \{x\}$. Nakprasit[16] proved the following result, the proof of which follows from the definitions of $\mathcal{R}, A, A', B$ and B.

Lemma 2.4[16]. *(i) $\mathcal{R} \subseteq \{V_1, V_2, \cdots, V_\delta\}$; (ii) $e(u, V_i) \geq 1$ for each $u \in B$ and $V_i \in \mathcal{R}$; (iii) $e(A, B) \geq r(m-r)t + r$ and $e(A', B') \geq r(m-r)t$.*

Denote the class of 1-planar graphs, NIC-planar graphs and IC-planar graphs by $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$, respectively. Let $q_{m,\Delta,t,k}$ be the largest integer such that each graph in $\mathcal{G}_k$ of order mt is equitably m-colorable if $\Delta(G) \leq \Delta$ and $e(G) \leq q_{m,\Delta,t,k}$. One can easily see that $q_{m,\Delta,t,3} \geq q_{m,\Delta,t,2} \geq q_{m,\Delta,t,1}$ since $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \mathcal{G}_3$.

Lemma 2.5[16]. *If G is an edge-minimal graph in $\mathcal{G}_k$ for some $k \in \{1, 2, 3\}$ with order mt and maximum degree Δ, then $e(G) \geq r(m-r)t + q_{r,\Delta,t,k} + 1$*

Lemma 2.6[16]. *If G is an edge-minimal graph in $\mathcal{G}_k$ for some $k \in \{1, 2, 3\}$ with order mt, maximum degree Δ and size at most $(r+1)(m-r)t - t + 2 + q_{r,\Delta,t,k}$, then B contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in V_1$.*

Lemma 2.7[16]. *Let G be an edge-minimal graph in $\mathcal{G}_k$ for some $k \in \{1, 2, 3\}$ with order mt and maximum degree Δ. If B contains two nonadjacent vertices α and β that are adjacent to exactly one and the same vertex $\gamma \in V_1$, then $e(G) \geq r(m-r)t + q_{r,\Delta,t,k} + q_{m-r,\Delta,t,k} - \Delta + 4$.*

Note that Lemmas 2.5, 2.6 and 2.7 are originally proved for edge-minimal planar graphs, but one can easily check that Nakprasit's proofs are also valid for 1-planar graphs, NIC-planar graphs and IC-planar graphs, since his proofs do not rely on the class of graphs. Combining Lemmas 2.1, 2.3, 2.4, 2.6 and 2.7, we have the following lemma.

Lemma 2.8. *If G is an edge-minimal graph in $\mathcal{G}_k$ for some $k \in \{1, 2, 3\}$ with order mt and maximum degree Δ, then $e(G) \geq r(m-r)t + q_{r,\Delta,t,k} + q_{m-r,\Delta,t,k} - \Delta + 4$ if one of the following conditions are satisfied:*

(i) $(m-r)t + 1 > (t-1)(\Delta + 5)/2$ and $k = 1$;

(ii) $(m-r)t + 1 > (t-1)(\Delta + 4)/2$ and $k = 2, 3$;

(iii) $e(G) \leq (r+1)(m-r)t - t + 2 + q_{r,\Delta,t,k}$ and $k = 1, 2, 3$, where $r \leq 7$ if $k = 1$ and $r \leq 6$ if $k = 2, 3$.

It is easy to prove that if $m \geq \lceil \frac{\Delta+1}{2} \rceil + r + 2$, then (i) holds, and if $m \geq \lceil \frac{\Delta}{2} \rceil + r + 2$, then

(ii) holds. Lemma 2.8 (iii) implies

$$e(G) \geq \min\{r(m-r)t + q_{r,\Delta,t,k} + q_{m-r,\Delta,t,k} - \Delta + 4, (r+1)(m-r)t - t + 3 + q_{r,\Delta,t,k}\}. \quad (2.1)$$

This lower bounds for $e(G)$ along with the one in Lemma 2.5 are frequently used for the estimations of the lower bounds for $q_{m,\Delta,t,k}$ in the next section.

To complete the main proofs of this paper, the following known results are also useful.

Lemma 2.9[3,7,23] *The size of a 1-planar graph, an NIC-planar graph and an IC-planar graph is at most $4n-8$, $\frac{18}{5}n - \frac{36}{5}$ and $\frac{13}{4}n - 6$, respectively, where n denotes the order of the graph.*

Lemma 2.10[20]**.** *If G is a graph with order mt, size at most $(m-1)t$ and chromatic number at most m, then G has an equitable m-coloring.*

Lemma 2.11[10]**.** *If G does not contain $K_{\Delta,\Delta}$, is not a complete graph and is not an odd cycle, then G has an equitable Δ-coloring whenever $\Delta(G) := \Delta \geq \frac{1}{4}|G|$.*

A directly corollary of Lemma 2.11 is as follows.

Lemma 2.12. *If G is 1-planar and $\Delta(G) := \Delta \geq \max\{7, \frac{1}{4}|G|\}$, or G is NIC-planar and $\Delta(G) := \Delta \geq \max\left\{6, \frac{1}{4}|G|\right\}$, then G has an equitable Δ-coloring.*

3 Lower Bounds for $q_{m,\Delta,t,k}$

Lemma 3.1. *$q_{1,\Delta,t,k} = 0$, $q_{2,\Delta,t,k} \geq 2$, $q_{3,\Delta,t,k} \geq 3$, $q_{4,\Delta,t,k} \geq 4$ for $k = 1,2,3$ and $q_{5,\Delta,t,k} \geq 5$ for $k = 1,2$*

Proof. Those results are obvious. □

Lemma 3.2. *$q_{5,\Delta,t,3} \geq 4t$ and $q_{6,\Delta,t,k} \geq 5t$ for $k = 1,2$.*

Proof. Those results follow from Lemma 2.10 and the fact that every IC-planar graph is 5-colorable and every 1-planar graph is 6-colorable. □

In the remaining lemmas of this section, we always assume that $t \geq 5$. Moreover, when estimating the lower bounds for $q_{m,\Delta,t,1}$, $q_{m,\Delta,t,2}$ and $q_{m,\Delta,t,3}$, we just consider the cases with

$$m+1 \leq \Delta \leq 16, m+1 \leq \Delta \leq 14, \qquad m+1 \leq \Delta \leq 12,$$

respectively.

Lemma 3.3. *$q_{6,\Delta,t,3} \geq 6t+5$ and $q_{7,\Delta,t,1} \geq 7t+6$.*

Proof. We just prove the first reslut since another two can be dealt with similarly. Let H be an IC-planar graph with $e(H) \leq 6t+5$. If H is not equitably 6-colorable, then choose a subgraph $G \subseteq H$ so that G is edge-minimal. Since G is IC-planar, $\delta(G) := \delta \leq 6$ by Lemma 2.1. If $\delta = 6$, then we color the non-isolate vertices with 6 colors. Since $e(G) \leq 6t+5$, every color class has at most t non-isolate vertices. Hence we can easily construct an equitable 6-coloring of G by adding isolated vertices to each color class of the above partial coloring, a contradiction. We now suppose $r \leq \delta \leq 5$ (recall $r = |\mathcal{R}|$). If $r \geq 2$, then by Lemmas 2.5, 3.1 and 3.2, $e(G) \geq 8t+3 > 6t+5$. If $r = 1$, then by (2.1), $e(G) \geq 9t - \Delta + 4 > 6t+5$. All are contradictions since $e(G) \leq e(H) \leq 6t+5$. □

Lemma 3.4. *$q_{7,\Delta,t,3} \geq 9t+2$ for $\Delta \geq 9$ and $q_{7,8,t,3} \geq 12t$.*

Proof. Let H be an IC-planar graph with $e(H) \leq 9t+2$ if $\Delta \geq 9$ and $e(H) \leq 12t$ if $\Delta = 8$. If H is not equitably 7-colorable, then choose a subgraph $G \subseteq H$ so that G is edge-minimal. Since G is IC-planar, $\delta(G) \leq 6$ and $r \leq 6$. We estimate the lower bounds for $e(G)$ by splitting the proof into cases according to the possible value of r. From Table 1, we have $e(G) \geq 9t+3$ if $\Delta \geq 9$ and $e(G) \geq 12t+1$ if $\Delta = 8$. This is a contradiction since $e(G) \leq e(H)$. □

Table 1. The Proof of Lemma 3.4

m	k	r	lower bounds for $e(G)$	Reasons
7	3	1	$12t+1$ for $\Delta = 8$	Lemmas 2.8 (ii), 3.1, 3.3
		1	$9t+3$ for $\Delta \geq 9$	(2.1), Lemmas 3.1, 3.3
		2	$14t-\Delta+6$	(2.1), Lemmas 3.1, 3.2
		3	$12t-\Delta+11$	(2.1), Lemma 3.1
		4	$12t-\Delta+11$	(2.1), Lemma 3.1
		5	$14t-\Delta+6$	(2.1), Lemmas 3.1, 3.2
		6	$12t-\Delta+9$	(2.1), Lemmas 3.1, 3.3

Lemma 3.5. *(i)* $q_{8,\Delta,t,1} \geq \min\{14t-\Delta+9, 13t+2\}$ *for* $\Delta \geq 10$ *and* $q_{8,9,t,1} \geq 14t$*;*

(ii) $q_{8,\Delta,t,2} \geq 13t-\Delta+8$*;*

(iii) $q_{8,\Delta,t,3} \geq 13t+2$ *for* $\Delta \geq 11$ *and* $q_{8,\Delta,t,3} \geq 16t-\Delta+5$ *for* $9 \leq \Delta \leq 10$.

Proof. Use Table 2 for an argument similar to the proof of Lemma 3.4. □

Table 2. The Proof of Lemma 3.5

m	k	r	lower bounds for $e(G)$	Reasons
8	1	1	$14t+1$ for $\Delta = 9$	Lemmas 2.8 (i), 3.1, 3.3
		1	$14t-\Delta+10$ or $13t+3$ for $\Delta \geq 10$	(2.1), Lemmas 3.1, 3.3
		2	$17t-\Delta+6$	(2.1), Lemmas 3.1, 3.2
		3	$15t-\Delta+12$	(2.1), Lemma 3.1
		4	$16t-\Delta+12$	(2.1), Lemma 3.1
		5	$15t-\Delta+12$	(2.1), Lemma 3.1
		6	$17t-\Delta+6$	(2.1), Lemmas 3.1, 3.3
		7	$14t-\Delta+10$	(2.1), Lemmas 3.1, 3.3
8	2	1	$13t-\Delta+9$	(2.1), Lemmas 3.1, 3.3
		2	$17t-\Delta+6$	(2.1), Lemmas 3.1, 3.2
		3	$15t-\Delta+12$	(2.1), Lemma 3.1
		4	$16t-\Delta+12$	(2.1), Lemma 3.1
		5	$15t-\Delta+12$	(2.1), Lemma 3.1
		6	$17t-\Delta+6$	(2.1), Lemmas 3.1, 3.3
8	3	1	$16t-\Delta+6$ for $9 \leq \Delta \leq 10$	Lemmas 2.8(ii), 3.1, 3.4
		1	$13t+3$ for $\Delta \geq 11$	(2.1), Lemmas 3.1, 3.4
		2	$18t-\Delta+11$ or $17t+5$	(2.1), Lemmas 3.1, 3.3
		3	$19t-\Delta+7$	(2.1), Lemmas 3.1, 3.2
		4	$16t-\Delta+12$	(2.1), Lemma 3.1
		5	$19t-\Delta+7$	(2.1), Lemmas 3.1, 3.2
		6	$18t-\Delta+11$	(2.1), Lemmas 3.1, 3.3

Lemma 3.6. *(i)* $q_{9,\Delta,t,1} \geq 15t+2$ *for* $\Delta \geq 12$ *and* $q_{9,\Delta,t,1} \geq \min\{21t-\Delta+5, 20t-\Delta+12\}$ *for* $10 \leq \Delta \leq 11$*;*

(ii) $q_{9,\Delta,t,2} \geq 15t+2$ *for* $\Delta \geq 13$ *and* $q_{9,\Delta,t,2} \geq \min\{21t-2\Delta+11, 20t-\Delta+10\}$ *for* $10 \leq \Delta \leq 12$*;*

(iii) $q_{9,\Delta,t,3} \geq \min\{21t-\Delta+5, 20t+4\}$ *for* $11 \leq \Delta \leq 12$ *and* $q_{9,10,t,3} \geq \min\{24t-12, 23t-3\}$.

Proof. Use Table 3 for an argument similar to the proof of Lemma 3.4. □

Table 3. The Proof of Lemma 3.6

m	k	r	lower bounds for $e(G)$	Reasons
9	1	1	$21t-\Delta+6$ for $10\le\Delta\le 11$	Lemmas 2.8(i), 3.1, 3.5
		1	$15t+3$ for $\Delta\ge 12$	(2.1), Lemmas 3.1, 3.5
		2	$21t-\Delta+12$ or $20t+5$	(2.1), Lemmas 3.1, 3.3
		3	$23t-\Delta+7$	(2.1), Lemmas 3.1, 3.2
		4	$20t-\Delta+13$	(2.1), Lemma 3.1
		5	$20t-\Delta+13$	(2.1), Lemma 3.1
		6	$23t-\Delta+7$	(2.1), Lemmas 3.1, 3.2
		7	$21t-\Delta+12$	(2.1), Lemmas 3.1, 3.3
9	2	1	$21t-2\Delta+12$ for $10\le\Delta\le 12$	Lemmas 2.8(ii), 3.1, 3.5
		1	$15t+3$ for $\Delta\ge 13$	(2.1), Lemmas 3.1, 3.5
		2	$20t-\Delta+11$	(2.1), Lemmas 3.1, 3.3
		3	$23t-\Delta+7$	(2.1), Lemmas 3.1, 3.2
		4	$20t-\Delta+13$	(2.1), Lemma 3.1
		5	$20t-\Delta+13$	(2.1), Lemma 3.1
		6	$23t-\Delta+7$	(2.1), Lemmas 3.1, 3.2
9	3	1	$24t-11$ for $\Delta=10$	Lemmas 2.8(ii), 3.1, 3.5
		1	$21t-\Delta+6$ for $\Delta\ge 11$	Lemmas 2.8(ii), 3.1, 3.5
		2	$23t-2$ for $\Delta=10$	Lemmas 2.8(ii), 3.1, 3.4
		2	$20t+5$ for $\Delta\ge 11$	(2.1), Lemmas 3.1, 3.4
		3	$24t-\Delta+12$ or $23t+6$	(2.1), Lemmas 3.1, 3.3
		4	$24t-\Delta+8$	(2.1), Lemmas 3.1, 3.2
		5	$24t-\Delta+8$	(2.1), Lemmas 3.1, 3.2
		6	$24t-\Delta+12$	(2.1), Lemmas 3.1, 3.3

Table 4. The Proof of Lemma 3.7

m	k	r	lower bounds for $e(G)$	Reasons
10	1	1	$30t-13$ or $29t-6$ for $\Delta=11$	Lemmas 2.8(i), 3.1, 3.6
		1	$24t-\Delta+6$ for $12\le\Delta\le 13$	Lemmas 2.8(i), 3.1, 3.6
		1	$17t+3$ for $\Delta\ge 14$	(2.1), Lemmas 3.1, 3.6
		2	$30t-7$ or $29t-3$ for $\Delta=11$	Lemmas 2.8(i), 3.1, 3.5
		2	$23t+5$ for $\Delta\ge 12$	(2.1), Lemmas 3.1, 3.5
		3	$28t-\Delta+13$ or $27t+6$	(2.1), Lemmas 3.1, 3.3
		4	$29t-\Delta+8$	(2.1), Lemmas 3.1, 3.2
		5	$25t-\Delta+14$	(2.1), Lemma 3.1
		6	$29t-\Delta+8$	(2.1), Lemmas 3.1, 3.2
		7	$28t-\Delta+13$	(2.1), Lemmas 3.1, 3.4
10	2	1	$30t-3\Delta+15$ or $29t-2\Delta+16$ for $11\le\Delta\le 12$	Lemmas 2.8(ii), 3.1, 3.6
		1	$24t-\Delta+6$ for $\Delta\ge 13$	Lemmas 2.8(ii), 3.1, 3.6
		2	$29t-2\Delta+14$ for $11\le\Delta\le 12$	Lemmas 2.8(ii), 3.1, 3.5
		2	$23t+5$ for $\Delta\ge 13$	(2.1), Lemmas 3.1, 3.5
		3	$27t-\Delta+12$	(2.1), Lemmas 3.1, 3.4
		4	$29t-\Delta+8$	(2.1), Lemmas 3.1, 3.2
		5	$25t-\Delta+14$	(2.1), Lemma 3.1
		6	$29t-\Delta+8$	(2.1), Lemmas 3.1, 3.4
10	3	1	$30t-2\Delta+9$ or $29t-\Delta+8$	Lemmas 2.8(ii), 3.1, 3.6
		2	$29t-\Delta+8$	Lemmas 2.8(ii), 3.1, 3.5
		3	$27t+6$	(2.1), Lemmas 3.1, 3.4
		4	$30t-\Delta+13$ or $29t+7$	(2.1), Lemmas 3.1, 3.2
		5	$33t-\Delta+4$	(2.1), Lemmas 3.1,3.2
		6	$30t-\Delta+13$	(2.1), Lemmas 3.1, 3.4

Lemma 3.7. *(i)* $q_{10,\Delta,t,1} \geq 17t+2$ *for* $\Delta \geq 14$, $q_{10,\Delta,t,1} \geq \min\{24t-\Delta+5, 23t+4\}$ *for* $12 \leq \Delta \leq 13$ *and* $q_{10,11,t,1} \geq 25t+2$*;*

(ii) $q_{10,\Delta,t,2} \geq \min\{24t-\Delta+5, 23t+4\}$ *for* $13 \leq \Delta \leq 14$ *and* $q_{10,\Delta,t,2} \geq 25t-\Delta+13$ *for* $11 \leq \Delta \leq 12$*;*

(iii) $q_{10,\Delta,t,3} \geq \min\{30t-2\Delta+8, 29t-\Delta+7, 27t+5\}$ *for* $11 \leq \Delta \leq 12$.

Proof. Use Table 4 for an argument similar to the proof of Lemma 3.4. □

Lemma 3.8. *(i)* $q_{11,16,t,1} \geq 19t+2$, $q_{11,\Delta,t,1} \geq \min\{27t-\Delta+5, 26t+4\}$ *for* $14 \leq \Delta \leq 15$ *and* $q_{11,\Delta,t,1} \geq \min\{34t-2\Delta+8, 33t-\Delta+7, 31t+5\}$ *for* $12 \leq \Delta \leq 13$*;*

(ii) $q_{11,\Delta,t,2} \geq \min\{34t-2\Delta+8, 33t-\Delta+7, 31t+5\}$ *for* $13 \leq \Delta \leq 14$ *and* $q_{11,12,t,2} \geq \min\{35t-8, 34t\}$*;*

(iii) $q_{11,12,t,3} \geq \min\{40t-25, 34t+6\}$.

Proof. Use Table 5 for an argument similar to the proof of Lemma 3.4. □

Table 5. The Proof of Lemma 3.8

m	k	r	lower bounds for $e(G)$	Reasons
11	1	1	$34t-2\Delta+9$ or $33t-\Delta+8$ for $12 \leq \Delta \leq 13$	Lemmas 2.8(i), 3.1, 3.7
		1	$27t-\Delta+6$ for $14 \leq \Delta \leq 15$	Lemmas 2.8(i), 3.1, 3.7
		1	$19t+3$ for $\Delta=16$	(2.1), Lemmas 3.1, 3.7
		2	$33t-\Delta+8$ for $12 \leq \Delta \leq 13$	Lemmas 2.8(i), 3.1, 3.6
		2	$26t+5$ for $\Delta \geq 14$	(2.1), Lemmas 3.1, 3.6
		3	$31t+6$	(2.1), Lemmas 3.1, 3.5
		4	$35t-\Delta+14$ or $34t+7$	(2.1), Lemmas 3.1, 3.3
		5	$35t-\Delta+9$	(2.1), Lemmas 3.1, 3.2
		6	$35t-\Delta+9$	(2.1), Lemmas 3.1, 3.2
		7	$35t-\Delta+14$	(2.1), Lemma 3.3
11	2	1	$35t-7$ for $\Delta=12$	Lemmas 2.8(ii), 3.1, 3.7
		1	$34t-2\Delta+9$ or $33t-\Delta+8$ for $\Delta \geq 13$	Lemmas 2.8(ii), 3.1, 3.7
		2	$39t-19$ or $38t+4$ for $\Delta=12$	Lemmas 2.8(ii), 3.1, 3.6
		2	$33t-\Delta+8$ for $\Delta \geq 13$	Lemmas 2.8(ii), 3.1, 3.6
		3	$37t-9$ for $\Delta=12$	Lemmas 2.8(ii), 3.1, 3.5
		3	$31t+6$ for $\Delta \geq 13$	Lemmas (2.1), 3.1, 3.5
		4	$34t-\Delta+13$	(2.1), Lemmas 3.1, 3.3
		5	$35t-\Delta+9$	(2.1), Lemmas 3.1, 3.2
		6	$35t-\Delta+9$	(2.1), Lemmas 3.1, 3.2
11	3	1	$40t-24$ or $37t-3$	Lemmas 2.8(ii), 3.1, 3.7
		2	$39t-13$ or $38t-2$	Lemmas 2.8(ii), 3.1, 3.6
		3	$37t-3$	Lemmas 2.8(ii), 3.1, 3.5
		4	$34t+7$	(2.1), Lemmas 3.1, 3.4
		5	$40t-3$ or $39t+3$	(2.1), Lemmas 3.1,3.2, 3.3
		6	$40t-3$ or $39t+8$	(2.1), Lemmas 3.1,3.2, 3.3

Lemma 3.9. *(i)* $q_{12,16,t,1} \geq \min\{29t+4, 30t-11\}$, $q_{12,\Delta,t,1} \geq \min\{38t-2\Delta+8, 37t-\Delta+7, 35t+5\}$ *for* $14 \leq \Delta \leq 15$ *and* $q_{12,13,t,1} \geq \min\{45t-28, 39t+6\}$*;*

(ii) $q_{12,\Delta,t,2} \geq \min\{45t-3\Delta+11, 39t+6\}$ *for* $13 \leq \Delta \leq 14$.

Proof. Use Table 6 for an argument similar to the proof of Lemma 3.4. □

Table 6. The Proof of Lemma 3.9

m	k	r	lower bounds for $e(G)$	Reasons
12	1	1	$45t-27$ or $44t-15$ or $42t-4$ for $\Delta=13$	Lemmas 2.8(i), 3.1, 3.8
		1	$38t-2\Delta+9$ or $37t-\Delta+8$ for $14\le\Delta\le15$	Lemmas 2.8(i), 3.1, 3.8
		1	$30t-10$ for $\Delta=16$	Lemmas 2.8(i), 3.1, 3.8
		2	$44t-2\Delta+11$ or $43t-\Delta+10$ for $13\le\Delta\le15$	Lemmas 2.8(i),3.1, 3.7
		2	$29t+5$ for $\Delta=16$	(2.1), Lemmas 3.1, 3.7
		3	$42t-4$ for $\Delta=13$	Lemmas 2.8(i), 3.1, 3.6
		3	$35t+6$ for $14\le\Delta\le16$	(2.1), Lemmas 3.1, 3.6
		4	$39t+7$	(2.1), Lemmas 3.1, 3.5
		5	$42t-\Delta+15$ or $41t+8$	(2.1), Lemmas 3.1, 3.3
		6	$46t-\Delta+4$	(2.1), Lemma 3.2
		7	$42t-\Delta+15$	(2.1), Lemmas 3.1, 3.3
12	2	1	$45t-3\Delta+12$ or $44t-2\Delta+11$ or $42t-\Delta+9$	Lemmas 2.8(ii), 3.1, 3.8
		2	$44t-2\Delta+11$ or $43t-\Delta+10$	Lemmas 2.8(ii), 3.1, 3.7
		3	$42t-\Delta+9$	Lemmas 2.8(ii), 3.1, 3.6
		4	$39t+7$	(2.1), Lemmas 3.1, 3.5
		5	$41t-\Delta+14$	(2.1), Lemmas 3.1, 3.3
		6	$46t-\Delta+4$	(2.1), Lemma 3.2

Lemma 3.10. *(i)* $q_{13,16,t,1}\ge\min\{41t-9,42t-24,39t+5\}$ *and* $q_{13,\Delta,t,1}\ge\min\{50t-3\Delta+11,49t-2\Delta+10,44t+6\}$ *for* $14\le\Delta\le15$*;*
(ii) $q_{13,14,t,2}\ge 47t+7$.

Proof. Use Table 7 for an argument similar to the proof of Lemma 3.4. □

Table 7. The Proof of Lemma 3.10

m	k	r	lower bounds for $e(G)$	Reasons
13	1	1	$50t-3\Delta+12$ or $49t-2\Delta+11$ or $47t-\Delta+9$ for $14\le\Delta\le15$	Lemmas 2.8(i), 3.1, 3.9
		1	$41t-8$ or $42t-23$ for $\Delta=16$	Lemmas 2.8(i), 3.1, 3.9
		2	$49t-2\Delta+11$ or $48t-\Delta+10$ for $14\le\Delta\le15$	Lemmas 2.8(i), 3.1, 3.8
		2	$41t-8$ for $\Delta=16$	Lemmas 2.8(i), 3.1, 3.8
		3	$47t-\Delta+9$ for $14\le\Delta\le15$	Lemmas 2.8(i), 3.1, 3.7
		3	$39t+6$ for $\Delta=16$	(2.1), Lemmas 3.1, 3.7
		4	$44t+7$	(2.1), Lemmas 3.1, 3.6
		5	$47t+8$	(2.1), Lemmas 3.1, 3.5
		6	$54t-\Delta+10$ or $53t+3$	(2.1), Lemmas 3.2, 3.3
		7	$54t-\Delta+10$	(2.1), Lemmas 3.2, 3.3
13	2	1	$57t-41$ or $51t-4$	Lemmas 2.8(ii), 3.1, 3.9
		2	$56t-28$ or $55t-15$ or $53t-3$	Lemmas 2.8(ii), 3.1, 3.8
		3	$54t-16$ or $53t-3$	Lemmas 2.8(ii), 3.1, 3.7
		4	$51t-4$	Lemmas 2.8(ii), 3.1, 3.6
		5	$47t+8$	(2.1), Lemmas 3.2, 3.5
		6	$53t-5$	(2.1), Lemmas 3.3, 3.4

Lemma 3.11. $q_{14,15,t,1}\ge\min\{63t-46,53t+7\}$ *and* $q_{14,16,t,1}\ge\min\{54t-22,53t-37,49t+6\}$.

Proof. Use Table 8 for an argument similar to the proof of Lemma 3.4. □

Table 8. The Proof of Lemma 3.11

m	k	r	lower bounds for $e(G)$	Reasons
14	1	1	$63t-45$ or $62t-31$ or $57t-5$ for $\Delta=15$	Lemmas 2.8(i), 3.1, 3.10
		1	$54t-21$ or $53t-36$ or $52t-7$ for $\Delta=16$	Lemmas 2.8(i), 3.1, 3.10
		2	$62t-31$ or $61t-17$ or $59t-4$ for $\Delta=15$	Lemmas 2.8(i), 3.1, 3.9
		2	$53t-6$ or $54t-5$ for $\Delta=16$	Lemmas 2.8(i), 3.1, 3.9
		3	$60t-18$ or $59t-4$ for $\Delta=15$	Lemmas 2.8(i), 3.1, 3.8
		3	$52t-7$ for $\Delta=16$	Lemmas 2.8(i), 3.1, 3.8
		4	$57t-5$ for $\Delta=15$	Lemmas 2.8(i), 3.1, 3.7
		4	$49t+7$ for $\Delta=16$	(2.1), Lemmas 3.1, 3.7
		5	$53t+8$	(2.1), Lemmas 3.1, 3.6
		6	$60t+3$	(2.1), Lemmas 3.2, 3.5
		7	$63t-\Delta+16$ or $62t+9$	(2.1), Lemma 3.3

Lemma 3.12. $q_{15,16,t,1} \geq \min\{67t-50, 59t+7\}$.

Proof. Use Table 9 for an argument similar to the proof of Lemma 3.4. □

Table 9. The Proof of Lemma 3.12

m	k	r	lower bounds for $e(G)$	Reasons
15	1	1	$68t-34$ or $67t-49$ or $63t-6$	Lemmas 2.8(i), 3.1, 3.11
		2	$67t-3$ or $68t-34$ or $65t-5$	Lemmas 2.8(i), 3.1, 3.10
		3	$65t-5$ or $66t-20$	Lemmas 2.8(i), 3.1, 3.9
		4	$63t-6$	Lemmas 2.8(i), 3.1, 3.8
		5	$59t+8$	(2.1), Lemmas 3.1, 3.7
		6	$64t-10$	(2.1), Lemmas 3.2, 3.6
		7	$70t+9$	(2.1), Lemmas 3.3, 3.5

4 Results

In this section, we prove the main results of this paper.

Theorem 4.1. *Each 1-planar graph with maximum degree at most Δ has an equitable Δ-coloring if $\Delta \geq 15$.*

Proof. Since Zhang[22] proved the result for $\Delta \geq 17$, it is suffice to consider the cases with $\Delta = 15$ or $\Delta = 16$. By Lemmas 2.2 and 2.12, we assume that the considered 1-planar graph G has order Δt with $t \geq 5$. Suppose, to the contrary, that G does not satisfy this result, and moreover, G is edge-minimal.

If $r \leq 5$, then $e(G) \geq r(\Delta-r)t+q_{r,\Delta,t,1}+q_{\Delta-r,\Delta,t,1}-\Delta+4$ by Lemma 2.8(i), since $(\Delta-r)t+1 > (t-1)(\Delta+5)/2$. If $6 \leq r \leq 7$, then $(r+1)(\Delta-r)t-t+2+q_{r,\Delta,t,1} > 4\Delta t-8 \geq e(G)$, thus by Lemma 2.8(iii), it still holds that $e(G) \geq r(\Delta-r)t+q_{r,\Delta,t,1}+q_{\Delta-r,\Delta,t,1}-\Delta+4$. For each case, we use the following table to estimate the lower bounds for $r(\Delta-r)t+q_{r,\Delta,t,1}+q_{\Delta-r,\Delta,t,1}-\Delta+4$, thus for $e(G)$, by Lemmas 3.1, 3.2, 3.3 and 3.5–3.12.

	$\Delta = 15$	$\Delta = 16$
$r = 1$	$\min\{77t - 57, 76t - 43, 67t - 4\}$	$\min\{82t - 62, 74t - 5\}$
$r = 2$	$\min\{76t - 43, 75t - 29, 70t - 3\}$	$\min\{82t - 32, 81t - 47, 77t - 4\}$
$r = 3$	$\min\{74t - 30, 73t - 16, 71t - 3\}$	$\min\{80t - 18, 81t - 33, 78t - 4\}$
$r = 4$	$\min\{71t - 17, 70t - 3\}$	$\min\{77t - 4, 78t - 19\}$
$r = 5$	$67t - 4$	$74t - 5$
$r = 6$	$74t - 9$	$82t - 10$
$r = 7$	$\min\{77t - 11, 76t - 3\}$	$85t - 4$

From the above table, one can see that $e(G) > 4\Delta t - 8$ for $\Delta = 15$ or $\Delta = 16$, which contradicts Lemma 2.9. Hence we proved the required result. □

Theorem 4.2. *Each NIC-planar graph with maximum degree at most Δ has an equitable Δ-coloring if $\Delta \geq 13$.*

Proof. Since every NIC-planar graph is 1-planar, by Theorem 4.1, it is suffice to consider the cases with $\Delta = 13$ or $\Delta = 14$. By Lemmas 2.2 and 2.12, we assume that the considered NIC-planar graph G has order Δt with $t \geq 5$. Suppose, to the contrary, that G does not satisfy this result, and moreover, G is edge-minimal.

By similar argument as the one in the proof of Theorem 4.1, we have $e(G) \geq r(\Delta - r)t + q_{r,\Delta,t,2} + q_{\Delta-r,\Delta,t,2} - \Delta + 4$ by Lemmas 2.8(ii) and 2.8(iii). Again, we give, by Lemmas 3.1, 3.2, 3.3 and 3.5–3.12, the lower bounds for $e(G)$ in the following table.

	$\Delta = 13$	$\Delta = 14$
$r = 1$	$\min\{57t - 37, 51t - 3\}$	$60t - 3$
$r = 2$	$\min\{56t - 25, 55t - 13, 53t - 2\}$	$\min\{69t - 39, 63t - 2\}$
$r = 3$	$\min\{54t - 14, 53t - 2\}$	$\min\{67t - 27, 66t - 14, 64t - 2\}$
$r = 4$	$51t - 3$	$\min\{65t - 15, 63t - 2\}$
$r = 5$	$53t - 9$	$60t - 3$
$r = 6$	$56t - 7$	$66t - 8$

From the above table, one can see that $e(G) > \frac{18}{5}\Delta t - \frac{36}{5}$ for $\Delta = 13$ or $\Delta = 14$, which contradicts Lemma 2.9. Hence we proved the required result. □

Theorem 4.3. *Each IC-planar graph with maximum degree at most Δ has an equitable Δ-coloring if $\Delta \geq 12$.*

Proof. Since every IC-planar graph is NIC-planar, by Theorem 4.1, it is suffice to consider the case with $\Delta = 12$. By Lemmas 2.2 and 2.12, we assume that the considered NIC-planar graph G has order Δt with $t \geq 5$. Suppose, to the contrary, that G does not satisfy this result, and moreover, G is edge-minimal.

By similar argument as the one in the proof of Theorem 4.1, we have $e(G) \geq r(\Delta - r)t + q_{r,\Delta,t,3} + q_{\Delta-r,\Delta,t,3} - \Delta + 4$ by Lemmas 2.8(ii) and 2.8(iii). The lower bounds for $e(G)$ in each case are shown in the following table, which is implied by Lemmas 3.1–3.12.

	$\Delta = 12$
$r = 1$	$\min\{51t - 33, 45t - 2\}$
$r = 2$	$\min\{50t - 22, 49t - 11, 47t - 1\}$
$r = 3$	$\min\{48t - 12, 47t - 1\}$
$r = 4$	$45t - 2$
$r = 5$	$48t - 6$
$r = 6$	$48t + 2$

From the above table, one can see that $e(G) > 39t - 6$, which contradicts Lemma 2.9. Hence we proved the required result. □

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