On r-equitable colorings of bipartite graphs*

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Abstract

An r-equitable k-coloring of a graph G is a proper k-coloring of G so that the size of any two color classes differ by at most r. The least k such that G is r-equitably k-colorable is the r-equitable chromatic number of G. In this paper, we prove that the r-equitable chromatic number of a connected bipartite graph $G(X, Y)$ with $|X| = m \geq n = |Y|$ is at most $\lceil \frac{m}{n+r} \rceil + 1$ provided that G satisfies a restriction on the number of edges. This generalizes a result of K.-W. Lih and P.-L. Wu [Discrete Math., 151 (1996) 155–160].
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1 Introduction

All graphs considered in this paper are finite, simple and undirected unless otherwise stated. By $V(G)$ and $E(G)$, we denote the *vertex set* and the *edge set* of a graph G, respectively. For a vertex $v \in V(G)$, $\mathtt{deg}(v)$ is the *degree* of v in G, which is the number of edges that are incident with v in G. For a subset of U of $V(G)$, by $e(U)$ we denote the number of edges in G which have at least one end vertex in U. Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote, respectively, the smallest integer not less than x and the largest integer not greater than x. A connected *bipartite graph* (i.e., *2-colorable graph*) $G(X, Y)$ is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y and there always exists a path between every pair of vertices.

If the vertices of a graph G are partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set with vertices colored by one single color and $\big||V_i| - |V_j|\big| \leq r$ for all $i \neq j$, then G is *r-equitably k-colorable*. The least integer k

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such that a graph G is r-equitably k-colorable is the *r-equitable chromatic number* of G and denoted by $\chi_{r=}(G)$. It is obvious that an r-equitably k-colorable graph is certainly $(r+1)$-equitably k-colorable. Although the concept of r-equitable colorability seems a natural generalization of usual equitable colorability (corresponding to r=1) introduced by Meyer [4] in 1973, it was first proposed in a recent paper by Hertz and Ries [1, 2], which gives a complete characterization of r-equitably k-colorable trees for any given integer $r \geq 1$. Actually, the study on the r-equitable colorability of graphs is still at the early stage. As far as we know, Wang, Yan, and Zhang [5] considered the r-equitable colorings of Kronecker products of complete graphs, and Yen [6] proposed a necessary and sufficient condition for a complete multipartite graph $G := K_{n_1,n_2,\ldots,n_t}$ to have an r-equitable k-coloring, and gave exact value of $\chi_{r=}(G)$ as follows.

Theorem 1. [6] *For any $r \geq 1$, $\chi_{r=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} \lceil n_i/(\theta + r)\rceil$, where $\theta = \max\{s \in \mathbb{N} : \lfloor n_i/s \rfloor \geq \lceil n_i/(s+r)\rceil\}$.*

Using Theorem 1, we can easily deduce a similar result on complete bipartite graphs.

Theorem 2. *Let $K_{m,n}$ be a compete bipartite graph with $m \geq n \geq 2$. If $r \geq n-1$, then*

$$\chi_{r=}(K_{m,n}) = \left\lceil \frac{m}{n+r} \right\rceil + 1.$$

Proof. Let $m = a(n+r) - b$ with $a = \lceil m/(n+r)\rceil$ and $0 \leq b < n+1$. Since $\lfloor n/s \rfloor = 0 < 1 \leq \lceil n/(s+r)\rceil$ for any $s > n$ and $\lfloor n/n \rfloor = 1 = \lceil n/(n+r)\rceil$, $\theta \leq n$. On the other hand, if $a \geq 2$, then $\lfloor m/n \rfloor = \lfloor (ar-b)/n \rfloor + a \geq \lfloor (ar-n)/n \rfloor + a \geq a$, since $ar - n \geq a(n-1) - n \geq n-2 \geq 0$, and if $a = 1$, then $\lfloor m/n \rfloor \geq 1 = a$. In each case we have $\lfloor m/n \rfloor \geq \lceil m/(n+r)\rceil$. Therefore, $\theta = n$ and thus $\chi_{r=}(K_{m,n}) = \lceil m/(n+r)\rceil + \lceil n/(n+r)\rceil = \lceil m/(n+r)\rceil + 1$ by Theorem 1. □

In this paper, we consider the r-equitable colorings of bipartite graphs which may be not complete. The aim of this paper is to generate the following result of Lih and Wu [3] to its r-equitable colorability version.

Theorem 3. [3] *Let $G(X,Y)$ be a connected bipartite graph with ε edges. If $|X| = m \geq n = |Y|$ and $\varepsilon < \lfloor m/(n+1)\rfloor(m-n)+2m$, then $\chi_{1=}(G) \leq \lceil m/(n+1)\rceil + 1$.*

In the next section, we give the detailed proof of the following main result of this paper.

Theorem 4. *Let $G(X,Y)$ be a connected bipartite graph with ε edges. If $|X| = m \geq n = |Y|$ and*

$$\varepsilon < \left\lfloor \frac{n(q+2)}{(q+1)(n+r)-m} \right\rfloor (m-n-r+1) + 2n, \tag{1}$$

where $q = \lfloor \frac{m}{n+r} \rfloor$, then

$$\chi_{r=}(G) \leq \left\lceil \frac{m}{n+r} \right\rceil + 1. \tag{2}$$

Now we use Theorem 4 to show the result of Lih and Wu (Theorem 3). Let $q' = \lfloor \frac{m}{n+1} \rfloor$ and $m = q'(n+1) + p$ with $0 \leq p < n+1$. If $p' = 0$, then we partition X into q' independent subsets $X_1, X_2, \ldots, X_{q'}$ of size $n+1$, and the partition $\{X_1, X_2, \ldots, X_{q'}, Y\}$ of $V(G)$ implies a $(q'+1)$-equitable coloring of G. If $p' \geq 1$, then $\varepsilon < \lfloor \frac{n(q'+2)}{(q'+1)(n+1)-m} \rfloor (m-n) + 2n$, because otherwise $\varepsilon \geq \lfloor \frac{n(q'+2)}{(q'+1)(n+1)-m} \rfloor (m-n) + 2n = \lfloor \frac{n(q'+2)}{n+1-p'} \rfloor (m-n) + 2n \geq (q'+2)(m-n) + 2n = q'(m-n) + 2m$, a contradiction to the condition for Theorem 3. Hence $\chi_{1=}(G) \leq \lceil \frac{m}{n+1} \rceil + 1$ by Theorem 4.

To end this section, we show that the upper bound in (2) of Theorem 4 cannot be reduced in the general case. Choose r to be an integer no less than $n-1$. For example, let $r = n$ (other values of r can be similarly discussed). One can check that if

$$n \leq \frac{m + 2 + \sqrt{4m^3 + 5m^2 + 4m + 4}}{4m},$$

then

$$\left\lfloor \frac{n(q+2)}{(q+1)(n+r) - m} \right\rfloor (m - n - r + 1) + 2n > mn.$$

Therefore, the complete bipartite graph $G := K_{m,n}$ satisfies the restriction (1) on the number of edges, and thus $\chi_{r=}(G) = \left\lceil \frac{m}{n+r} \right\rceil + 1$ by Theorem 2.

2 The proof of Theorem 4

Let $q = \lfloor m/(n+r) \rfloor$. It follows that $m = q(n+r) + p$ with $0 \leq p < n+r$, and $\lceil m/(n+r) \rceil$ is q if $p = 0$, and is $q+1$ if $p \neq 0$. Therefore, we just generate that $\chi_{r=}(G) \leq q+1$ if $p = 0$, and $\chi_{r=}(G) \leq q+2$ if $p \neq 0$. If $q = 0$, then $m = p < n+r$ and G is r-equitably 2-colorable (coloring X with one color and Y with the other color). Hence in the following we always assume that $q \geq 1$.

Case 1: $p = 0$**.**

In this case, we have $|X| = q(n+r)$. Dividing X into q independent subsets of size $n+r$, and recognizing Y as a single independent subset of G, we obtain an r-equitable $(q+1)$-coloring of G.

Case 2: $n \leq p < n+r$**.**

We divide X into $q+1$ independent subsets so that q of them have size $n+r$ and one of them has size p. Those $q+1$ independent subsets along with Y form an r-equitable $(q+2)$-coloring of G.

Case 3: $0 < p < n$**.**

We generate that $\chi_{r=}(G) \leq q+2$. Hence if we can find a scheme which can r-equitably color G with $q+2$ colors, then we prove the theorem.

To find the scheme, we reclassify the vertices first by moving a set B consisting of k vertices from Y to X, where

$$k = \left\lfloor \frac{n - p + r}{q + 2} \right\rfloor.$$

By the definition of k, we know that

$$
\begin{aligned}
n-k \geq n-\frac{n-p+r}{q+2} &= \frac{n(q+2)-((q+1)(n+r)-m)}{q+2} \\
&\geq \frac{(q+1)(n+r)-m}{q+2}\cdot\left\lfloor\frac{n(q+2)}{(q+1)(n+r)-m}-1\right\rfloor \\
&> \frac{(q+1)(n+r)-m}{q+2}\cdot\left(\frac{\varepsilon-2n}{m-n-r+1}-1\right) \\
&\geq \frac{(q+1)(n+r)-m}{q+2}\cdot\left(\frac{m-n-1}{m-n-r+1}-1\right) \\
&= \frac{(q+1)(n+r)-m}{q+2}\cdot\left(\frac{r-2}{m-n-r+1}\right) \geq 0
\end{aligned}
$$

if $r \geq 2$,

$$
n-k \geq n-\frac{n-p+1}{q+2} \geq n-\frac{n}{3} > 0
$$

if $r = 1$, and

$$
n-2k+r \geq n-\frac{2(n-p+r)}{q+2}+r \geq n-\frac{2(n-p+r)}{3}+r > \frac{1}{3}n+\frac{1}{3}r > 0.
$$

Let $n-p+r = k(q+2)+t$ with $0 \leq t < q+2$. Since

$$
(m+k)-t(n-k+r-1)-(q+1-t)(n-k+r) = k(q+2)+t-n+p-r = 0 \quad (3)
$$

we can partition $m+k$ elements into t classes of size $n-k+r-1$ and $q+1-t$ classes of size $n-k+r$.

If $k = 0$, then we divide X into t independent subsets of size $n+r-1$, $q+1-t$ independent subsets of size $n+r$, and then recognize Y as a single independent subset of G. This implies an r-equitable $(q+2)$-coloring of G. Therefore, we assume that $k > 0$.

Moving Lemma: *If $k > 0$, then there exist $A \subseteq X$ and $B \subseteq Y$ such that $|A| = n-2k+r$, $|B| = k$ and $A \cup B$ is an independent set of size $n-k+r$.*

Proof. Let $n = ak+b$, where $a = \lfloor \frac{n}{k} \rfloor$ and $0 \leq b < k$. Suppose Y consists of vertices $v_1, v_2, \ldots, v_n$ with $\mathtt{deg}(v_1) \geq \mathtt{deg}(v_2) \geq \ldots \geq \mathtt{deg}(v_n)$.

If $b \neq 0$, then choose $U = \{v_1, v_2, \ldots, v_b\}$. If U contains no vertex of degree 1, then it is clear that $e(U) \geq 2b$. If U contains at least one vertex of degree 1, then $\mathtt{deg}(v_i) = 1$ for every $b < i \leq n$, which implies that $e(U) = \varepsilon-(n-b) \geq (m+n-1)-(n-b) = m+b-1 \geq n+b > 2b$. Note that we have assumed that $m > n$ here, since it is trivial that $\chi_{r=}(G) \leq 2 = \lceil \frac{m}{n+r} \rceil + 1$ if $m = n$. If $b = 0$, then choose $U = \emptyset$ and then $e(U) = 0 = 2b$. Indeed, in any case we have that $e(U) \geq 2b$.

Next, we partition $Y-U$ into a independent subsets $Y_1, Y_2, \ldots, Y_a$ so that $|Y_i| = k$ for any $1 \leq i \leq a$.

If $e(Y_i) \geq m - n + 2k - r + 1$ for any $1 \leq i \leq a$, then

$$\begin{aligned}
\varepsilon &= \sum_{i=1}^{a} E(Y_i) + e(U) \\
&\geq a(m - n + 2k - r + 1) + 2b \\
&= a(m - n - r + 1) + 2n \\
&= \left\lfloor \frac{n}{k} \right\rfloor (m - n - r + 1) + 2n \\
&\geq \left\lfloor \frac{n(q+2)}{n - p + r} \right\rfloor (m - n - r + 1) + 2n.
\end{aligned}$$

However, we have

$$\begin{aligned}
\varepsilon &< \left\lfloor \frac{n(q+2)}{(q+1)(n+r) - m} \right\rfloor (m - n - r + 1) + 2n \\
&= \left\lfloor \frac{n(q+2)}{n - (m - q(n+r)) + r} \right\rfloor (m - n - r + 1) + 2n \\
&= \left\lfloor \frac{n(q+2)}{n - p + r} \right\rfloor (m - n - r + 1) + 2n.
\end{aligned}$$

This contradiction implies that there exists a set Y_i with $e(Y_i) \leq m - n + 2k - r$ for some $1 \leq i \leq a$. Since there are only m vertices in X, X contains at least $m - (m - n + 2k - r) = n - 2k + r$ vertices which are independent of Y_i. Hence we are able to choose the required sets A and B from X and Y, respectively. □

Let $A \subseteq X$ and $B \subseteq Y$ be the vertex sets found by the moving lemma. By (3), we can divide $X - A$ into t independent subsets of size $n - k + r - 1$ and $q - t$ independent subsets of size $n - k + r$. Those q independent subsets along with $A \cup B$ (an independent subset of size $n - k + r$) and $Y - B$ (an independent subset of size $n - k$) imply an r-equitable $(q + 2)$-coloring of G.

This completes the proof of Theorem 4. □

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