Discrete Mathematics 339 (2016) 1393–1399

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

The edge chromatic number of outer-1-planar graphs[☆]

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ARTICLE INFO

Article history:
Received 5 August 2014
Received in revised form 31 October 2015
Accepted 9 December 2015

Keywords:
Outer-1-planar graph
Pseudo-outerplanar graph
Edge coloring

ABSTRACT

A graph is outer-1-planar if each block has an embedding in the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this paper, we completely determine the edge chromatic number of outer 1-planar graphs.

1. Introduction

All graphs considered in this paper are simple and undirected. By $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, we denote the set of vertices, the set of edges, the maximum degree and the minimum degree of a graph G, respectively. In any figure of this paper, the degree of a solid or hollow vertex is exactly or at least the number of edges that are incident with it, respectively. Moreover, solid vertices are distinct but two hollow vertices may be identified unless stated otherwise.

A graph is *outer-1-planar* if each block has an embedding in the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. Outer-1-planar graphs were first introduced by Eggleton [3] who called them *outerplanar graphs with edge crossing number one*, and were also investigated under the notion of *pseudo-outerplanar graphs* by Zhang, Liu and Wu [11]. The notion of outer-1-planarity is a natural generalization of outer-planarity, and is also a combination of 1-planarity and outer-planarity. The definition of outer-1-planarity implies that outer-1-planar graphs are all planar.

It has been recently shown by Dehkordi and Eades [2] that every outer-1-planar graph has a right angle crossing drawing, and by Auer et al. [1] that the recognition of outer-1-planarity can process in linear time. On the other hand, the class of outer-1-planar graphs is used as a special graph class for verifying interesting conjectures on graph coloring. For instance, the list edge coloring conjecture and the list total coloring conjecture are verified for outer-1-planar graphs with maximum degree at least 5 [7,9], and the total coloring conjecture and the equitable Δ-coloring conjecture are confirmed for all outer-1-planar graphs [10,7].

An *edge k-coloring* of a graph G is an assignment $f : E(G) \to \{1, 2, \dots, k\}$ so that $f(e_1) \neq f(e_2)$ whenever e_1 and e_2 are two adjacent edges. The minimum integer k so that G has an edge k-coloring, denoted by $\chi'(G)$, is the *edge chromatic number* of G. The well-known Vizing's theorem states that $\Delta(G) \le \chi'(G) \le \Delta(G)+1$ for every simple graph G. Therefore, to determine the edge chromatic number of a graph is interesting. However, this problem is NP-complete, and deciding whether a simple graph with maximum degree 3 has edge chromatic number 3 is still NP-complete [4]. As far as we know, the edge chromatic numbers of only few classes of graphs have been determined. For example, $\chi'(G) = \Delta(G)$ if G is a 1-planar graph with

☆ Supported by the National Natural Science Foundation of China (Nos. 11301410, 11201440), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130203120021), and the Fundamental Research Funds for the Central Universities (No. JB150714).

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http://dx.doi.org/10.1016/j.disc.2015.12.009

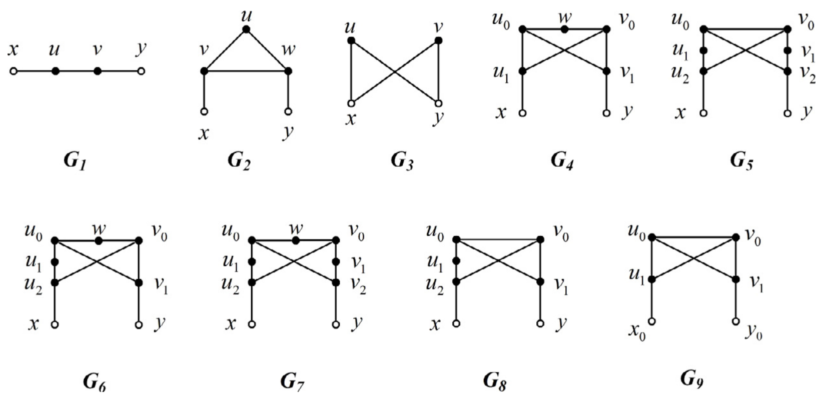

Fig. 1. Unavoidable structures in outer-1-planar graph with maximum degree at most 3.

maximum degree at least 10 [12], or a planar graph with maximum degree at least 7 [6], or a series–parallel graph (thus also an outer-planar graph) with maximum degree at least 3 [5].

The edge coloring of outer-1-planar graphs was first considered by Zhang, Liu and Wu [11]. They proved that the edge chromatic number of an outer-1-planar graph with maximum degree at least 4 is equal to the maximum degree, and announced that there are outer-1-planar graphs with maximum degree 3 and edge chromatic number 4 (the graph derived from K_4 by subdividing an edge is such an example). In this paper, we follow their work. First, we give structural results for outer-1-planar graphs with maximum degree at most 3. Next, we use the structural theorems to determine the edge chromatic number of all such outer-1-planar graphs.

2. Local structures of subcubic outer-1-planar graphs

Any outer-1-planar graph considered in this section is drawn in the plane so that its outer-1-planarity is satisfied and the number of crossings is as few as possible. We call such an outer-1-planar drawing an *outer-1-plane graph*. We follow the notation in [11].

Let G be 2-connected outer-1-plane graph. Denote by $v_1, v_2, \ldots, v_{|G|}$ the vertices of G that lie clockwise on the circle. Let $\mathcal{V}[v_i, v_j] := \{v_i, v_{i+1}, \ldots, v_j\}$ and let $\mathcal{V}(v_i, v_j) := \mathcal{V}[v_i, v_j] \setminus \{v_i, v_j\}$, where the subscripts are taken modulo $|G|$. Set $\mathcal{V}[v_i, v_i] := V(G)$. A vertex set $\mathcal{V}[v_i, v_j]$ with $i \neq j$ is *non-edge* if $j - i = 1 \pmod{|G|}$ and $v_iv_j \notin E(G)$, is *path* if $v_kv_{k+1} \in E(G)$ for all $i \leq k < j$, and is *subpath* if $j - i \neq 1 \pmod{|G|}$ and some edge in the form v_kv_{k+1} with $i \leq k < j$ is missing. An edge v_iv_j in G is *chord* if $j - i \neq 1 \pmod{|G|}$. By $\mathcal{C}[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$.

Lemma 2.1 ([11]). *Let v_i and v_j be vertices of a 2-connected outer-1-plane graph G. If there are no crossed chords in $\mathcal{C}[v_i, v_j]$ and no edges between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then $\mathcal{V}[v_i, v_j]$ is either non-edge or path.*

Subdividing an edge xy of a graph G means replacing xy with a path xzy with $d(z) = 2$. By K_4^+, K_4^{2a+}, K_4^{2b+} and K_4^{3+}, we denote the graph K_4 with one edge subdivided, two adjacent edges subdivided, two nonadjacent edges subdivided and a path of length 3 subdivided, respectively.

Theorem 2.2. *Every 2-connected outer-1-planar graph G with maximum degree at most 3 contains one of the configurations among $G_1, G_2, \ldots, G_8$ and G_9 as a subgraph, see Fig. 1, unless G is isomorphic to any of the graphs among K_4, K_4^+, K_4^{2a+}, K_4^{2b+} and K_4^{3+}. Moreover,*

(a) *if G contains G_2 with $x \neq y$ as a subgraph, then the graph derived from G by deleting u and identifying v with w is outer-1-planar;*

(b) *if G contains G_4 with $x \neq y$ as a subgraph, then the graph derived from G by deleting u_0, v_0, w and identifying u_1 with v_1 is outer-1-planar;*

(c) *if G contains G_8 with $x \neq y$ as a subgraph, then the graph derived from G by deleting u_0, u_1, v_0 and identifying u_2 with v_1 is outer-1-planar.*

Further, in the cyclic ordering of the vertices of G, the vertices of the configuration occur consecutively in the same order as drawn in the figure (up to symmetry in G_6 and G_8).

Proof. We prove this result by contradiction. If there are no crossings in G, then G is outer-planar and the result holds (cf. [8]). Therefore, there is at least one crossing in G.

Let v_iv_j and v_kv_l be two mutually crossed chords in G with $1 \leq i < k < j < l$. Without loss of generality, assume that there is no other pair of mutually crossed chords in $\mathcal{C}[v_i, v_l]$. By the outer-1-planarity of G and Lemma 2.1, each of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ is either non-edge or path.

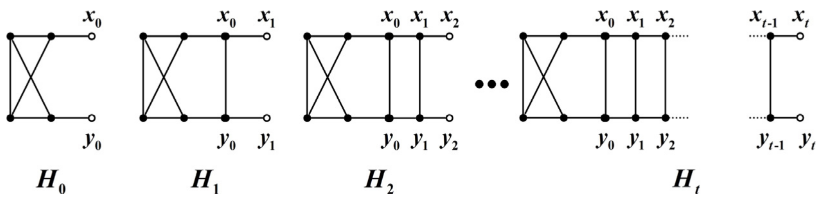

Fig. 2. The definition of H_t with integer $t \geq 0$.

If $k - i \geq 3$, then $\mathcal{V}[v_i, v_k]$ is path. If there are no chords in $\mathcal{C}[v_i, v_k]$, then the vertices $v_{i+1}, \ldots, v_{k-1}$ are all of degree two and G_1 appears. Therefore, there is a chord $v_r v_s$ with $i \leq r < s \leq k$. If $s - r \geq 3$, then the vertices $v_{r+1}, \ldots, v_{s-1}$ are all divalent vertices again and G_1 appears. Thus, $s - r = 2$, which implies that $d(v_{r+1}) = 2$. If $d(v_r) = 2$ or $d(v_s) = 2$, then G_1 appears. If $d(v_r) = 3$ and $d(v_s) = 3$, then G_2 appears and (a) holds.

We now assume that $k - i \leq 2$, and similarly, assume that $j - k \leq 2$ and $l - j \leq 2$. If two of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are non-edges, then we either find a 1-valent vertex in G or obtain another drawing of G with the number of crossings reduced by one. Hence, at least two of $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are paths.

Case 1. $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ are paths and $\mathcal{V}[v_j, v_l]$ is non-edge (the case when $\mathcal{V}[v_i, v_k]$ is non-edge and $\mathcal{V}[v_k, v_j]$, $\mathcal{V}[v_j, v_l]$ are paths is similar).

If $j - k = k - i = 1$, then $d(v_j) = 2$ and $d(v_k) = 3$, which imply either G_1 or G_2 and that (a) holds.

If $j - k = 1$ and $k - i = 2$, then $d(v_{i+1}) = d(v_j) = 2$, which implies the appearance of G_3. Further, we can reversing the ordering of v_j and v_k so that in this new cyclic ordering of the vertices of G, the vertices of the configuration G_3 occur consecutively in the same order as drawn in the figure.

If $j - k = 2$, then $d(v_{j-1}) = d(v_j) = 2$ and G_1 appears.

Case 2. $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_j, v_l]$ are paths and $\mathcal{V}[v_k, v_j]$ is non-edge.

If $k - i = l - j = 1$, then G_3 occurs.

If $k - i = 2$ (the case when $l - j = 2$ is similar), then $d(v_{k-1}) = 2$, which implies either G_1 or G_2, and that (a) holds once G_2 appears.

Case 3. $\mathcal{V}[v_i, v_k]$, $\mathcal{V}[v_k, v_j]$ and $\mathcal{V}[v_j, v_l]$ are all paths.

If $j - k = 2$ and $k - i = l - j = 1$, then G_4 occurs and (b) holds unless G is isomorphic to K_4^+.

If $k - i = 2$ and $j - k = l - j = 1$, or $l - j = 2$ and $k - i = j - k = 1$, then G_8 appears and (c) holds unless G is isomorphic to K_4^+.

If $k - i = j - k = 2$ and $l - j = 1$, or $j - k = l - j = 2$ and $k - i = 1$, then G_6 appears unless G is isomorphic to the K_4^{2a+}. If $k - i = l - j = 2$ and $j - k = 1$, then G_5 appears unless G is isomorphic to K_4^{2b+}.

If $k - i = j - k = l - j = 2$, then G_7 appears unless G is isomorphic to K_4^{3+}.

If $k - i = j - k = l - j = 1$, then $d(v_k) = d(v_j) = 3$. If $d(v_l) = 2$, then v_i is a cut vertex unless G is isomorphic to the graph K_4 with one edge removed, which contains G_2 as a subgraph and satisfies (a). Hence, $d(v_l) = 3$, and $d(v_i) = 3$ by symmetry. This implies the appearance of G_9 unless G is isomorphic to K_4.

In each of the above three major cases, one can easily check that the last conclusion in the theorem holds. □

Next, we prove a slightly stronger structural result than Theorem 2.2. Before stating it, we define a series of configurations as follows.

First, set $H_0 := G_9$. Next, construct H_t with integer $t \geq 1$ by adding two vertices x_t and y_t and three edges $x_{t-1}y_{t-1}$, $x_{t-1}x_t$ and $y_{t-1}y_t$ to H_{t-1}. Note that x_{t-1} and y_{t-1} are 3-valent vertices and x_t, y_t have uncertain degrees in H_t, see Fig. 2.

Let G be a 2-connected outer-1-plane graph with maximum degree 3. Any subgraph of G that is isomorphic to (in terms of drawing) any of the structures in Fig. 3(I) or Fig. 3(II) is an *A-cluster* or *B-cluster* in G, respectively. The *length* of a cluster is $R - L \pmod{|G|}$, where R and L are the subscripts of the far right vertex and the far left vertex on the circle (see in a clockwise direction from left to right). Note that A-clusters and B-clusters are defined based on the structure H_i in Fig. 2.

Theorem 2.3. *Let G be a 2-connected outer-1-planar graph with maximum degree 3. If G is not isomorphic to any of the graphs among K_4, K_4^+, K_4^{2a+}, K_4^{2b+} and K_4^{3+}, and it does not contain any of the configurations among $G_1, \ldots, G_8$ as a subgraph, then G contains H_t with $x_t y_t \notin E(G)$ for some integer $t \geq 0$ as a subgraph and the graph derived from G by deleting all solid vertices of H_t and add an edge $x_t y_t$ is outer-1-planar.*

Proof. Let $v_i v_j$ and $v_k v_l$ be two mutually crossed chords in G with $1 \leq i < k < j < l$. Without loss of generality, assume that $i = 1$ and that there is no other pair of mutually crossed chords in $\mathcal{C}[v_i, v_l]$. By the proof of Theorem 2.2, we have $k = 2, j = 3, l = 4$ and $v_1v_2, v_2v_3, v_3v_4 \in E(G)$. Note that $d(v_1) = d(v_4) = 3$, because otherwise we have a copy of G_2, a contradiction. Let v_r be the third neighbor of v_4. We distinguish two major cases.

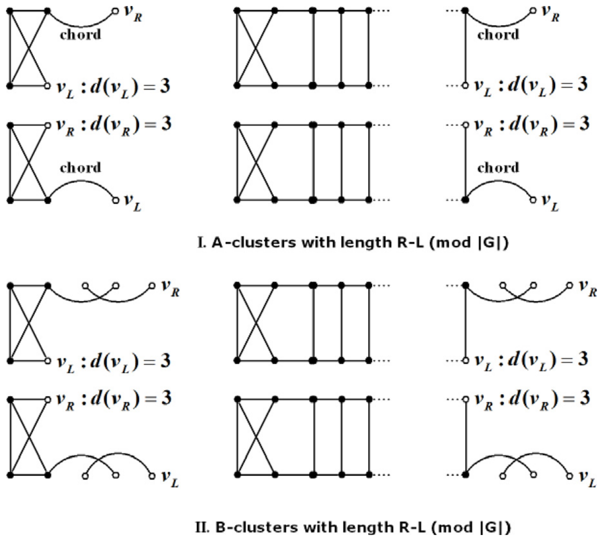

Fig. 3. Definitions of A-clusters and B-clusters.

Case 1. v_4v_r is chord, i.e., $r \geq 6$.

If v_4v_r is non-crossed, then it is easy to see that v_r disconnects the set $S = \{v_5, \ldots, v_{r-1}\} \neq \emptyset$ and $V(G) \setminus (S \cup \{v_r\})$. This implies that v_r is a cut-vertex, a contradiction. Hence, v_4v_r is crossed by another chord v_xv_y with $x < r < y$. Note that the graph induced by the edges $v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4$ and v_4v_r is an A-cluster and the graph induced by the edges $v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_4v_r$ and v_xv_y is a B-cluster.

Without loss of generality, assume that

(1) *there are no A-clusters with length less than* $r - 1$ *in the graph induced by* $\mathcal{V}[v_1, v_r]$,

(2) *there are no B-clusters with length less than* $y - 1$ *in the graph induced by* $\mathcal{V}[v_1, v_y]$.

Subcase 1.1. There is a pair of crossed chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $4 \leq i' < k' < j' < l' \leq x$.

By the proof of Theorem 2.2, we have $k' - i' = j' - k' = l' - j' = 1$ and $d(v_{i'}) = d(v_{l'}) = 3$. Note that $i' \neq 4$. If $l' = x$, then v_x disconnects the set $S = \{v_5, \ldots, v_{x-1}\} \neq \emptyset$ and $V(G) \setminus (S \cup \{v_x\})$. This implies that v_x is a cut-vertex. This contradiction guarantees that $l' < x$.

Subcase 1.1.1. There is a chord $v_{l'}v_{r'}$ with $r' \neq i', k'$.

By (1), $4 < r' < i'$. If $v_{l'}v_{r'}$ is non-crossed, then $v_{r'}$ is a cut-vertex, a contradiction. Hence $v_{l'}v_{r'}$ is crossed.

Since $d(v_{i'}) = 3$, there is an edge $v_{i'}v_{s'}$. If $s' > l'$, then redraw the graph by reversing the order of $v_{i'}, v_{k'}, v_{j'}$ and $v_{l'}$ on the boundary of the circle. After doing so, we avoid the crossing that generates by $v_{i'}v_{s'}$ crossing $v_{l'}v_{r'}$, which contradicts the basic assumption that the drawing of G minimizes the number of possible crossings. If $s' < i' - 1$, then $v_{i'}v_{s'}$ is a chord and an A-cluster with length less than $r - 1$ appears, a contradiction to (1). Hence $s' = i' - 1$.

Since $v_{l'}v_{r'}$ is crossed, $s' \neq r'$. Consider the subgraph induced by $\mathcal{V}[v_{i'}, v_{l'}]$. If $v_{r'}v_{s'} \notin E(G)$, then we have a copy of H_0 satisfying the conclusion of the theorem. If $v_{r'}v_{s'} \in E(G)$, then $v_{l'}v_{r'}$ is crossed by a chord $v_{s'}v_{t'}$ with $4 < t' \leq x$, and moreover, either $s' = r' + 1$ or $v_{r'}v_{s'}$ is non-crossed chord. Note that the graph induced by $v_{i'}v_{j'}, v_{i'}v_{k'}, v_{j'}v_{l'}, v_{k'}v_{j'}, v_{k'}v_{l'}$ and $v_{l'}v_{r'}$ is an A-cluster. Without loss of generality, assume that

(3) *there are no A-clusters contained in the graph induced by* $\mathcal{V}[v_{r'}, v_{s'}]$.

Suppose that $v_{r'}v_{s'}$ is chord. If there are no crossed chords in $\mathcal{C}[v_{r'}, v_{s'}]$, then by Lemma 2.1, $\mathcal{V}[v_{r'}, v_{s'}]$ is path, which implies the appearance of G_1 or G_2, a contradiction. If there is a pair of crossed chords $v_{i''}v_{j''}$ and $v_{k''}v_{l''}$ with $r' \leq i'' < k'' < j'' < l'' \leq s'$, then $v_{i''}v_{k''}, v_{k''}v_{j''}, v_{j''}v_{l''} \in E(G)$, and furthermore, we have $i'' \neq r', l'' \neq s'$ and $v_{i''-1}v_{i''}, v_{l''}v_{l''+1} \in E(G)$ by (3). Now we see a copy of an H_0.

If $v_{i''-1}v_{l''+1} \notin E(G)$, then adding an edge $v_{i''-1}v_{l''+1}$ to G do not disturb its outer-1-planarity, hence the conclusion of the theorem holds.

If $v_{i''-1}v_{l''+1} \in E(G)$, then by (3), we have $v_{l''+1}v_{l''+2}, v_{i''-2}v_{i''-1} \in E(G)$ and thus a copy of H_1. We then discuss accordingly whether $v_{i''-2}v_{l''+2}$ is an edge of G or not, and the remaining arguments are similar and iterative (i.e., if $v_{i''-2}v_{l''+2} \notin E(G)$ then we find a copy of H_1 satisfying the conclusion of the theorem, and otherwise we find a copy of H_2 that is similarly considered at this stage). Since the chord $v_{r'}v_{s'}$ is non-crossed and $\Delta(G) = 3$, we would finally find a copy of H_t for some integer $t \geq 1$ so that the conclusion of the theorem satisfies. Hence, $v_{r'}v_{s'}$ is not chord, which implies $r' = s' - 1$.

Recall that $v_{s'}v_{t'}$ crosses $v_{l'}v_{r'}$. If $4 < t' < r'$, then $v_{t'}$ is a cut-vertex, a contradiction. If $t' > l'$, then redraw the graph by reversing the order of $v_{s'}, v_{i'}, v_{k'}, v_{j'}$ and $v_{l'}$ on the boundary of the circle. The new drawing avoids the crossing generated by $v_{l'}v_{r'}$ crossing $v_{s'}v_{t'}$, contradicting the fact the drawing of G minimizes the number of crossings.

Subcase 1.1.2. There are no chords $v_{l'}v_{r'}$ with $r' \neq i', k'$.

Since $d(v_{l'}) = 3$ and $v_{i'}v_{l'} \notin E(G)$ (otherwise G has an isolated K_4 as a subgraph), we have, by the assumption of this subcase, that $v_{l'}v_{l'+1} \in E(G)$. By symmetry, we also assume that $v_{i'-1}v_{i'} \in E(G)$. We now find a copy of an H_0. If $v_{i'-1}v_{l'+1} \notin E(G)$, then adding an edge $v_{i'-1}v_{l'+1}$ to G do not disturb its outer-1-planarity, hence the conclusion of the theorem holds. If $v_{i'-1}v_{l'+1} \in E(G)$, then by similar arguments as in subcase 1.1.1, we have $v_{l'+1}v_{l'+2}, v_{i'-2}v_{i'-1} \in E(G)$ and thus a copy of H_1. We then discuss depending upon whether $v_{i'-2}v_{l'+2}$ is an edge of G or not. The remaining arguments are again similar and iterative. Since there are finite vertices in $\mathcal{V}[v_4, v_x]$ and v_xv_y is crossed, we would finally find a copy of H_t for some integer $t \geq 1$ so that the conclusion of the theorem satisfies.

Subcase 1.2. There are no crossed chords in $\mathcal{C}[v_4, v_x]$.

By Lemma 2.1, $\mathcal{V}[v_4, v_x]$ is either non-edge or path. Since v_4 has no neighbors in $\mathcal{V}[v_4, v_x]$, $\mathcal{V}[v_4, v_x]$ can only be non-edge and thus $x = 5$. By similar arguments as the one in Subcase 1.1, one can also conclude that there are no crossed chords in $\mathcal{C}[v_5, v_r]$ unless the graph induced by $[v_5, v_r]$ and the edges v_4v_6, v_5v_y are just a copy of H_t for some integer $t \geq 0$. If this H_t exists, then it satisfies the conclusion of the theorem, since $v_4v_y \notin E(G)$. Therefore, $\mathcal{V}[v_5, v_r]$ is either non-edge or path by Lemma 2.1.

If $\mathcal{V}[v_5, v_r]$ is non-edge, then v_5 is a 1-valent vertex, a contradiction. If $\mathcal{V}[v_5, v_r]$ is path and there is a chord in $\mathcal{C}[v_5, v_r]$, then either G_1 or G_2 appears, a contradiction. If $\mathcal{V}[v_5, v_r]$ is a path and there are no chords in $\mathcal{C}[v_5, v_r]$, then $r = 6$, since otherwise we have $d(v_5) = d(v_6) = 2$ and that G_1 appears. Note now that $v_5v_6 \in E(G)$ and $d(v_5) = 2$.

If there are no crossed chords in $\mathcal{C}[v_6, v_y]$, then by Lemma 2.1, $\mathcal{V}[v_6, v_y]$ is either non-edge or path. If it is non-edge, then $d(v_6) = 2$ and G_1 appears, a contradiction. If $\mathcal{V}[v_6, v_y]$ is path, then G_1 or G_2 appears while $y \geq 9$, G_3 appears while $y = 8$, and G_2 appears while $y = 7$. All are contradictions.

Hence, there is a pair of crossed chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $6 \leq i' < k' < j' < l' \leq y$. Again, we shall have $k' - i' = j' - k' = l' - j' = 1$ and $d(v_{i'}) = d(v_{l'}) = 3$. Since $\Delta(G) = 3$, $i' \neq 6$. By the 2-connectivity of G, $l' \neq y$, otherwise v_6 is a cut-vertex.

Subcase 1.2.1. There is a chord $v_{l'}v_{r'}$ with $r' \neq k', i'$.

An important fact is that $v_{l'}v_{r'}$ is crossed because otherwise $v_{r'}$ would be a cut-vertex.

If $r' > l'$, then by (2), $v_{l'}v_{r'}$ can only be crossed by a chord $v_{x'}v_{y'}$ with $l' < x' < t'$ and $6 \leq y' \leq i'$. If $y' = i'$, then $v_{r'}$ is a cut-vertex, a contradiction, thus $6 \leq y' < i'$. Since $d(v_{i'}) = 3$, there is an edge $v_{i'}v_{s'}$ with $s' < i'$. If $v_{i'}v_{s'}$ is chord, then it is crossed by a chord v_av_b with $y' \leq b < s' < a < i'$, which implies a B-cluster with length less than $y - 1$ in $G[\mathcal{V}[v_1, v_y]]$, a contradiction to (2). Hence $s' = i' - 1$.

If $s' \neq y'$, or $s' = y'$ and $v_{y'}v_{r'} \notin E(G)$, then $v_{s'}v_{r'} \notin E(G)$. In this case a copy of H_1 appears, and moreover, the graph derived from G by adding a new edge $v_{s'}v_{r'}$ and removing the edge $v_{l'}v_{r'}$ is outer-1-planar, thus the conclusion of the theorem holds. If $s' = y'$ and $v_{y'}v_{r'} \in E(G)$, then $v_{r'}$ becomes a cut-vertex, a contradiction. Hence $6 \leq r' < i'$.

Since $d(v_{i'}) = 3$, there is an edge $v_{i'}v_{s'}$ with $s' \neq k', j'$. If $v_{i'}v_{s'}$ is chord, then by similar arguments as above, we shall have $s' > l'$. At this stage we obtain a drawing of G with less crossings by reversing the order of $v_{i'}$, $v_{k'}$, $v_{j'}$ and $v_{l'}$ on the boundary of the circle, a contradiction. Hence $s' = i' - 1$. This follows that $r' \neq s'$, otherwise $v_{r'}$ is a cut-vertex. If $v_{s'}v_{r'} \notin E(G)$, then we have a copy of H_0 and the graph obtained from G by adding an edge $v_{s'}v_{r'}$ and removing the edge $v_{r'}v_{l'}$ is outer-1-planar, so the conclusion of the theorem is satisfied. If $v_{s'}v_{r'} \in E(G)$, then $v_{l'}v_{r'}$ is crossed by an edge that is incident with $v_{s'}$, say $v_{s'}v_{t'}$.

If $t' < r'$, then $v_{t'}$ is a cut-vertex, a contradiction. Hence $t' > l'$. Assume, without loss of generality, that there are no A-clusters contained in the graph induced by $\mathcal{V}[v_{r'}, v_{s'}]$. This assumption implies that there are no crossed chords in $\mathcal{C}[v_{r'}, v_{s'}]$. Hence by Lemma 2.1, $\mathcal{V}[v_{r'}, v_{s'}]$ is path, which implies that $r' = s' - 1 = i' - 2$. Redraw the graph by reversing the order of $v_{s'}$, $v_{i'}$, $v_{k'}$, $v_{j'}$ and $v_{l'}$ on the boundary of the circle and we then get a drawing of G with less crossings, a contradiction.

Subcase 1.2.2. There are no chords $v_{l'}v_{r'}$ with $r' \neq k', i'$.

Note that $v_{i'}v_{l'} \notin E(G)$, because otherwise we have an isolated K_4, a contradiction. By the assumption of this subcase and the fact that $d(v_{i'}) = d(v_{l'}) = 3$, we immediately have $v_{l'}v_{l'+1} \in E(G)$, and by symmetry, we also have $v_{i'-1}v_{i'} \in E(G)$. At this stage, we find a copy of H_0. If $v_{i'-1}v_{l'+1} \notin E(G)$, then adding an edge $v_{i'-1}v_{l'+1}$ to G do not disturb its outer-1-planarity, hence the conclusion of the theorem holds. If $v_{i'-1}v_{l'+1} \in E(G)$, then by similar arguments as above, we have $v_{l'+1}v_{l'+2}, v_{i'-2}v_{i'-1} \in E(G)$ and thus a copy of H_1. We then condition based on whether $v_{i'-2}v_{l'+2}$ is an edge or not. The remaining arguments are similar and iterative. Since there are finite vertices in $\mathcal{V}[v_6, v_y]$ and v_5 has no neighbors in $\mathcal{V}(v_6, v_y)$, we would finally find a copy of H_t for some integer $t \geq 1$ so that the conclusion of the theorem satisfies unless the graph induced by $\mathcal{V}[v_7, v_y]$ and the edges v_6v_7, v_5v_y are a copy of H_t for some integer $t \geq 1$. However, this excluded case would not occur, since otherwise we obtain a drawing of G with less crossings by changing the order of $v_4, v_5, v_6, \ldots, v_y$ to $v_4, v_6, \ldots, v_y, v_5$ on the boundary of the circle.

Case 2. v_4v_r is not chord, i.e., $r = 5$.

Assume, by symmetry, that $v_{|G|}v_1 \in E(G)$. We now have a copy of H_0. If $v_{|G|}v_5 \notin E(G)$, then adding an edge $v_{|G|}v_5$ to G does not disturb its outer-1-planarity, hence the conclusion of the theorem holds. If $v_{|G|}v_5 \in E(G)$, then by similar arguments as the one in Case 1, we have $v_5v_6, v_{|G|-1}v_{|G|} \in E(G)$ and a copy of H_1.

By the absences of the configurations G_1 and G_2, G is not isomorphic to the graph K_4 with one edge removed or the graph H_t with the vertices x_t and y_t removed for some integer $t \geq 1$. Therefore, we would finally find a copy of H_t satisfying the conclusion of the theorem for some integer $t \geq 1$ by iterative arguments. □

3. The edge chromatic number of outer-1-planar graphs

In this section, we investigate the edge coloring of outer-1-planar graphs with maximum degree 3. It is easy to see that the smallest (in terms of the order) outer-1-planar graph with $\Delta(G) = 3$ and $\chi'(G) = 4$ is the graph K_4 with one edge subdivided, say K_4^+.

Definition 3.1. A graph G belongs to the class $\mathcal{P}$, if it is isomorphic to K_4^+ or derives from K_4^+ by a sequence of the following operations:

- Remove a vertex z of degree two and paste a copy of G_2, or G_4, or G_8 on the current graph by identifying x and y with z_1 and z_2, respectively, where z_1 and z_2 are the neighbors of z;
- Remove an edge z_1z_2 and paste a copy of H_t for some integer $t \geq 0$ on the current graph by identifying x_t and y_t with z_1 and z_2, respectively.

One can easily check that any graph $G \in \mathcal{P}$ has maximum degree 3 and minimum degree 2.

Theorem 3.2. *If $G \in \mathcal{P}$, then $\chi'(G) = 4$.*

Proof. Let F be a graph in $\mathcal{P}$. If there is a vertex z of degree two with neighbors z_1 and z_2 in F, then remove it and paste a copy of G_2 (or G_4, or G_8, respectively) on $F - z$ by identifying x and y with z_1 and z_2, respectively. Denote the current graph by F_2 (or F_4, or F_8, respectively).

If F_2 (or F_4, or F_8, respectively) admits an edge 3-coloring c, then one can see that $c(vx) \neq c(wy)$ (or $c(u_1x) \neq c(v_1y)$, or $c(u_2x) \neq c(v_1y)$, respectively). Hence we can construct an edge 3-coloring of F by restricting c to $F - z$ and coloring the remaining two edges zz_1 and zz_2 with $c(vx)$ and $c(wy)$ (or $c(u_1x)$ and $c(v_1y)$, or $c(u_2x)$ and (v_1y), respectively). Therefore, if $\chi'(F) = 4$ then $\chi'(F_2) = 4$ (or $\chi'(F_4) = 4$, or $\chi'(F_8) = 4$, respectively).

Let R be the graph derived from F by applying the second operation in Definition 3.1 once. If R has an edge 3-coloring c, then one can check that $c(x_{t-1}x_t) = c(y_{t-1}y_t)$. Hence we can construct an edge 3-coloring of F by restricting c to $F - z_1z_2$ and coloring z_1z_2 with $c(x_{t-1}x_t)$. Therefore, if $\chi'(F) = 4$ then $\chi'(R) = 4$.

We conclude that any graph derived from a graph F in $\mathcal{P}$ with edge chromatic number four by a sequence of the operations in Definition 3.1 has edge chromatic number four. Since $\chi'(K_4^+) = 4$ and $K_4^+ \in \mathcal{P}$, $\chi'(G) = 4$ for any $G \in \mathcal{P}$. □

Theorem 3.3. *Let $\mathcal{O}_3$ be the class of outer-1-planar graphs with maximum degree 3. If $G \in \mathcal{O}_3 \setminus \mathcal{P}$, then $\chi'(G) = 3$.*

Proof. Let G be a minimal counterexample to this statement. One can see that G is 2-connected.

If G is isomorphic to any of the graphs among K_4, K_4^{2a+}, K_4^{2b+} and K_4^{3+}, then it is easy to check that $\chi'(G) = 3$.

If G contains G_1 as a subgraph, then $G - uv$ is an outer-1-planar graph with maximum degree 3 and minimum degree 1, thus $G - uv \in \mathcal{O}_3 \setminus \mathcal{P}$. By the minimality of G, $G - uv$ has an edge 3-coloring c, which can be extended to G by coloring uv with a color different form $c(ux)$ and $c(uy)$.

If G contains G_2 as a subgraph, then $x \neq y$, otherwise G is isomorphic to the graph K_4 with one edge removed by its 2-connectivity and we are done. Delete u, vw and identify v with w as a common vertex z. Denote the resulting graph by M_2. If $\Delta(M_2) \leq 2$, then $\chi'(M_2) \leq 3$ by Vizing's theorem. If $\Delta(M_2) = 3$, then by Theorem 2.2(a) and Definition 3.1, $M_2 \in \mathcal{O}_3 \setminus \mathcal{P}$, which implies that $\chi'(M_2) = 3$ by the minimality of G. Let c be an edge 3-coloring of M_2. Assume, without loss of generality, that $c(zx) = 1$ and $c(zy) = 2$. We construct an edge 3-coloring of G by restricting c to $G - \{u, v, w\}$ and coloring vx, uw with 1, uv, wy with 2 and vw with 3.

If G contains G_3 as a subgraph, then assume that $d(x) = d(y) = 3$, otherwise we come back to the case while G contains G_1 as a subgraph. Since either $G - \{u, v\} \in \mathcal{O}_3 \setminus \mathcal{P}$ or $G - \{u, v\}$ has maximum degree at most 2, $G - \{u, v\}$ has an edge 3-coloring c. Let x_1 and y_1 be the third neighbor of x and y, respectively. If xx_1 and yy_1 receive same color under c, say 1, then extend c to an edge 3-coloring of G by coloring ux, vy with 2 and vx, uy with 3. If xx_1 and yy_1 receive different colors under c, say 1 and 2, respectively, then extend c to an edge 3-coloring of G by coloring vy with 1, ux with 2 and vx, uy with 3.

If G contains G_4 as a subgraph, then $x \neq y$, otherwise G is isomorphic to K_4^{2b+} by its 2-connectivity and we are done. Delete u_0, v_0, w and identify u_1 with v_1 as a common vertex z. Denote the resulting graph by M_4. If $\Delta(M_4) \leq 2$, then $\chi'(M_4) \leq 3$. If $\Delta(M_4) = 3$, then by Theorem 2.2(b) and Definition 3.1, $M_4 \in \mathcal{O}_3 \setminus \mathcal{P}$, which implies that $\chi'(M_4) = 3$ by the minimality of G. Let c be an edge 3-coloring of M_4. Assume, without loss of generality, that $c(zx) = 1$ and $c(zy) = 2$. We construct an edge 3-coloring of G by restricting c to $G - \{u_0, u_1, v_0, v_1, w\}$ and coloring u_1x, u_0w, v_0v_1 with 1, u_0u_1, wv_0, v_1y with 2 and u_0v_1, u_1v_0 with 3.

If G contains G_5 as a subgraph, then for similar reason as previous lines, $G - \{u_0, u_1, v_0, v_1\}$ has an edge 3-coloring c. If u_2x and v_2y receive same color under c, say 1, then extend c to an edge 3-coloring of G by coloring u_0u_1, v_0v_1 with 1, u_1u_2, v_1v_2 with 2 and u_0v_2, v_0u_2 with 3. If u_2x and v_2y receive different colors under c, say 1 and 2, respectively, then extend c to an edge 3-coloring of G by coloring u_0v_2, v_0v_1 with 1, u_0v_0 with 2 and u_0u_1, v_1v_2, u_2v_0 with 3.

If G contains G_6 as a subgraph, then for similar reason as before, $G - \{u_0, u_1, v_0, w\}$ has an edge 3-coloring c. If u_2x and v_1y receive same color under c, say 1, then extend c to an edge 3-coloring of G by coloring u_0u_1, wv_0 with 1, u_1u_2, u_0w, v_0v_1

with 2 and u_0v_1, u_2v_0 with 3. If u_2x and v_1y receive different colors under c, say 1 and 2, respectively, then extend c to an edge 3-coloring of G by coloring u_0u_1, v_0v_1 with 1, u_0w, u_2v_0 with 2 and wv_0, u_1u_2, u_0v_1 with 3.

If G contains G_7 as a subgraph, then by a similar argument as before, $G - \{u_0, u_1, v_0, v_1, w\}$ has an edge 3-coloring c. If u_2x and v_2y receive same color under c, say 1, then extend c to an edge 3-coloring of G by coloring u_0u_1, v_0v_1 with 1, u_0w, u_2v_0, v_1v_2 with 2 and wv_0, u_0v_2, u_1u_2 with 3. If u_2x and v_2y receive different colors under c, say 1 and 2, respectively, then extend c to an edge 3-coloring of G by coloring wv_0, u_0v_2 with 1, u_0w, v_0v_1, u_1u_2 with 2 and u_0u_1, v_1v_2, u_2v_0 with 3.

If G contains G_8 as a subgraph, then $x \neq y$, otherwise G is isomorphic to K_4^{2a+} by its 2-connectivity and we are done. Delete u_0, u_1, v_0 and identify u_2 with v_1 as a common vertex z. Denote the resulting graph by M_8. If $\Delta(M_8) \leq 2$, then $\chi'(M_8) \leq 3$. If $\Delta(M_8) = 3$, then by Theorem 2.2(c) and Definition 3.1, $M_8 \in \mathcal{O}_3 \setminus \mathcal{P}$, which implies that $\chi'(M_8) = 3$ by the minimality of G. Let c be an edge 3-coloring of M_8. Assume, without loss of generality, that $c(zx) = 1$ and $c(zy) = 2$. We construct an edge 3-coloring of G by restricting c to $G - \{u_0, u_1, u_2, v_0, v_1\}$ and coloring u_2x, u_0u_1, v_0v_1 with 1, u_1u_2, u_0v_0, v_1y with 2 and u_0v_1, u_2v_0 with 3.

Therefore, by Theorem 2.3, G contains H_t with $x_ty_t \notin E(G)$ for some integer $t \geq 0$ as a subgraph.

Since $G \notin \mathcal{P}$, $x_t \neq y_t$. Delete all solid vertices of H_t and add an edge x_ty_t. Denote the resulting graph by M_t. If $\Delta(M_t) \leq 2$, then $\chi'(M_t) \leq 3$ by Vizing's theorem. If $\Delta(M_t) = 3$, then by Theorem 2.3 and Definition 3.1, $M_i \in \mathcal{O}_3 \setminus \mathcal{P}$, which implies that $\chi'(M_t) = 3$ by the minimality of G. Since the configuration H_t is edge 3-colorable (seen as a partial coloring of G) if and only if $x_{t-1}x_t$ and $y_{t-1}y_t$ receive same color, any edge 3-coloring c of M_t can be extended to an edge 3-coloring of G by restricting c to $M_t - x_ty_t$, coloring $x_{t-1}x_t$, $y_{t-1}y_t$ with $c(x_ty_t)$ and assigning proper colors to the remaining uncolored edges of G, or H_t. □

4. Conclusions and open problems

When Theorems 3.2 and 3.3 are combined with the result of Zhang, Liu and Wu [11] that every outer-1-planar graph with maximum degree $\Delta \geq 4$ has edge chromatic number Δ, we have the following theorem, which completely determines the edge chromatic number of outer 1-planar graphs.

Theorem 4.1. *If G is an outer-1-planar graph, then*

$$\chi'(G) = \begin{cases} \Delta(G), & \text{if } G \notin \mathcal{P} \text{ and } G \text{ is not an odd cycle;} \\ \Delta(G) + 1, & \text{otherwise.} \end{cases}$$

Since every graph $G \in \mathcal{P}$ has minimum degree 2, the following is immediate.

Corollary 4.2. *If G is a cubic outer-1-planar graph, then $\chi'(G) = \Delta(G) = 3$.*

As we know, any graph in $\mathcal{P}$ has maximum degree 3, minimum degree 2 and edge chromatic number 4. But Definition 3.1 cannot guarantee that any graph in $\mathcal{P}$ is outer-1-planar. Hence a natural open problem is

Problem 1. Can we construct a class $\mathcal{Q}$ of outer-1-planar graphs so that Theorem 4.1 holds if the class $\mathcal{P}$ is replaced by $\mathcal{Q}$?

Actually, this problem is equivalent to the following

Problem 2. What is $\mathcal{P} \setminus \mathcal{O}_3$?

Although $\mathcal{P} \setminus \mathcal{O}_3$ may be not empty, a result of Auer et al. [1] implies that whether a graph in $\mathcal{P}$ is outer-1-planar can be tested in linear-time. Naturally, we can consider the other hand.

Problem 3. Is there a linear-time algorithm that tests whether an outer-1-planar graph G is in the class $\mathcal{P}$?

If the answer is positive, then deciding whether an outer-1-planar graph with maximum degree Δ has edge chromatic number Δ is a P-problem.

Acknowledgments

The author thanks the referees for their careful reading and valuable suggestions.

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