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Equitable colorings of Cartesian products with balanced complete multipartite graphs[☆]

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ABSTRACT

A proper vertex coloring of a graph is equitable if the sizes of any two color classes differ by at most one. The equitable chromatic number of a graph G, denoted by $\chi_=(G)$, is the minimum k such that G is equitably k-colorable. Lin and Chang conjectured that for any (nontrivial) connected graphs G and H, $\chi_=(G\Box H) \le \chi(G)\chi(H)$, where $\Box$ denotes the Cartesian product. In this paper, we prove the conjecture when G or H is a balanced complete multipartite graph. More precisely, we show a stronger result that for any graph H with $\chi(H) \ge 2$, $\chi_=(K_{r(n)}\Box H) \le r\left\lceil \frac{\chi(H)-1}{r-1} \right\rceil$, where $r \ge 2$, $n \ge 1$ and $K_{r(n)}$ denotes the balanced complete r-partite graph with part size n.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Given a positive integer k and a graph G, a *k-coloring* of G is a mapping c: $V(G) \to [k] = \{1, 2, \dots, k\}$ such that $c(x) \ne c(y)$ whenever $xy \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number k for which G has a k-coloring. An *equitable k-coloring* is a k-coloring for which any two color classes differ in size by at most 1. The *equitable chromatic number* of G, denoted by $\chi_=(G)$, is the smallest number k for which G has an equitable k-coloring. It is obvious that $\chi_=(G) \ge \chi(G)$. Note that $\chi_=(G)$ and $\chi(G)$ can vary a lot. For example, $\chi(K_{1,n}) = 2 < 1 + \lceil n/2 \rceil = \chi_=(K_{1,n})$ for $n \ge 3$. One can refer to a survey by Lih [3] for the progresses on the equitable coloring of graphs since it was first introduced by Meyer [5] in 1973.

For graphs G and H, the *Cartesian product* of G and H is the graph $G\Box H$ with vertex set $V(G\Box H) = V(G) \times V(H) = \{(x, y): x \in V(G), y \in V(H)\}$, and edge set $E(G\Box H) = \{(x, u)(y, v): x = y \text{ with } uv \in E(H), \text{ or } xy \in E(G) \text{ with } u = v\}$. The following result on the usual chromatic number of the Cartesian product is due to Sabidussi [6].

Theorem 1. *For any two graphs G and H,* $\chi(G\Box H) = \max\{\chi(G), \chi(H)\}$.

The equitable colorability of Cartesian products of graphs was first investigated by Chen et al. [1] and Furmańzyk [2]. Chen et al. [1] proved the following general result.

Theorem 2. *If G and H are equitably k-colorable, then so is* $G\Box H$.

Since the empty graph E_n with n vertices is equitably k-colorable for any $k \ge 1$, the following corollary is immediate.

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Corollary 3. *If G is equitably k-colorable, then so is $E_n \square G$ for any $n \geq 1$.*

Recently, Lin and Chang [4] proved that if G and H are (nontrivial) bipartite graphs then $G \square H$ is equitably 4-colorable and hence $\chi_=(G \square H) \leq 4$. Furthermore, Yan, Lin and Wang [7] proved that $G \square H$ is equitably k-colorable for any $k \geq 4$, which settled a conjecture of Lin and Chang [4]. Instead of bounding $\chi_=(G \square H)$ by equitable colorability of its factors as in Theorem 2, Lin and Chang believed that it is possible to bound $\chi_=(G \square H)$ by usual colorability of its factors. At the end of [4], they raised the following conjecture.

Conjecture 4. $\chi_=(G \square H) \leq \chi(G)\chi(H)$ *for connected graphs G and H.*

Remark. Since K_1 is a unit for Cartesian product (that is, $K_1 \square G = G \square K_1 = G$), the conjecture may not hold when one factor is K_1. Hence we assume that neither G nor H is the trivial graph K_1.

By $K_{r(n)}$ we denote the balanced complete r-partite graph whose each partite set contains n vertices. In this paper we settle Conjecture 4 for the case when one factor, say G, is $K_{r(n)}$ with $r \geq 2$ and $n \geq 1$. Actually, we prove a better result.

Theorem 5. *Let $r \geq 2$, $n \geq 1$. For any graph H with $\chi(H) \geq 2$, $\chi_=(K_{r(n)} \square H) \leq r\left\lceil \frac{\chi(H)-1}{r-1} \right\rceil$.*

Let $V_1, \ldots, V_{\chi(H)}$ with $\chi(H) \geq 2$ be a partition of $V(H)$ into independent sets. Since adding edges between different parts V_i and V_j does not increase $\chi(H)$ or decrease $\chi_=(K_{r(n)} \square H)$, it suffices to prove Theorem 5 for the case when H is a complete multipartite graph. We may restate Theorem 5 as the following.

Theorem 6. *For $r \geq 2$, $s \geq 2$ and $n, m_1, \ldots, m_s \geq 1$, $\chi_=(K_{r(n)} \square K_{m_1,\ldots,m_s}) \leq r\left\lceil \frac{s-1}{r-1} \right\rceil$.*

2. Proof of Theorem 6

We shall prove Theorem 6 by showing that the graph $K_{r(n)} \square K_{m_1,\ldots,m_s}$ is equitably $r\left\lceil \frac{s-1}{r-1} \right\rceil$-colorable. For a complete multipartite graph $K_{m_1,\ldots,m_s}$, it is custom to assume that each m_i is positive. However, for technical reasons, we allow some m_i's to take the value of zero.

Lemma 7. *Let r and s be integers with $s \geq r \geq 2$. For any nonnegative integers $m_1, m_2, \ldots, m_s$, there exist an $(r-1)$-subset I and an r-subset J of $[s]$ with $I \subset J$ such that*

$$\sum_{i \in I} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \leq \sum_{i \in J} m_i. \quad (1)$$

Proof. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_s$. If we can show that there exists an integer $p \in [s-r+1]$ such that

$$\sum_{i=p}^{p+r-2} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \leq \sum_{i=p}^{p+r-1} m_i, \quad (2)$$

then the lemma holds by taking $I = \{p, \ldots, p+r-2\}$ and $J = \{p, \ldots, p+r-1\}$.

From the assumption that $m_1 \leq m_2 \leq \cdots \leq m_s$,

$$\frac{1}{r-1} \sum_{i=1}^{r-1} m_i \leq \frac{1}{s} \sum_{i=1}^{s} m_i \leq \frac{1}{r} \sum_{i=s-r+1}^{s} m_i. \quad (3)$$

Since $s \geq r$, we see that $\frac{r-1}{s-1} \leq \frac{r}{s}$. Note $\sum_{i=1}^{s} m_i \geq 0$. These facts along with (3) lead to

$$\sum_{i=1}^{r-1} m_i \leq \frac{r-1}{s} \sum_{i=1}^{s} m_i \leq \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \leq \frac{r}{s} \sum_{i=1}^{s} m_i \leq \sum_{i=s-r+1}^{s} m_i. \quad (4)$$

Since each m_i is an integer, from (4),

$$\sum_{i=1}^{r-1} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \leq \sum_{i=s-r+1}^{s} m_i. \quad (5)$$

We define

$$S = \left\{ j\colon 1 \leq j \leq s-r+1 \text{ and } \sum_{i=j}^{j+r-2} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \right\}.$$

By the left inequality in (5), $1 \in S$ and hence S is nonempty. Let p be the maximum integer in S. We show that p satisfies the desired relation (2).

Since $p \in S$, the definition of S implies the left inequality in (2). If $p = s-r+1$ then the right inequality in (2) follows from the right inequality in (5). Now assume $p \le s-r$. Since p is the maximum integer in S, $p+1 \notin S$. Since $1 \le p+1 \le s-r+1$ and $m_p \ge 0$, the definition of S implies

$$\left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor < \sum_{i=p+1}^{p+r-1} m_i \le \sum_{i=p}^{p+r-1} m_i,$$

as desired. □

For $X \subset V(G)$, let $\langle X \rangle$ denote the subgraph of G induced by X. For n graphs $G_1, \ldots, G_n$ with pairwise disjoint vertex sets, the *disjoint union* of $G_1, \ldots, G_n$, denoted by $G_1 \cup \cdots \cup G_n$, is the graph with vertex set $V(G_1) \cup \cdots \cup V(G_n)$ and edge set $E(G_1) \cup \cdots \cup E(G_n)$.

Lemma 8. *Let $H = K_{m_1,\ldots,m_s}$, $s \ge 2$ and $m_i \ge 0$ for each $i \in [s]$. Denote partite sets of H by $V_1, \ldots, V_s$ with $|V_i| = m_i$ for each $i \in [s]$. For any $r \ge 2$, there exists a partition $\Pi = (\pi_1, \ldots, \pi_r)$ of $[s]$ such that the disjoint union*

$$U = \left\langle \bigcup_{i\in\pi_1} V_i \right\rangle \cup \left\langle \bigcup_{i\in\pi_2} V_i \right\rangle \cup \cdots \cup \left\langle \bigcup_{i\in\pi_r} V_i \right\rangle \tag{6}$$

is equitably $\lceil \frac{s-1}{r-1} \rceil$-colorable.

Proof. As $K_{m_1,\ldots,m_s} = K_{m_1,\ldots,m_s,0} = K_{m_1,\ldots,m_s,0,0} = \cdots$, we may always assume that $s-1$ is divisible by $r-1$. Set $k = \lceil \frac{s-1}{r-1} \rceil = \frac{s-1}{r-1}$. We fix r and prove the lemma by induction on k. If $k = 1$ then $s = r$. Let

$$\Pi = (\pi_1, \ldots, \pi_r) = (\{1\}, \{2\}, \ldots, \{s\}).$$

Since all graphs $\langle \cup_{j\in\pi_i} V_j \rangle$ are empty and so is their disjoint union, the lemma holds for $k = 1$. Assume now that $k \ge 2$ and the lemma holds for $k-1$. By Lemma 7, there exist an $(r-1)$-subset I and an r-subset J of $[s]$ with $I \subset J$ such that

$$\sum_{i\in I} m_i \le \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \le \sum_{i\in J} m_i. \tag{7}$$

By rearranging $m_1, \ldots, m_s$, we may assume $I = \{s-r+2, \ldots, s\}$ and $J = I \cup \{s-r+1\}$. As $k = \frac{s-1}{r-1}$, (7) becomes

$$\sum_{i=s-r+2}^{s} m_i \le \left\lfloor \frac{1}{k} \sum_{i=1}^{s} m_i \right\rfloor \le \sum_{i=s-r+1}^{s} m_i. \tag{8}$$

Set $s' = s-r+1$ and

$$q = \left\lfloor \frac{1}{k} \sum_{i=1}^{s} m_i \right\rfloor - \sum_{i=s-r+2}^{s} m_i. \tag{9}$$

By (8), $0 \le q \le m_{s'}$. Let $V_i' = V_i$ for $1 \le i < s'$ and let $V_{s'}'$ be any subset of $V_{s'}$ with $m_{s'} - q$ vertices. Since $\lceil \frac{s'-1}{r-1} \rceil = \lceil \frac{s-1}{r-1} \rceil - 1 = k-1$, by the induction assumption, there exists a partition $\Pi' = (\pi_1', \ldots, \pi_r')$ of $[s']$ such that the disjoint union

$$U' = \left\langle \bigcup_{i\in\pi_1'} V_i' \right\rangle \cup \left\langle \bigcup_{i\in\pi_2'} V_i' \right\rangle \cup \cdots \cup \left\langle \bigcup_{i\in\pi_r'} V_i' \right\rangle$$

is equitably $(k-1)$-colorable. Without loss of generality, we may assume $s' \in \pi_1'$. Let

$$\Pi = (\pi_1, \ldots, \pi_r) = (\pi_1', \pi_2' \cup \{s'+1\}, \ldots, \pi_r' \cup \{s'+r-1\}). \tag{10}$$

It is clear that Π is a partition of $[s]$. We claim that the graph U defined by (6) is equitably k-colorable. First, use $k-1$ colors to color the subgraph U' equitably. Now, by (9), the number of uncolored vertices is exactly

$$|V_{s'} \setminus V_{s'}'| + |V_{s'+1}| + \cdots + |V_{s'+r-1}| = q + \sum_{i=s-r+2}^{s} m_i = \left\lfloor \frac{1}{k} \sum_{i=1}^{s} m_i \right\rfloor.$$

Finally, by (10), the subgraph of U induced by these uncolored vertices is a disjoint union of r empty graphs and hence is empty. Assigning a new color to these uncolored vertices, we obtain an equitable k-coloring of U. This proves the claim and hence the lemma holds. □

Proof of Theorem 6. Let $U_1, U_2, \ldots, U_r$ and $V_1, V_2, \ldots, V_s$ be the partite sets of $K_{r(n)}$ and $K_{m_1,\ldots,m_s}$, respectively. By Lemma 8, there exists a partition $\Pi = (\pi_1, \ldots, \pi_r)$ of $[s]$ such that the disjoint union

$$U = \left\langle \bigcup_{i\in\pi_1} V_i \right\rangle \cup \left\langle \bigcup_{i\in\pi_2} V_i \right\rangle \cup \cdots \cup \left\langle \bigcup_{i\in\pi_r} V_i \right\rangle$$

is equitably $\lceil \frac{s-1}{r-1} \rceil$-colorable. For each $k \in [r]$ and $i \in [r]$ we define

$$W_{k,i} = U_{i+k} \times \bigcup_{j \in \pi_i} V_j \quad \text{and} \quad W_k = \bigcup_{i=1}^{r} W_{k,i},$$

where the additions on the indices are taken modulo r. If $i \neq i'$, $(x, y) \in W_{k,i}$ and $(x', y') \in W_{k,i'}$, then $x \neq x'$ and $y \neq y'$, implying that (x, y) and (x', y') are not adjacent in $K_{r(n)} \Box K_{m_1,\ldots,m_s}$. Hence,

$$\begin{aligned} \langle W_k \rangle &= \left\langle \bigcup_{i=1}^{r} W_{k,i} \right\rangle \\ &= \bigcup_{i=1}^{r} \langle W_{k,i} \rangle \\ &= \bigcup_{i=1}^{r} \left(E_n \Box \left\langle \bigcup_{j \in \pi_i} V_j \right\rangle \right) \\ &= E_n \Box \bigcup_{i=1}^{r} \left\langle \bigcup_{j \in \pi_i} V_j \right\rangle \\ &= E_n \Box U. \end{aligned}$$

Since U is equitably $\lceil \frac{s-1}{r-1} \rceil$-colorable, Corollary 3 implies that $\langle W_k \rangle = E_n \Box U$ is also equitably $\lceil \frac{s-1}{r-1} \rceil$-colorable. Note that $(W_1, \ldots, W_r)$ is a partition of $V(K_{r(n)} \Box K_{m_1,\ldots,m_s})$ and all classes have equal sizes. By partitioning each W_k equitably into $\lceil \frac{s-1}{r-1} \rceil$ independent sets, we obtain an equitable $r\lceil \frac{s-1}{r-1} \rceil$-coloring of $V(K_{r(n)} \Box K_{m_1,\ldots,m_s})$. This proves the theorem.

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References

[1] B.-L. Chen, K.-W. Lih, J.-H. Yan, Equitable coloring of interval graphs and products of graphs, arXiv:0903.1396v1.
[2] H. Furmańzyk, Equitable colorings of graph products, Opuscula Math. 26 (1) (2006) 31–44.
[3] K.-W. Lih, Equitable coloring of graphs, in: P.M. Pardalos, D.-Z. Du, R. Graham (Eds.), Handbook of Combinatorial Optimization, second ed., Springer, 2013, pp. 1199–1248.
[4] W.-H. Lin, G.J. Chang, Equitable colorings of Cartesian products of graphs, Discrete Appl. Math. 160 (2012) 239–247.
[5] W. Meyer, Equitable coloring, Amer. Math. Monthly 80 (1973) 920–922.
[6] G. Sabidussi, Graphs with given group and given graph-theoretical properties, Canad. J. Math. 9 (1957) 512–525.
[7] Z. Yan, W.-H. Lin, W. Wang, The equitable chromatic threshold of the Cartesian product of bipartite graphs is at most 4, Taiwanese J. Math. 18 (2014) 773–780.