Note

Equitable colorings of Cartesian products with balanced complete multipartite graphs

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Given a positive integer $k$ and a graph $G$, a $k$-coloring of $G$ is a mapping $c : V(G) \rightarrow [k] = \{1, 2, \ldots, k\}$ such that $c(x) \neq c(y)$ whenever $xy \in E(G)$. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest number $k$ for which $G$ has a $k$-coloring. An equitable $k$-coloring is a $k$-coloring for which any two color classes differ in size by at most one. The equitable chromatic number of $G$, denoted by $\chi_=(G)$, is the smallest number $k$ for which $G$ has an equitable $k$-coloring. It is obvious that $\chi_=(G) \leq \chi(G)$. Note that $\chi_=(G)$ and $\chi(G)$ can vary a lot. For example, $\chi((K_{1,n}) = 2 < 1 + \lceil n/2 \rceil = \chi_((K_{1,n})$ for $n \geq 3$. One can refer to a survey by Lih [3] for the progresses on the equitable coloring of graphs since it was first introduced by Meyer [5] in 1973.

For graphs $G$ and $H$, the Cartesian product of $G$ and $H$ is the graph $G \square H$ with vertex set $V(G \square H) = V(G) \times V(H) = \{(x, y) : x \in V(G), y \in V(H)\}$, and edge set $E(G \square H) = \{(x, u)(y, v) : x = y \text{ with } uv \in E(H), \text{ or } xy \in E(G) \text{ with } u = v\}$. The following result on the usual chromatic number of the Cartesian product is due to Sabidussi [6].

**Theorem 1.** For any two graphs $G$ and $H$, $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$.

The equitable colorability of Cartesian products of graphs was first investigated by Chen et al. [1] and Furmańczyk [2]. Chen et al. [1] proved the following general result.

**Theorem 2.** If $G$ and $H$ are equitably $k$-colorable, then so is $G \square H$.

Since the empty graph $E_n$ with $n$ vertices is equitably $k$-colorable for any $k \geq 1$, the following corollary is immediate.
Corollary 3. If $G$ is equitably $k$-colorable, then so is $E_n \Box G$ for any $n \geq 1$.

Recently, Lin and Chang [4] proved that if $G$ and $H$ are (nontrivial) bipartite graphs then $G \Box H$ is equitably 4-colorable and hence $\chi_\omega(G \Box H) \leq 4$. Furthermore, Yan, Lin and Wang [7] proved that $G \Box H$ is equitably $k$-colorable for any $k \geq 4$, which settled a conjecture of Lin and Chang [4]. Instead of bounding $\chi_\omega(G \Box H)$ by equitable colorability of its factors as in Theorem 2, Lin and Chang believed that it is possible to bound $\chi_\omega(G \Box H)$ by usual colorability of its factors. At the end of [4], they raised the following conjecture.

Conjecture 4. $\chi_\omega(G \Box H) \leq \chi(G) \chi(H)$ for connected graphs $G$ and $H$.

Remark. Since $K_1$ is a unit for Cartesian product (that is, $K_1 \Box G = G \Box K_1 = G$), the conjecture may not hold when one factor is $K_1$. Hence we assume that neither $G$ nor $H$ is the trivial graph $K_1$.

By $K_{r(n)}$ we denote the balanced complete $r$-partite graph whose each partite set contains $n$ vertices. In this paper we settle Conjecture 4 for the case when one factor, say $G$, is $K_{r(n)}$ with $r \geq 2$ and $n \geq 1$. Actually, we prove a better result.

Theorem 5. Let $r \geq 2$, $n \geq 1$. For any graph $H$ with $\chi(H) \geq 2$, $\chi_\omega(K_{r(n)} \Box H) \leq r \left\lfloor \frac{\chi(H)-1}{r-1} \right\rfloor$.

Let $V_1, \ldots, V_{\chi(H)}$ with $\chi(H) \geq 2$ be a partition of $V(H)$ into independent sets. Since adding edges between different parts $V_i$ and $V_j$ does not increase $\chi(H)$ or decrease $\chi_\omega(K_{r(n)} \Box H)$, it suffices to prove Theorem 5 for the case when $H$ is a complete multipartite graph. We may restate Theorem 5 as the following.

Theorem 6. For $r \geq 2$, $s \geq 2$ and $n, m_1, \ldots, m_s \geq 1$, $\chi_\omega(K_{r(n)} \Box K_{m_1, \ldots, m_s}) \leq r \left\lfloor \frac{s-1}{r-1} \right\rfloor$.

2. Proof of Theorem 6

We shall prove Theorem 6 by showing that the graph $K_{r(n)} \Box K_{m_1, \ldots, m_s}$ is equitably $r \left\lfloor \frac{s-1}{r-1} \right\rfloor$-colorable. For a complete multipartite graph $K_{m_1, \ldots, m_s}$, it is custom to assume that each $m_i$ is positive. However, for technical reasons, we allow some $m_i$’s to take the value of zero.

Lemma 7. Let $r$ and $s$ be integers with $s \geq r \geq 2$. For any nonnegative integers $m_1, m_2, \ldots, m_s$, there exist an $(r-1)$-subset $I$ and an $r$-subset $J$ of $[s]$ with $I \subset J$ such that

$$\sum_{i \in I} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \leq \sum_{i \in J} m_i. \quad (1)$$

Proof. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_s$. If we can show that there exists an integer $p \in [s-r+1]$ such that

$$\sum_{i=p}^{p+r-2} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \leq \sum_{i=p}^{p+r-1} m_i, \quad (2)$$

then the lemma holds by taking $I = \{p, \ldots, p+r-2\}$ and $J = \{p, \ldots, p+r-1\}$.

From the assumption that $m_1 \leq m_2 \leq \cdots \leq m_s$, we have

$$\frac{1}{r-1} \sum_{i=1}^{r-1} m_i \leq \frac{1}{s} \sum_{i=1}^{s} m_i \leq \frac{1}{r} \sum_{j=s-r+1}^{s} m_i.$$ \hspace{1cm} (3)

Since $s \geq r$, we see that $\frac{r-1}{s-1} \leq \frac{r}{s}$. Note $\sum_{i=1}^{s} m_i \geq 0$. These facts along with (3) lead to

$$\sum_{i=1}^{r-1} m_i \leq \frac{r-1}{s} \sum_{i=1}^{s} m_i \leq \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \leq \frac{r}{s} \sum_{i=1}^{s} m_i \leq \sum_{i=s-r+1}^{s} m_i.$$ \hspace{1cm} (4)

Since each $m_i$ is an integer, from (4),

$$\sum_{i=1}^{r-1} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \leq \sum_{i=s-r+1}^{s} m_i.$$ \hspace{1cm} (5)

We define

$$S = \left\{j: 1 \leq j \leq s - r + 1 \text{ and } \sum_{i=j}^{j+r-1} m_i \leq \left\lfloor \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right\rfloor \right\}.$$ \hspace{1cm}

By the left inequality in (5), $1 \in S$ and hence $S$ is nonempty. Let $p$ be the maximum integer in $S$. We show that $p$ satisfies the desired relation (2).
Since \( p \in S \), the definition of \( S \) implies the left inequality in (2). If \( p = s - r + 1 \) then the right inequality in (2) follows from the right inequality in (5). Now assume \( p \leq s - r \). Since \( p \) is the maximum integer in \( S, p + 1 \not\in S \). Since \( 1 \leq p + 1 \leq s - r + 1 \) and \( m_p \geq 0 \), the definition of \( S \) implies

\[
\frac{r - 1}{s - 1} \sum_{i=1}^{s} m_i < \sum_{i=p+1}^{p+r-1} m_i \leq \sum_{i=p}^{p+r-1} m_i.
\]

as desired. \( \square \)

For \( X \subseteq V(G) \), let \( (X) \) denote the subgraph of \( G \) induced by \( X \). For \( n \) graphs \( G_1, \ldots, G_n \) with pairwise disjoint vertex sets, the disjoint union of \( G_1, \ldots, G_n \), denoted by \( G_1 \cup \cdots \cup G_n \), is the graph with vertex set \( V(G_1) \cup \cdots \cup V(G_n) \) and edge set \( E(G_1) \cup \cdots \cup E(G_n) \).

**Lemma 8.** Let \( H = K_{m_1, \ldots, m_s} \), \( s \geq 2 \) and \( m_i \geq 0 \) for each \( i \in [s] \). Denote partite sets of \( H \) by \( V_1, \ldots, V_s \) with \( |V_i| = m_i \) for each \( i \in [s] \). For any \( r \geq 2 \), there exists a partition \( \Pi = (\pi_1, \ldots, \pi_r) \) of \([s]\) such that the disjoint union

\[
U = \left( \bigcup_{i \in \pi_1} V_i \right) \cup \left( \bigcup_{i \in \pi_2} V_i \right) \cup \cdots \cup \left( \bigcup_{i \in \pi_r} V_i \right)
\]

is equivalently \( \left[ \frac{s}{r-1} \right] \)-colorable.

**Proof.** As \( K_{m_1, \ldots, m_s} = K_{m_1, \ldots, m_s, 0} = K_{m_1, \ldots, m_s, 0, 0} = \cdots \), we may always assume that \( s - 1 \) is divisible by \( r - 1 \). Set \( k = \left[ \frac{s-1}{r-1} \right] = \frac{s-1}{r-1} \). We fix \( r \) and prove the lemma by induction on \( k \). If \( k = 1 \) then \( s = r \). Let

\[
\Pi = (\pi_1, \ldots, \pi_r) = (\{1\}, \{2\}, \ldots, \{s\}).
\]

Since all graphs \( \left( \bigcup_{i \in \{1\}} V_i \right) \) are empty and so is their disjoint union, the lemma holds for \( k = 1 \). Assume now that \( k \geq 2 \) and the lemma holds for \( k - 1 \). By Lemma 7, there exist an \((r - 1)\)-subset \( I \) and an \( r \)-subset \( J \) of \([s]\) with \( I \subset J \) such that

\[
\sum_{i \in I} m_i \leq \left[ \frac{r-1}{s-1} \sum_{i=1}^{s} m_i \right] \leq \sum_{i \in J} m_i.
\]

By rearranging \( m_1, \ldots, m_s \), we may assume \( I = \{s-r+2, \ldots, s\} \) and \( J = I \cup \{s-r+1\} \). As \( k = \frac{s-1}{r-1} \), (7) becomes

\[
\sum_{i=s-r+2}^{s} m_i \leq \left[ \frac{1}{k} \sum_{i=1}^{s} m_i \right] \leq \sum_{i=s-r+1}^{s} m_i.
\]

Set \( s' = s - r + 1 \) and

\[
q = \left[ \frac{1}{k} \sum_{i=1}^{s} m_i \right] - \sum_{i=s-r+2}^{s} m_i.
\]

By (8), \( 0 \leq q \leq m_s \). Let \( V_i' = V_i \) for \( 1 \leq i < s' \) and let \( V_{s'}' \) be any subset of \( V_{s'} \) with \( m_{s'} - q \) vertices. Since \( \left[ \frac{s-1}{r-1} \right] - 1 = k - 1 \), by the induction assumption, there exists a partition \( \Pi' = (\pi_1', \ldots, \pi_{s'}) \) of \([s']\) such that the disjoint union

\[
U' = \left( \bigcup_{i \in \pi_1} V_i' \right) \cup \left( \bigcup_{i \in \pi_2} V_i' \right) \cup \cdots \cup \left( \bigcup_{i \in \pi_{s'}} V_i' \right)
\]

is equivalently \((k - 1)\)-colorable. Without loss of generality, we may assume \( s' \in \pi_1 \). Let

\[
\Pi = (\pi_1, \ldots, \pi_r) = (\pi_1', \pi_2' \cup \{s' + 1\}, \ldots, \pi_{s'}' \cup \{s' + r - 1\}).
\]

It is clear that \( \Pi \) is a partition of \([s]\). We claim that the graph \( U \) defined by (6) is equivalently \( k \)-colorable. First, use \( k - 1 \) colors to color the subgraph \( U' \) equivalently. Now, by (9), the number of uncolored vertices is exactly

\[
|V_{s'} \setminus V_{s'}'| + |V_{s'+1}| + \cdots + |V_{s'+r-1}| = q + \sum_{i=s-r+2}^{s} m_i = \left[ \frac{1}{k} \sum_{i=1}^{s} m_i \right] - q.
\]

Finally, by (10), the subgraph of \( U \) induced by these uncolored vertices is a disjoint union of \( r \) empty graphs and hence is empty. Assigning a new color to these uncolored vertices, we obtain an equivalently \( k \)-coloring of \( U \). This proves the claim and hence the lemma holds. \( \square \)

**Proof of Theorem 6.** Let \( U_1, U_2, \ldots, U_r \) and \( V_1, V_2, \ldots, V_s \) be the partite sets of \( K_{r(t)} \) and \( K_{m_1, \ldots, m_s} \), respectively. By Lemma 8, there exists a partition \( \Pi = (\pi_1, \ldots, \pi_r) \) of \([s]\) such that the disjoint union

\[
U = \left( \bigcup_{i \in \pi_1} V_i \right) \cup \left( \bigcup_{i \in \pi_2} V_i \right) \cup \cdots \cup \left( \bigcup_{i \in \pi_r} V_i \right)
\]
is equitably \( \left\lceil \frac{i+1}{r-1} \right\rceil \)-colorable. For each \( k \in [r] \) and \( i \in [r] \) we define

\[
W_{k,i} = U_{i+k} \times \bigcup_{j \in \pi_i} V_j \quad \text{and} \quad W_k = \bigcup_{i=1}^{r} W_{k,i},
\]

where the additions on the indices are taken modulo \( r \). If \( i \neq i' \), \((x, y) \in W_{k,i}\) and \((x', y') \in W_{k,i'}\), then \( x \neq x' \) and \( y \neq y' \), implying that \((x, y)\) and \((x', y')\) are not adjacent in \( K_r \square K_{m_1, \ldots, m_s} \). Hence,

\[
\langle W_k \rangle = \left\lceil \frac{r}{r-1} \right\rceil \bigcup_{i=1}^{r} (E_n \square \bigcup_{j \in \pi_i} V_j) = E_n \square \left( \bigcup_{i=1}^{r} \left( \bigcup_{j \in \pi_i} V_j \right) \right) = E_n \square \bigcup_{i=1}^{r} (E_n \cup \bigcup_{j \in \pi_i} V_j) = E_n \cup \bigcup_{i=1}^{r} \bigcup_{j \in \pi_i} V_j = E_n \cup U.
\]

Since \( U \) is equitably \( \left\lceil \frac{i+1}{r-1} \right\rceil \)-colorable, Corollary 3 implies that \( \langle W_k \rangle = E_n \square U \) is also equitably \( \left\lceil \frac{i+1}{r-1} \right\rceil \)-colorable. Note that \((W_1, \ldots, W_r)\) is a partition of \( V(K_r \square K_{m_1, \ldots, m_s}) \) and all classes have equal sizes. By partitioning each \( W_k \) equitably into \( \left\lceil \frac{i+1}{r-1} \right\rceil \) independent sets, we obtain an equitable \( r \left\lceil \frac{i+1}{r-1} \right\rceil \)-coloring of \( V(K_r \square K_{m_1, \ldots, m_s}) \). This proves the theorem.

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References