Light triangles in plane graphs with near-independent crossings *

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Abstract

It is proved that every plane graph with near-independent crossings and with minimum degree at least five contains a light triangle.

1 Introduction

All graphs considered in the paper are finite, simple and undirected. By $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$, we denote the set of vertices, the set of edges, the set of faces, the minimum degree and the maximum degree of a graph G, respectively. A k-, k^+- and k^--*vertex* (resp. *face*) is a vertex (resp. face) of degree k, at least k and at most k, respectively. For other undefined concepts we refer the reader to [1].

A graph is 1-*planar* if it can be drawn on a plane so that each edge is crossed by at most one other edge. The concept of the 1-planarity was introduced by Ringel [7] in 1965 when he considered the vertex-face coloring of plane graphs (corresponding to the vertex coloring of 1-planar graphs). Although nearly fifty years past, the class of 1-planar graphs is still litter explored compared to the well-established planar graphs.

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We now turn the attention to the drawing of 1-planar graphs. A 1-planar drawing (of a 1-planar graph) is *good* if it contains the minimum number of crossings, and normally, we assume that every 1-planar drawing considered in this paper is good. Note that every crossing in a 1-planar drawing is generalized by two mutually crossed edges, thus for every crossing c there exists a vertex set $N(c)$ of size four consisting of the end-vertices of the two edges that generalize c. It is easy to see that $|N(c_1) \cap N(c_2)| \leq 2$ for any two distinct crossings c_1 and c_2 in a good 1-planar drawing. In view of this, we can define two subclasses of 1-planar graphs. Let G be a 1-planar graph. If $|N(c_1) \cap N(c_2)| = 0$ for any two distinct crossings c_1 and c_2, then G is a *plane graph with independent crossings* (see [6]) or *IC-planar graph* for short (see [8]). If $|N(c_1) \cap N(c_2)| \leq 1$ for any two distinct crossings c_1 and c_2, then G is a *plane graph with near-independent crossings* or *NIC-planar graph* for short. The notion of NIC-planarity was introduced by Zhang [9] very recently, and also by Czap and Šugerek [3]. Let $\mathcal{G}, \mathcal{G}_0, \mathcal{G}_1$ and $\mathcal{G}_2$ be the classes of planar graphs, IC-planar graphs, NIC-planar graphs and 1-planar graphs, respectively. It is easy to see that $\mathcal{G} \subset \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2$.

Let H be a connect graph and let $\mathcal{G}$ be a family of graphs. If for any graph $G \in \mathcal{G}$, G contains a subgraph $K \simeq H$ such that $\max_{x \in V(K)}\{d_G(x)\}$ is bounded by a constant independent of G, then we say that H is *light* in $\mathcal{G}$, and otherwise *heavy* in $\mathcal{G}$. Seeking light small graphs in a giving graph class is a classic problem in the structural graph theory. A famous result by Borodin [2] states that every planar graph with minimum degree 5 contains a triangle uvw with $d(u)+d(v)+d(w) \leq 17$ and the bound 17 is sharp, thus triangle is light in the class of planar graphs with minimum degree 5. For the class of 1-planar graphs with the same minimum degree, the result is surprisedly opposite. Actually, for any positive integer m there is a 1-planar graph with minimum degree at least 5 that contains isomorphic copies of triangles and every triangle contains an m-vertex (see [5]). Hence triangle is heavy in the class of 1-planar graphs with minimum degree at least 5.

In view of this, an interesting problem is to find subclasses of 1-planar graphs with minimum degree at least 5 in which triangle is light. A recent result by Zhang [10] states that triangle is light in the class of 1-planar graphs with minimum degree at least 5 and with minimum edge degree at least 12. In this paper, we consider the lightness of triangle in the class of NIC-planar graphs with high minimum degree. The following is the main result, which implies that triangle is light in the class of NIC-planar graphs with minimum degree at least 5, and thus in the class of IC-planar graphs with minimum degree at least 5.

Theorem 1.1. Every plane graph with near-independent crossings and with minimum degree at least 5 contains a triangle uvw with $\max\{d(u), d(v), d(w)\} \leq 26$.

Let G be an NIC-planar drawing of a graph. The *associated plane graph* of G, denoted by $G^\times$, is the graph obtained from G by turning all crossings of G into new 4-vertices, and those new 4-vertices are called *false vertices*. The face that is incident with no false vertex in $G^\times$ is called *true face*, and otherwise, we call it *false face*. Let v be a k-vertex in $G^\times$ and let $v_1, \ldots, v_k$ be the neighbors of v that lies clockwise. By f_i with $1 \leq i \leq k$, we denote the face that is incident with vv_i

and vv_{i+1} in $G^\times$. Here the addition on the subscripts are taken modulo k. Note that for an NIC-planar graph with minimum degree at least 5, $d_G(v) = d_{G^\times}(v)$ for any vertex $v \in V(G)$ and $d_{G^\times}(v) = 4$ if and only if v is false in $G^\times$. Hence we do not distinguish $d_G(v)$ and $d_{G^\times}(v)$ in the following arguments.

2 Discharging: the proof of Theorem 1.1

Suppose that G is a counterexample to Theorem 1.1 and that $G^\times$ is the associated plane graph of G. Assign an initial charge c to each element $x \in V(G^\times) \bigcup F(G^\times)$ as follows:

$$c(x) = \begin{cases} d(x) - 6, & \text{if } x \in V(G^\times); \\ 2d(x) - 6, & \text{if } x \in F(G^\times), \end{cases}$$

By Euler's formula on $G^\times$, $\sum_{x \in V(G^\times) \bigcup F(G^\times)} c(x) = -12$. We now redistribute the charges among $V(G^\times) \bigcup F(G^\times)$ according to the rules defined below.

Rule 1 Every 27^+-vertex sends $\frac{7}{9}$ to each of its incident faces;
Rule 2 Let $f = xyz$ be a 3-face with a 27^+-vertex x.
Rule 2.1 If y and z are 5^--vertices, then f sends $\frac{7}{18}$ to each of y and z;
Rule 2.2 If y is a 27^+-vertex and z is a 4- or 5-vertex, then f sends $\frac{14}{9}$ to z;
Rule 2.3 If y is a 4-vertex and z is a 5-vertex, then f sends $\frac{1}{2}$ to y and $\frac{5}{18}$ to z;
Rule 2.4 If y is a 4- or 5-vertex and z is a vertex of degree between 6 and 26, then f sends $\frac{7}{9}$ to y.
Rule 3 Every 4^+-face sends 1 to each of its incident 4-vertices and $\frac{1}{n_5}$ with $n_5 \geq 1$ to each of its incident 5-vertices, where n_5 is the number of 5-vertices that are incident with f.

Let $c'(x)$ be the final charge of $x \in V(G^\times) \bigcup F(G^\times)$ after applications of the above rules. We now prove that $c'(x) \geq 0$ for each $x \in V(G^\times) \bigcup F(G^\times)$, therefore,

$$-12 = \sum_{x \in V(G^\times) \bigcup F(G^\times)} c(x) = \sum_{x \in V(G^\times) \bigcup F(G^\times)} c'(x) \geq 0,$$

which is a contradiction.

Let f be a face in $G^\times$. If $d(f) = 3$, then Rules 1 and 2 guarantee that $c'(f) \geq 0$. If $d(f) = 4$, then the number of 4-vertices that are incident with f is at most 1 by the drawing of G, thus by Rule 3, $c'(f) \geq 2 \times 4 - 6 - 1 \times 1 - 3 \times \frac{1}{3} = 0$. If $d(f) = 5$, then f is incident with at most two 4-vertices by the drawing of G and $c'(f) \geq 2 \times 5 - 6 - 2 \times 1 - 3 \times \frac{1}{3} = 1 > 0$ by Rule 3. If $d(f) \geq 6$, then $c'(f) \geq 2d(f) - 6 - d(f) = d(f) - 6 \geq 0$ by Rule 3. Let v be a vertex in $G^\times$. If $d(v) \geq 27$, then $c'(v) \geq 27 - 6 - 27 \times \frac{7}{9} = 0$ by Rule 1. If $6 \leq d(v) \leq 26$, then $c'(v) = c(v) \geq 0$ since v is not involved in the rules. Until now, we are left only two cases.

Case 1. $d(v) = 4$.

<u>*Subcase 1.1*</u>. *v is incident with at least two 4^+-faces.*

It is easy to see that $c'(v) \geq 4 - 6 + 2 \times 1 = 0$ by Rule 3.

Subcase 1.2. *v is incident with exactly one 4^+-face.*

Without loss of generality, assume that $d(f_4) \geq 4$ and $d(f_1) = d(f_2) = d(f_3) = 3$. If $d(v_2) \geq 27$, then by Rules 2.2, 2.3 and 2.4, each of f_1 and f_2 sends at least $\min\{\frac{14}{9}, \frac{1}{2}, \frac{7}{9}\} = \frac{1}{2}$ to v. By Rule 3, f_4 sends 1 to v. Hence $c'(v) \geq 4-6+2\times\frac{1}{2}+1 = 0$. If $d(v_3) \geq 27$, then we still have $c'(v) \geq 0$ similarly. If $d(v_2) \leq 26$ and $d(v_3) \leq 26$, then $d(v_1) \geq 27$ and $d(v_4) \geq 27$, otherwise a light triangle with all 26^--vertices occurs in G, a contradiction. In this case, each of f_1 and f_3 sends at least $\frac{1}{2}$ to v by Rules 2.2, 2.3 and 2.4 and f_4 sends 1 to v by Rule 3. This implies that $c'(v) \geq 4 - 6 + 1 + 2 \times \frac{1}{2} = 0$.

Subcase 1.3. *v is incident with only 3^+-faces.*

Since G is a counterexample, v is incident with at least two 27^+-vertices. If v is incident with four faces that are incident with a 27^+-vertices, then $c'(v) \geq 4 - 6 + 2 \times \frac{1}{2} = 0$ by Rules 2.2, 2.3 and 2.4. Hence, we assume, without loss of generality, that v_1 and v_4 are 27^+-vertices. By Rules 2.2, 2.3 and 2.4, each of f_1 and f_3 sends at least $\frac{1}{2}$ to v, and f_4 sends $\frac{14}{9}$ to v. Therefore, $c'(v) \geq 4-6+2\times\frac{1}{2}+\frac{14}{9} > 0$.

Case 2. $d(v) = 5$.

Subcase 2.1. *v is incident with at least three 4^+-faces.*

By Rule 3, each 4^+-face sends $\frac{1}{3}$ to v. This implies that $c'(v) \geq 5-6+3\times\frac{1}{3} = 0$.

Subcase 2.2. *v is incident with exactly two 4^+-faces.*

If v is incident with two adjacent 4^+-faces, say f_1 and f_2, then f_3, f_4 and f_5 are 3-faces, and moreover, at least one of them, say f_3, is true and incident with a 27^+-vertex by the choice of G. This implies by Rules 2.1, 2.2 and 2.4 that f_3 sends at least $\min\{\frac{7}{18}, \frac{14}{9}, \frac{7}{9}\} = \frac{7}{18}$ to v. By Rule 3, each of f_1 and f_2 sends $\frac{1}{3}$ to v. Hence, $c'(v) \geq 5 - 6 + 2 \times \frac{1}{3} + \frac{7}{18} > 0$.

If v is incident with two nonadjacent 4^+-faces, say f_1 and f_3, then f_2, f_4 and f_5 are 3-faces. If one of them is true, then by same arguments as in Subcase 2.2 we have $c'(v) > 0$. Hence we assume that f_2, f_4 and f_5 are all false. By the definition of G, v_5 must be false. Since vv_1v_4 is a triangle in G, one of v_1 and v_4, say v_1, is a 27^+-vertex. By Rule 2.3, f_5 sends $\frac{5}{18}$ to v. If v_2 is false, then f_1 is incident with at most $d(f_1) - 2$ vertices of degree 5, which implies by Rule 3 that f_1 sends to v at least $\frac{2d(f_1)-6-\lfloor d(f_1)/2\rfloor}{d(f_1)-2} \geq \frac{1}{2}$ for $d(f_1) \geq 4$, since f_1 is incident with at most $\lfloor\frac{d(f_1)}{2}\rfloor$ false vertices. By Rule 3, f_3 sends at least $\frac{1}{3}$ to v. Therefore, $c'(v) \geq 5 - 6 + \frac{5}{18} + \frac{1}{2} + \frac{1}{3} > 0$. We now assume that v_2 is true and v_3 is false. If f_1 is a 5^+-face, then f_1 sends to v at least $\frac{2d(f_1)-6-\lfloor d(f_1)/2\rfloor}{d(f_1)-1} \geq \frac{1}{2}$ for $d(f_1) \geq 5$, since f_1 is incident with at most $d(f_1) - 1$ vertices of degree 5. If f_1 is a 4-face, then v is incident with at most one false vertex, in which case v is incident with at most two 5-vertices. Hence by Rule 3, f_1 sends to v at least $\min\{\frac{2\times4-6-1}{2}, \frac{2\times4-6}{3}\} \geq \frac{1}{2}$. In any case, f_1 sends at least $\frac{1}{2}$ to v and f_3 sends at least $\frac{1}{3}$ to v by Rule 3. Therefore, $c'(v) \geq 5 - 6 + \frac{5}{18} + \frac{1}{2} + \frac{1}{3} > 0$.

Subcase 2.3. *v is incident with only one 4^+-face, say f_1.*

By Rule 3, f_1 sends at least $\frac{1}{3}$ to v. If v is incident with at least two true 3-faces, then each of them is incident with a 27^+-vertex by the choice of G, from which v receives at least $\frac{7}{18}$ by Rules 2.1, 2.2 and 2.4. Hence $c'(v) \geq 5 - 6 + \frac{1}{3} + 2 \times \frac{7}{18} > 0$

and we assume that v is incident with at most one true 3-face.

If v_3 is false, then v_2, v_4 and v_5 are true by the choice of G and thus v_1 is false. Since vv_2v_4 is a triangle in G, one of v_2 and v_4 is a 27^+-vertices, which implies that either f_2 or f_3 sends $\frac{5}{18}$ by Rule 2.3. Note that f_4 is a true 3-face, from which v receives at least $\frac{7}{18}$ by Rules 2.1, 2.2 and 2.4. Hence $c'(v) \geq 5-6+\frac{5}{18}+\frac{7}{18}+\frac{1}{3} = 0$.

We now assume that v_3 is true, and moreover, that v_5 is true by symmetry. Since v is incident with at most one true 3-face, v_4 is false. However, in this case v_1 and v_2 cannot be false by the choice of G. This implies that f_2 and f_5 are true 3-faces, a contradiction.

Subcase 2.4. *v is incident only with 3-faces.*

By the choice of G, at most one of v_1, v_2, v_3, v_4 and v_5 is false, which implies that v is incident with at least three true 3-faces. Since every true 3-face incident with v contains a 27^+-vertex, from which v received at least $\frac{7}{18}$ by Rules 2.1, 2.2 and 2.4. Therefore, $c'(v) \geq 5-6+3\times\frac{7}{18} > 0$.

3 Remarks on Theorem 1.1

Fabrici, Hexel, Jendrol' and Walter [4] showed for every integer $m \geq 4$ that there is a 3-connected planar graph G with $\delta(G) \geq 4$ such that each subgraph of G isomorphic to a triangle has a vertex x with $d(x) \geq m$. Hence triangle is heavy in the class of planar graphs with minimum degree at least 4. Since the class of planar graphs is a subclass of NIC-planar graphs, triangle is also heavy in the class of NIC-planar graphs with minimum degree at least 4. Therefore, the condition on the minimum degree in Theorem 1.1 cannot be weakened, but whether the upper bound on the maximum degree of the triangle in the theorem is sharp is unknown. Actually, if we replace the condition on the minimum degree with $\delta(G) = 6$ (note that every NIC-planar graph contains a vertex of degree at most 6, see [9]), we have the following result with smaller upper bound on the maximum degree of the light triangle. Note that Theorem 3.1 is originally proved for 1-planar graphs, a larger class than NIC-planar graphs.

Theorem 3.1. ([5]) Every plane graph with near-independent crossings and with minimum degree 6 contains a triangle uvw with $\max\{d(u), d(v), d(w)\} \leq 10$.

4 An improvement of Theorem 3.1

In the section, we give the following theorem, which improves Theorem 3.1.

Theorem 4.1. Every plane graph with near-independent crossings and with minimum degree 6 contains a triangle uvw so that $\max\{d(u), d(v), d(w)\} \leq 7$.

Proof. The strategy of the proof of this result is same to the one of Theorem 1.1.

First, assign each element $x \in V(G^{\times}) \cup F(G^{\times})$ an initial charge

$$c(x) = \begin{cases} d(x) - 6, & \text{if } x \in V(G^{\times}); \\ 2d(x) - 6, & \text{if } x \in F(G^{\times}), \end{cases}$$

where $G^{\times}$ is the associated plane graph of the counterexample G to the result.

Second, define proper discharging rules. Before stating them, we need some more notions. A *false fan* that is incident with a true vertex v is a subgraph of $G^{\times}$ that consists of four vertices u, v, w and x so that (1) u, x and w are three neighbors of v in $G^{\times}$ that lies clockwise; (2) x is a false vertex; (3) $ux, wx \in E(G^{\times})$, i.e., $uw \in E(G)$. It is easy to see that v is incident with two false 3-faces if v is incident with a false fan. We call those false 3-faces derived from false fans *false 3^F-faces*.

Claim A. *Every true vertex v in $G^{\times}$ is incident with at most $\lfloor \frac{d(v)}{3} \rfloor$ false fans, thus at most $2\lfloor \frac{d(v)}{3} \rfloor$ false 3^F-faces.*

Proof. Otherwise, there are two adjacent false fans, that is, a subgraph of $G^{\times}$ that consists of six vertices v, u_1, u_2, w, x and y so that (1) u_1, x, w, y and u_2 are neighbors of v in $G^{\times}$ that lies clockwise and u_1xwyu_2 is a path in $G^{\times}$; (2) x and y are false vertices. Now, one can see that x and y are two crossings in G satisfying $|N(x) \cap N(y)| \geq 2$, a contradiction to the definition of NIC-planarity. □

The discharging rules are as follows.

Rule 1 Every 8^+-vertex sends $\frac{1}{2}$ to each of its incident false 3^F-faces;
Rule 2 Every false 3^F-face sends the positive charge saving after applying Rule 1 to its incident 4-vertex;
Rule 3 Every 4^+-face sends 2 to each of its incident 4-vertices.

Let $c'(x)$ be the final charge of $x \in V(G^{\times}) \bigcup F(G^{\times})$ after applications of the above rules. Since every 4-face is incident with at most one 4-vertex and every 5^+-face f is incident with at most $\lfloor \frac{d(f)}{2} \rfloor$ false vertices, $c'(f) \geq 2 \times 4 - 6 - 2 = 0$ for $d(f) = 4$ and $c'(f) \geq 2d(f) - 6 - 2\lfloor \frac{d(f)}{2} \rfloor \geq 0$ for $d(f) \geq 5$ by Rule 3. If f is a 3-face, then by Rules 1 and 2, $c'(f) = c(f) = 0$.

Let v be a vertex of $G^{\times}$. If $6 \leq d(v) \leq 7$, then $c'(v) = c(v) \geq 0$. If $d(v) \geq 8$, then by Rule 1 and Claim A, $c'(v) \geq d(v) - 6 - \lfloor \frac{d(v)}{3} \rfloor \geq 0$. If $d(v) = 4$ and v is incident with at least one 4^+-vertex, then by Rule 3, $c'(v) \geq 4 - 6 + 2 = 0$. If $d(v) = 4$ and v is incident only with 3-faces, then it is easy to see that all 3-faces that are incident with v are false 3^F-faces, and v is incident with at least two 8^+-vertices by the choice of G. We now end the proof by distinguishing two nonisomorphic cases. First, if v_1 and v_3 are 8^+-vertices, then by Rules 1 and 2, $c'(v) \geq 4 - 6 + 4 \times \frac{1}{2} = 0$, since each of f_1, f_2, f_3 and f_4 sends at least $\frac{1}{2}$ to v. Second, if v_1 and v_2 are 8^+-vertices, then f_1 sends $2 \times \frac{1}{2} = 1$ to v and each of f_2 and f_4 sends at least $\frac{1}{2}$ to v by Rules 2 and 3, which implies that $c'(v) \geq 4 - 6 + 1 + 2 \times \frac{1}{2} = 0$. □

As we know, the class of IC-planar graphs is a subclass of the one of NIC-planar graphs, and every IC-planar graph also contains a vertex of degree at most

6 (see [8]), where the bound 6 is sharp. We end this paper by the following result on the lightness of triangle in the class of IC-planar graphs.

Theorem 4.2. Every plane graph with independent crossings and with minimum degree 6 contains a triangle uvw so that $d(u) = d(v) = d(w) = 6$.

The proof of Theorem 4.2 is almost the same with the one of Theorem 4.1. One difference is that Claim A can be improved to the following Claim B for IC-planar graphs, the proof of which is trivial.

Claim B. *Every true vertex v in $G^{\times}$ is incident with at most one false fan, thus at most two false 3^F-faces.* □

Another difference is the estimation of the final charges of large vertices, which are 7^+-vertices here. Actually, after replacing 8^+-vertices with 7^+-vertices in Rule 1, we have $c'(v) \geq 7 - 6 - 2 \times \frac{1}{2} = 0$ for $d(v) \geq 7$ by Claim B.

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