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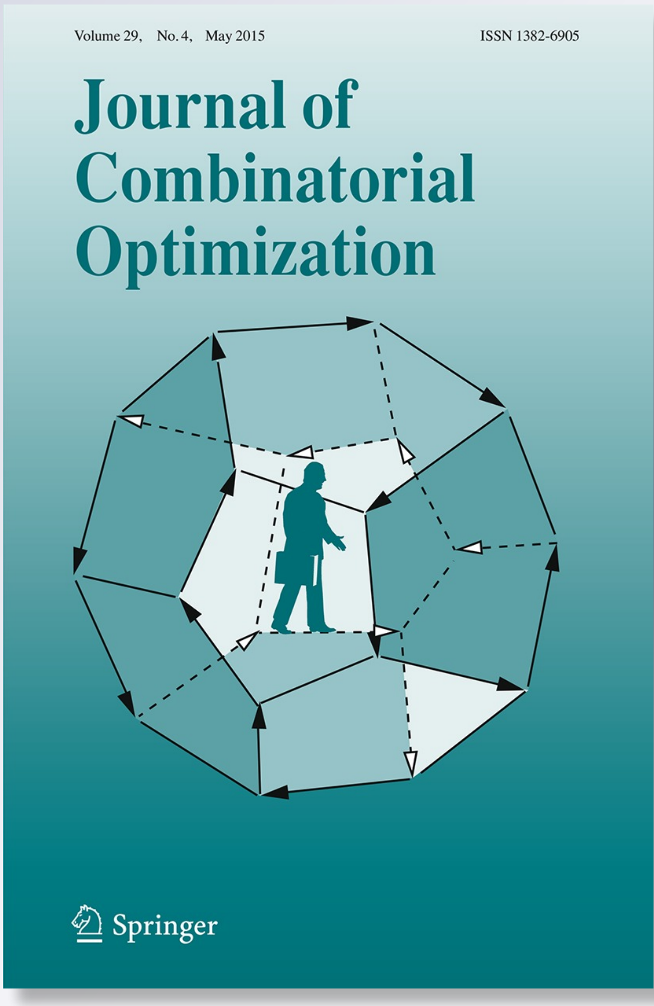

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The r-acyclic chromatic number of planar graphs

Guanghui Wang · Guiying Yan · Jiguo Yu · Xin Zhang

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Abstract A vertex coloring of a graph G is r-acyclic if it is a proper vertex coloring such that every cycle C receives at least $\min\{|C|, r\}$ colors. The r-acyclic chromatic number $a_r(G)$ of G is the least number of colors in an r-acyclic coloring of G. Let G be a planar graph. By Four Color Theorem, we know that $a_1(G) = a_2(G) = \chi(G) \leq 4$, where $\chi(G)$ is the chromatic number of G. Borodin proved that $a_3(G) \leq 5$. However when $r \geq 4$, the r-acyclic chromatic number of a class of graphs may not be bounded by a constant number. For example, $a_4(K_{2,n}) = n + 2 = \Delta(K_{2,n}) + 2$ for $n \geq 2$, where $K_{2,n}$ is a complete bipartite (planar) graph. In this paper, we give a sufficient

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condition for $a_r(G) \leq r$ when G is a planar graph. In precise, we show that if $r \geq 4$ and G is a planar graph with $g(G) \geq \frac{10r-4}{3}$, then $a_r(G) \leq r$. In addition, we discuss the 4-acyclic colorings of some special planar graphs.

Keywords Acyclic coloring · Planar graph · Girth

1 Introduction and notation

We use Bondy and Murty (1976) for terminology and notations not defined here and consider undirected graphs only. Let $G = (V, E)$ be a graph. A vertex coloring of a graph G is r-*acyclic* if it is a proper vertex coloring such that every cycle C receives at least $\min\{|C|, r\}$ colors. The r-*acyclic chromatic number* of G, $a_r(G)$, is the least number of colors in an r-acyclic coloring of G.

For $r \leq 2$, the r-acyclic coloring is actually the proper vertex coloring, so for any graph G with maximum degree Δ, its r-acyclic chromatic number is at most $\Delta + 1$. The 3-acyclic coloring, which is also known as *acyclic coloring* in the literature, has been studied extensively. It was proved by Skulrattanakulchai (2004) that $a_3(G) \leq 4$ for any graph of maximum degree 3. Burnstein (1979) showed that $a_3(G) \leq 5$ for any graph of maximum degree 4. Kostochka and Stocker (2011) proved that $a_3(G) \leq 7$ for any graph of maximum degree 5. Hocquard (2011) confirmed that $a_3(G) \leq 11$ for any graph of maximum degree 6. Dieng et al. (2010) showed for each graph G with maximum degree $\Delta \geq 7$ that $a_3(G) \leq f(\Delta)$, where

$$f(\Delta) = \begin{cases} 17 & \text{if } \Delta = 7. \\ \frac{\Delta^2 - 5\Delta}{2} + 2 \times \left[\frac{\Delta - 1}{2}\right] + 3 & \text{if } \Delta \geq 8. \end{cases} \quad (1.1)$$

Yadav et al (2009) showed for any graph G with maximum degree Δ that $a_3(G) \leq \frac{3\Delta^2+4\Delta+8}{8}$. Alon et al. (1991) gave upper and lower bounds for $a_3(G)$ by using the probabilistic method; they proved that for some constants $c_1, c_2 > 0$,

$$\frac{c_1 \Delta^{\frac{4}{3}}}{(log \Delta)^{\frac{1}{3}}} \leq a_3(G) \leq c_2 \Delta^{\frac{4}{3}}.$$

For $r \geq 4$, it was shown in Greenhill and Pikhurko (2005) that there exist positive constants c, c' such that $c\Delta^{\lfloor \frac{r}{2} \rfloor} \leq a_r(G) \leq c' \Delta^{\lfloor \frac{r}{2} \rfloor}$. Cai et al. (2013) proved that for a graph G with maximum degree Δ and girth $g \geq 2(r-1)\Delta$, $a_r(G) \leq 6(r-1)\Delta$, where $r \geq 4$ is a positive integer. For more references, we refer to (Albertson and Berman 1976; Fertin and Raspaud 2005, 2008; Zhang et al. 2012).

Now we focus on planar graphs. First of all, the Four Color Theorem implies $a_1(G) \leq 4$ and $a_2(G) \leq 4$. (Grünbaum 1973) conjectured that 5 colors are sufficient to acyclically color any planar graph; this conjecture was confirmed by Borodin (1979). However, for $r \geq 4$ and a class of graphs $\mathcal{G}$, $a_r(\mathcal{G}) := \max\{a_r(G) \mid G \in \mathcal{G}\}$ may not be bounded by a constant number, the class of planar complete bipartite graphs is such an example, since $a_r(K_{2,n}) = n + 2$ is dependent of the maximum degree, where $r \geq 4$ and $n \geq 2$.

In this paper, we consider r-acyclic colorings of planar graphs. First, we discuss the 4-acyclic colorings of some special planar graphs. Whereafter, we give a sufficient condition for $a_r(G) \leq r$ for planar graphs.

2 4-acyclic colorings of outerplanar graphs

A graph is *outerplanar* if it can be drawn in the plane so that all vertices are lying on the outside face. It is known that any outerplanar graphs contains no $K_{2,3}$-minors and K_4-minors. In this and the next section, multiple edges are allowed. The following structural lemma for outerplanar graphs proved by Borodin and Woodall (1995) is a useful start.

Lemma 2.1 *Borodin and Woodall (1995) Every outerplanar graph with minimum degree at least two contains one of the following configurations:*

(a) *two adjacent 2-vertices u and v;*
(b) *a 3-cycle uvw with $d(u) = 2$ and $d(v) = 3$;*
(c) *two intersecting 3-cycles uvw and xvy with $d(u) = d(x) = 2$ and $d(v) = 4$.*

Theorem 2.1 *For each outerplanar graph G, $a_4(G) \leq 4$ and the bound 4 is sharp.*

Proof Suppose that G is a minimal counterexample with the smallest number of $|V(G)| + |E(G)|$. Clearly, $\delta(G) \geq 2$. By Lemma 2.1, we need consider two cases.

First, suppose that G contains two adjacent 2-vertices u and v. Denote the other neighbor of u and v by w and z, respectively. If $zw \in E(G)$, then let $H := G - \{u, v\}$; otherwise, let $H := G - \{u, v\} + zw$. In each case one can check that H is still outerplanar and thus by the minimality of G, H admits a 4-acyclic coloring φ with $\varphi(z) \neq \varphi(w)$. Extend φ to a coloring of G by coloring u and v with two distinct colors that are different from $\varphi(z)$ and $\varphi(w)$. Since every cycle of length at least 4 in G passing through u and v contains the path $wuvz$, the resulting coloring of G is 4-acyclic as required.

Second, suppose that G contains a 2-vertex u that is incident with a triangle uvw. Let $H := G - u$. By the minimality of G, H admits a 4-acyclic coloring φ. Since G is outerplanar, $|N(v) \cap N(w)| \leq 2$, because otherwise one can find a $K_{2,3}$-minor in G. This implies that u is incident with at most one 4-cycle in G. If $N(v) \cap N(w) = \{u, s\}$, then extend φ to a coloring of G by coloring u with a color different from $\varphi(v)$, $\varphi(w)$ and $\varphi(s)$. If C is a cycle of length at least 5 in G passing through u, then $C - u + vw$ is a cycle of length at least 4 in H. Since H has already been 4-acyclically colored, the cycle $C - u + vw$ in H is incident with at least 4 colors under φ. Thus, the cycle C in G is also incident with at least 4 colors after the extension of φ. By the choice of the coloring on u, the vertices in the unique 4-cycle that passes through u are colored all distinctly. Therefore, the extended coloring of G is 4-acyclic, a contradiction. If $N(v) \cap N(w) = \{u\}$, then extend φ to a coloring of G by coloring u with a color different from $\varphi(v)$ and $\varphi(w)$. If C is a cycle of length at least 4 in G passing through u, then $C - u + vw$ is a cycle of length at least 4 in H, which is incident with at least 4 colors under φ since H has already been 4-acyclically colored. Thus, the extended coloring of G is 4-acyclic, a contradiction.

For each cycle C_n with $n \geq 4$, $a_4(C_n) = 4$, so the bound 4 is sharp. □

3 4-acyclic colorings of series-parallel graphs

A graph is *series-parallel* if it has no K_4-minors. It is known that every series-parallel graph contains a vertex of degree at most 2 Duffin (1965). In this section, we aim to give a sharp upper bound for the 4-acyclic chromatic number of a series-parallel graph.

Theorem 3.1 *For each series-parallel graph* G, $a_4(G) \leq \Delta(G) + 2$ *and the bound* $\Delta(G) + 2$ *is sharp.*

Proof Suppose that G is a minimal counterexample with the smallest number of $|V(G)| + |E(G)|$. Clearly, $\delta(G) \geq 2$, so by the 2-degeneracy of G, G contains a vertex of degree 2, say u.

For convenience, let $\Delta = \Delta(G)$. Denote the neighbors of u in G by v and w. If $vw \in E(G)$, then let $H := G - \{u\}$; otherwise, let $H := G - \{u\} + vw$. In any case one can check that H is still a series-parallel graph with $\Delta(H) \leq \Delta$ and thus by the minimality of G, H admits a 4-acyclic $(\Delta + 2)$-coloring φ with $\varphi(v) \neq \varphi(w)$. Let S be the set of colors on the vertices of $N(v) \cap N(w)$ under φ. Since u is uncolored under φ and $|(N(v) \cap N(w)) \setminus \{u\}| \leq \Delta - 1$, $|S| \leq \Delta - 1$. Extend φ to a coloring of G by coloring u with a color different from any color in $F := S \cup \{\varphi(v), \varphi(w)\}$. Since $|F| \leq \Delta + 1$, there is one available color for u, so the above extension is exercisable. Note that $\varphi(v) \neq \varphi(w)$. By the choice of the color on u, one can see that the vertices of every 4-cycle in G that passes through u are colored all distinctly. Let C be a cycle of length at least 5 in G that passes through u. By the definition of H, $C - u + vw$ is a cycle of length at least 4 in H, and thus this cycle is incident with at least 4 colors under φ. This implies that the cycle C in G is also incident with at least 4 colors after the extension of φ. Therefore, the extended coloring of G is 4-acyclic, a contradiction.

Since $a_4(G) = \Delta(G) + 2$ when G is a complete bipartite graph $K_{2,n}$ with $n \geq 2$, the bound $\Delta(G) + 2$ in the theorem is sharp. □

4 *r*-acyclic colorings of planar graphs

In this section, we give an upper bound for the r-acyclic chromatic number of a planar graph when $r \geq 4$. First, we give some definitions and notations which will be used in our proof. The $kthpower$ G^k of a graph G is defined on the same set of vertices as G and has an edge between any pair of vertices of distance at most k in G. Agnarsson and Halldórsson (2003) showed the following theorem.

Theorem 4.1 *Let* G *be a planar graph with maximum degree* Δ. *For any fixed* $k \geq 1$, G^k *is* $O(\Delta^{\lfloor \frac{k}{2} \rfloor})$*-colorable. Also, there is a family of graphs that attains this bound.*

Clearly, $a_r(G) \leq \chi(G^{r-1})$. So by Theorem 4.1, we have the following corollary.

Corollary 4.1 *Let* G *be a planar graph with maximum degree* Δ. *For any fixed* $r \geq 4$, $a_r(G) \leq C(\Delta^{\lfloor \frac{r-1}{2} \rfloor})$*, where* C *is a constant number.*

For sparse planar graphs, we will get an improvement for the bound.

Theorem 4.2 *Let $r \geq 4$ and G be a graph with $g(G) \geq 3r-2$. If every subgraph of G has average degree less than $2+\frac{6}{5r-5}$, then $a_r(G) \leq r$.*

Theorem 4.2 gives the following immediate corollary.

Corollary 4.2 *If $r \geq 4$ and G is a planar graph with girth at least $\frac{10r-4}{3}$, or a graph embeddable on the torus or Klein bottle with girth greater than $\frac{10r-4}{3}$, then $a_r(G) \leq r$.*

Proof Note that $\frac{10r-4}{3} \geq 3r-2$. Clearly every subgraph of G has girth at least as large as the girth of G, thus if our conclusion fails, then by Theorem 4.2, the average degree of G is at least $2+\frac{6}{5r-5}$. It follows that $|E(G)| \geq \frac{5r-2}{5r-5}|V(G)|$. We use ν, e to denote the number of vertices and edges of G, respectively. Let f be the number of faces in an embedding of G on a surface of Euler characteristic N and $g(G)$ be the girth of G. By Euler's Formula, we have

$$2-N=\nu-e+f \leq e\left(\frac{5r-5}{5r-2}-1+\frac{2}{g(G)}\right)=e\left(\frac{2}{g(G)}-\frac{3}{5r-2}\right).$$

Since $N \leq 2$ for the surface mentioned in the corollary, we deduce that $\frac{2}{g(G)} \geq \frac{3}{5r-2}$, that is, $g(G) \leq \frac{10r-4}{3}$, and equality holds only when $N=2$. This contradiction completes the proof. □

5 Proof of Theorem 4.2

Suppose that Theorem 4.2 does not hold. We choose a minimal counterexample G to Theorem 4.2 in terms of $|V(G)|+|E(G)|$. Clearly, G has minimum degree at least two.

A *thread* in a graph G is a path whose internal vertices have degree 2 in G. Two vertices are *weak neighbors* or *weakly adjacent* if they are the endpoints of a thread (this includes adjacent vertices, since threads may have no internal vertices). For simplicity, let $[r]=\{1,2,\ldots,r\}$. We give the following claims.

Claim 1 *Every thread in G has length at most $r-1$.*

Proof of Claim 1 Otherwise, we assume that G has a thread $v_0v_1\cdots v_r$ of length r. Consider $H=G-\{v_1,\cdots,v_{r-1}\}$. By our assumption, $a_r(H) \leq r$. Suppose that π is an r-acyclic coloring of H by using the colors in $[r]$. Without loss of generality, assume that $\pi(v_0)=1$. Let $\pi(v_1)=2, \pi(v_2)=3, \ldots, \pi(v_{r-1})=r$. If $\pi(v_{r-1})=\pi(v_r)$, we recolor v_{r-1} with a color different from $\pi(v_r)$ and $\pi(v_{r+2})$, and then we get an r-acyclic coloring of G, which is a contradiction. □

Claim 2 *No three vertices of G with degree at least 3 are pairwise weakly adjacent, and no two threads have the same set of endpoints.*

Proof of Claim 2 Otherwise, by Claim 1, G has a cycle of length at most $3r-3$, which contradicts $g(G) \geq 3r-2$. □

When u and v are weakly adjacent, let l_{uv} denote the number of the internal vertices of a shortest u, v-thread. (Note that if u, v are adjacent, then $l_{uv} = 0$). Let $Y = \{v \in V(G) : d(v) \geq 3\}$. A weak neighbor u of v is a *weak Y-neighbor* of v if $u \in Y$; Otherwise it is a *weak*2- *neighbor* of v.

For $v \in V(G)$, let $N_Y(v)$ denote the set of weak Y-neighbors of v in G. For $v \in Y$, let

$$f(v) = -r + \sum_{u \in N_Y(v)} (r - l_{vu} - 1).$$

By Claim 1, $l_{uv} \leq r - 2$, so $r - l_{vu} - 1 \geq 1$ for each $u \in N_Y(G)$. Let P_{uv} denote a u, v-thread and L_{uv} denote a sub-thread of P_{uv} by deleting the vertices u, v. For any subgraph A of G, if π is a coloring of A, then let $C(A) = \{\pi(v) \mid v \in A\}$. A vertex coloring of A such that any two vertices of A have distinct colors is called a *rainbow* coloring of A.

Claim 3 *If $v \in Y$, then $f(v) \geq 1$.*

Proof of Claim 3 Clearly, if for some $u \in N_Y(v), l_{uv} = 0$, then $f(v) \geq 1$, so we assume that $l_{uv} \geq 1$ for $u \in N_Y(v)$. Let H be the graph obtained from G by deleting v and all its weak 2-neighbors. By our assumption, $a_r(H) \leq r$. Let π be an r-acyclic coloring of H by using the colors in $[r]$. Assume $d(v) = m$ and $N_Y(v) = \{u_1, u_2, \cdots, u_m\}$.

Now let $\pi(v) \in [r]\backslash\pi(u_1)$. If $\sum_{u \in N_Y(v)} (r - l_{uv} - 1) - 1 \leq r - 1$, then there exist $S_1, S_2, \cdots, S_m$ such that $\pi(u_1) \notin S_1$ and $S_1 \cup S_2 \cup \cdots S_m \subseteq [r]\backslash\pi(v)$, where $|S_1| = r - l_{u_1 v} - 2$, $|S_i| = r - l_{u_i v} - 1$ for $2 \leq i \leq m$ and $S_i \cap S_j = \phi$, for $i \neq j$. Now we give each thread $L_{u_{i v}}$ a rainbow coloring using the colors in $[r]\backslash(\pi(v) \cup S_i)$ for $i = 2, \ldots, m$. We then give $L_{u_{1v}}$ a rainbow matching using the colors in $[r]\backslash(\pi(v) \cup \pi(u_1) \cup S_1)$. For each thread $P(u_{iv})$, if $\pi(u_i) = \pi(x)$, where x is the neighbor of u_i in thread $P_{u_{iv}}$, then we recolor x to obtain a proper coloring of P_{u_i}. It follows that $C(P_{u_{iv}}) \cup C(P_{u_{jv}}) = [r]$, for $i \neq j$. Thus we get an r-acyclic coloring of G, which is a contradiction, so $\sum_{u \in N_Y(v)} (r - l_{uv} - 1) - 1 \geq r$ and thus $f(v) \geq 1$. □

Claim 4 *If $v \in Y$, then $\sum_{u \in N_Y(v)} f(u) \geq r + 2$.*

Proof of Claim 4 Suppose on the contrary, $\sum_{u \in N_Y(v)} f(u) \leq r + 1$. For a vertex $u \in N_Y(v)$, if it satisfies that $f(u) \leq r - l_{uv} - 1$, then we call it *v - good*, otherwise we call it $v - bad$. For convenience, let $N_Y^g(v)$ denote the set of $v - good$ vertices and $N_Y^b(v)$ denote the set of $v - bad$ vertices. Note that when u is $v - bad$, then $l_{uv} \geq 1$, since otherwise we have that $f(u) \geq r - l_{uv} = r$, hence $\sum_{u \in N_Y(v)} f(u) \geq r + 2$, which is a contradiction. Let H be the graph obtained from G by deleting the vertex v, the v-good neighbors, and all their weak 2-neighbors. Let H_1 denote the subgraph induced by $V - V(H)$. By assumption, H has an r-acyclic coloring π. First we color v such that $\pi(v) = r$.

Suppose that u is v-good. Consider each thread from u except u, v-thread. Let

$$s(u) = r - \sum_{w \in N_Y(u), w \neq v} (r - l_{uw} - 1) + 1.$$

We then have

$$\begin{aligned} s(u) &= 1 + r - \sum_{w \in N_Y(u)} (r - 1 - l_{uw}) + (r - 1 - l_{uv}) \\ &= 1 - f(u) + r - 1 - l_{uv} \\ &\leq 1. \end{aligned}$$

Fact *For any subset $S(u) \subseteq [r-1]$ of size $s(u)$, we have a proper coloring of each thread $P_{uw}(w \in N_Y(v) - v)$ such that*

A. $S(u) \subseteq C(P_{uw})$;
B. $C(P_{uw_i} \cup P_{uw_j}) = [r]$, *for $w_i, w_j \in N_Y(u) - v$ and $i \neq j$.*

Proof of Fact First suppose that there is a thread (say L_{uw_1}, where $w_1 \in N_Y(u) - v$) such that $l_{uw_1} = 0$, then we have that $s(u) = 1, d(u) = 3$ and $l_{uw_2} = r - 2$. Let $S(u) = \{c\}$. If $\pi(w_1) \neq c$, then let $\pi(u) = c$ and give L_{uw_2} a rainbow coloring by using the colors in $[r]\backslash(\pi(w_1) \cup \pi(u))$. In thread L_{uw_2}, if the neighbor of w_2 has the same color with w_2, we change it to get a proper coloring of P_{uw_2}. If $\pi(w_1) = c$, we choose a color from $[r-1]\backslash(\pi(w_1) \cup \pi(w_2))$ for u, then give a rainbow coloring to L_{uw_2} by using the colors from $[r]\backslash(\pi(u) \cup \pi(w_2))$. Such coloring satisfies A and B, so in the following we assume that $l_{uw} \geq 1$ for $w \in N_Y(u) - v$.

If $s(u) = 1$ and $C(N_Y(u) - v) = S(u)$, then we color u by a color in $[r-1]\backslash S(u)$ and give each thread L_{wu} for $w \in N_Y(u) - v$ a rainbow coloring using the colors in $[r]\backslash(S(u) \cup \pi(u))$. Let $T = [r]\backslash(S(u) \cup \pi(u))$ and let $M(L_{uw}) = T\backslash C(L_{uw})$ denote the colors which are in T and do not appear in thread L_{uw}. Clearly $|M(L_{uw})| = r - 2 - l_{uw}$. Since $\sum_{w \in N_Y(u)-v}(r - l_{uw} - 2) \leq r - 2 = r - s(u) - 1 = |T|$, we further assume that $M(L_{uw_i}) \cap M(L_{uw_j}) = \emptyset$, for $w_i, w_j \in N_Y(u) - v$ with $i \neq j$. It follows that $C(P_{uw_i} \cup P_{uw_j}) = [r]$, for $w_i, w_j \in N_Y(u) - v$ and $i \neq j$. Finally, in each thread P_{uw} for $w \in N_Y(u) - v$, we recolor the neighbor of w if necessary to get a proper coloring of P_{uw}. Now it is easy to see this coloring satisfies A and B.

Now we assume that $s(u) \geq 2$ or $C(N_Y(u) - v) \neq S(u)$. We color u such that $\pi(u) \in S(u)$ and $\pi(u) \neq \pi(w)$ for some $w \in N_Y(u) - v$. Without loss of generality, we assume that $\pi(u) \neq \pi(w_1)$. Let $T = [r]\backslash S(u)$. We give a rainbow coloring to each thread L_{uw} $(w \notin \{w_1, v\})$ using the colors in $[r]\backslash\pi(u)$ such that $(S(u)\backslash\pi(u)) \subseteq C(L_{uw})$. If $\pi(w_1) \in S(u)$, we give a rainbow coloring to L_{uw_1} using the colors in $[r]\backslash\{\pi(u), \pi(w_1)\}$ such that the colors in $S(u)\backslash(\pi(w_1) \cup S(u))$ appear on L_{uw_1}. If $\pi(w_1) \notin S(u)$, we give a rainbow coloring to L_{uw_1} using the colors in $[r]\backslash\{\pi(u), \pi(w_1)\}$ such that $(S(u)\backslash\pi(u)) \subseteq C(L_{uw_1})$. Next we give a rainbow coloring to each thread L_{uw} $(w \in N_Y(u) - \{w_1, v\})$ by using the colors in $[r]\backslash\pi(u)$ such that $(S(u)\backslash\pi(u)) \subseteq C(L_{uw})$.

Let $M(L_{uw}) = T\backslash C(L_{uw})$. If $\pi(w_1) \in S(u)$, then $|M(L_{uw_1})| = r - s(u) - (l_{uw_1} - s(u) + 2) = r - l_{uw_1} - 2$. If $\pi(w_1) \notin S(u)$, then $|M(L_{uw_1})| = r - s(u) - (l_{uw_1} - s(u) + 1) = r - l_{uw_1} - 1$. For $w \in N_Y(u) - \{w_1, v\}$, it holds that $|M(L_{uw})| = r - s(u) - (l_{uw} - s(u) + 1) = r - l_{uw} - 1$. Recall that $\sum_{w \in N_Y(u)-\{v\}}(r - l_{uw} - 1) = r - s(u) + 1 = |T| + 1$. If $\pi(w_1) \in S(u)$, we may assume that $M(L_{uw_i}) \cap M(P_{uw_j}) = \emptyset$ for $i \neq j$. If $\pi(w_1) \notin S(u)$, we may assume that $M(L_{uw_i}) \cap M(P_{uw_j}) = \emptyset$ for $i \neq j$

except $M_R(L_{uw_1}) \cap M(P_{uw_2}) = \pi(w_1)$, so $C(P_{uw_i} \cup P_{uw_j}) = [r]$, for $i \neq j$. At last, in each thread $P_{uw}(w \in N_Y(u) - v)$, we change the neighbor of w if necessary to get a proper coloring of P_{uw}. Hence A and B hold. This completes the proof of the fact. □

Assume that $d(v) = m \geq 3$ and $N_Y(v) = \{u_1, u_2, \cdots, u_m\}$. Since $\sum_{u \in N_Y(v)} f(u) \leq r$,

$$\begin{aligned}&\sum_{u \in N_Y^g(v)} (r - 1 - s(u) - l_{uv}) + \sum_{u \in N_Y^b(v)} (r - 1 - l_{uv})\\ &\leq \sum_{u \in N_Y^g(v)} (f(u) - 1) + \sum_{u \in N_Y^b(v)} (f(u) - 1)\\ &= \sum_{u \in N_Y(v)} f(u) - d(u)\\ &\leq r - 1.\end{aligned}$$

Thus, there exist $S_1, S_2, \cdots, S_m$ such that $S_1 \cup S_2 \cup \cdots S_m \subseteq [r-1]$, where $|S_i| = r - s(u_i) - l_{u_{iv}} - 1$ if u_i is v-good and $|S_i| = r - l_{u_{iv}} - 1$ if u_i is v-bad. Moreover, we assume that $S_i \cap S_j = \phi$, for $i \neq j$.

If $u_i \in N_Y(v)$ is v-bad, then we give a rainbow coloring of $L_{u_{iv}}$ using the colors in $[r-1] \backslash S_i$. If u_i is v-good, then we choose $S(u_i) \in [r-1] \backslash S_i$ such that $|S(u_i)| = s(u_i)$. By the above fact, we can properly color each thread of $P_{u_{iw}}(w \in N_Y(u_i) - v)$ such that $S(u_i) \subseteq C(P_{uiw})$ and $C(P_{u_{iwj}} \cup P_{u_{iwk}}) = [r]$, for $w_k, w_j \in N_Y(u_i) - v$ and $w_k \neq w_j$. Give a rainbow coloring of $L_{u_{iv}}$ using the colors in $[r-1] \backslash S_i \cup S(u_i)$. In each thread $P_{u_{iv}}$, we adjust the color of the neighbor of u_i if necessary to get a proper coloring of $P_{u_{iv}}$. Thus, we have a proper coloring of H_1, moreover for any two vertices $x, y \in V(H_1)$ and any path $Q(x, y)$ between x and y in H_1, $C(Q(x, y)) = [r]$. Thus, we get an r-acyclic coloring of G, which is a contradiction. Therefore, $\sum_{u \in N_Y(v)} f(u) \geq r + 2$. This completes the proof of Claim 4. □

We complete the proof using discharging method. Let $d(v)$ be the initial charge on the vertex $v \in V(G)$. We move charge from vertex to vertex, without changing the total according to the following rules:

a. Every $v \in Y$ gives each weak 2-neighbor the amount $\frac{3}{5r-1}$.
b. Every $v \in Y$ gives each weak Y-neighbor the amount $\frac{3f(v)+(r+2)(d(v)-3)}{(5r-1)d(v)}$.

Claim 5 *Every $v \in Y$ receives from its weak Y-neighbors at least $\frac{r+2}{5r-1}$.*

Proof of Claim 5 If every $u \in N_Y(v)$ sends v at least $\frac{f(u)}{5r-1}$, then v receives from $N_Y(v)$ at least $\frac{1}{5r-1} \sum_{u \in N_Y(v)} f(u) \geq \frac{r+2}{5r-1}$, by Claim 4.

Otherwise, for some $u \in N_Y(v)$, it holds that

$$\frac{3f(u) + (r+2)(d(u) - 3)}{(5r-1)d(u)} < \frac{f(u)}{5r-1},$$

that is,

$$(r+2)(d(u)-3) < f(u)(d(u)-3),$$

so we conclude that $d(u) \geq 4$ and $f(u) > r+2$. Thus, u gives to v at least

$$\frac{3f(u)+(r+2)(d(u)-3)}{(5r-1)d(u)} \geq \frac{3(r+2)+(r+2)(d(u)-3)}{(5r-1)d(u)} = \frac{r+2}{5r-1}.$$

Moreover, all other amounts to v are nonnegative, since if $y \in N_Y(v)$, then $d(w) \geq 3$ and $f(w) \geq 1$. □

Let $\hat{d}(v)$ denote the new charge of v after discharging.

Claim 6 *After the discharging, it holds that* $\hat{d}(v) \geq 2 + \frac{4d(v)-2}{5r-1}$, *for all* $v \in V(G)$.

Proof of Claim 6 If $d(v) = 2$, then v sends out zero and receives $\frac{3}{5r-1}$ from each of its two weak Y-neighbors, so $\hat{d}(v) = 2 + \frac{6}{5r-1} = 2 + \frac{4d(v)-2}{5r-1}$.

Now consider $v \in Y$. By the discharging rule, v sends out $\frac{3}{5r-1}\sum_{w\in N_Y(v)} l_{vw}$ to its weak 2-neighbors and $\frac{3f(v)+(r+2)(d(v)-3)}{5r-1}$ to its weak Y-neighbors. By Claim 5, v receives at least $\frac{r+2}{5r-1}$ from its weak Y-neighbors, so

$$\begin{aligned}\hat{d}(v) &\geq d(v) - \frac{3}{5r-1}\sum_{w\in N_Y(v)} l_{wv} - \frac{3f(v)+(r+2)(d(v)-3)}{5r-1} + \frac{r+2}{5r-1}\\ &= d(v) - \frac{3}{5r-1}\left[-r + \sum_{w\in N_Y(v)}(r - l_{wv} + l_{wv} - 1)\right] - \frac{(r+2)(d(v)-4)}{5r-1}\\ &= \frac{rd(v)+7r+8}{5r-1}.\end{aligned}$$

Since $d(v) \geq 3$ and $r \geq 4$, we have

$$rd(v)+7r+8 = (d(v)-3)r + 10 + 10r - 2 \geq 4d(v) - 2 + 10r - 2.$$

Therefore,

$$\frac{(r+2)d(v)+7r+4}{5r-1} \geq 2 + \frac{4d(v)-2}{5r-1},$$

and the proof of Claim 6 completes. □

Now we have that $\hat{d}(v) \geq 2 + \frac{4d(v)-2}{5r-1}$, for all $v \in V(G)$. It follows that

$$\begin{aligned}2|E(G)| &= \sum_{v\in V(G)} \hat{d}(v) \geq \sum_{v\in V(G)}\left(2 + \frac{4d(v)-2}{5r-1}\right)\\ &= 2\left(1 - \frac{1}{5r-1}\right)|V(G)| + \frac{8}{5r-1}|E(G)|,\end{aligned}$$

and hence

$$\frac{5r-2}{5r-1}|V(G)| \leq \frac{5r-5}{5r-1}|E(G)|.$$

Thus, the average degree of G is at least $2 + \frac{6}{5r-5}$, which gives a contradiction. This completes the proof of Theorem 4.2.

6 Remark

Let $r \geq 4$ be an integer. We propose the following problems for further research.

Problem 1 *What is the best upper bound for $a_r(G)$ when G is a planar graph ?*

Problem 2 *What is the best upper bound for $a_r(G)$ when G is a planar graph containing no copy of $K_{2,n}$ or even no C_4?*

References

Agnarsson G, Halldórsson MM (2003) Coloring powers of planar graphs. SIAM J Discret Math 16(4):651–662

Albertson MO, Berman DM (1976) The acyclic chromatic number. In Proceedings of Seventh Southeastern Conference on Combinatorics. Graph Theory and Computing, Utilitas Mathematicsa Inc., Winniper, pp 51–60

Alon N, McDiarmid C, Reed B (1991) Acyclic coloring of graphs. Random Struct Algorithms 2:277–288

Bondy JA, Murty USR (1976) Graph theory with applications. Macmillan Press[M], New York

Borodin OV (1979) On acyclic colorings of planar graphs. Discret Math 25:211–236

Borodin OV, Woodall DR (1995) Thirteen coloring numbers of outerplane graphs. Bull Inst Combin Appl 14:87–100

Burnstein MI (1979) Every 4-valent graph has an acyclic 5-coloring. Soobsc Akad Nauk Grucin 93:21–24 (in Russian)

Cai J, Wang G, Yan G (2013) The generalized acyclic chromatic number of graphs with large girth. Acta Math Sinica 56(1):27–30

Dieng Y, Hocquard H, Naserasr R (2010) Acyclic colorings of graph with maximum degree. LaBRI, Manuscript

Duffin J (1965) Topology of series-parallel networks. J Math Anal Appl 10:303–318

Fertin G, Raspaud A (2005) Acyclic colorings of graphs of maximum degree Δ, European conference on combinatorics graph theory and applications, pp 389–396

Fertin G, Raspaud A (2008) Acyclic colorings of graphs of maximum degree five: nine colors are enough. Inf Process Lett 105:65–72

Greenhill C, Pikhurko O (2005) Bounds on the generalized acyclic chromatic number of bounded degree graphs. Graphs Combin 21:407–419

Grünbaum B (1973) Acyclic colorings of planar graphs. Israel J Math 14:390–408

Hocquard H (2011) Graphs with maximum degree 6 are acyclically 11-colorabe. Inf Process Lett 111:748–753

Kostochka AV, Stocker C (2011) Graphs with maximum degree 5 are acyclically 7-colorable. Ars Mathematica Contemporanea 4:153–164

Skulrattanakulchai S (2004) Acyclic coloring of subcubic graphs. Inf Process Lett 92:161–167

Yadav K, Varagani S, Kothapalli K, Venkaiah VCh (2009) Acyclic vertex coloring of graphs of maximum degree Δ. Proceedings of Indian Mathematical Society

Zhang X, Wang G, Yu Y, Li J, Liu G (2012) On r-acyclic edge colorings of planar graphs. Discret Appl Math 160:2048–2053