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A Conjecture on Equitable Vertex Arboricity of Graphs

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Abstract. Wu, Zhang and Li [4] conjectured that the set of vertices of any simple graph G can be equitably partitioned into $\lceil(\Delta(G)+1)/2\rceil$ subsets so that each of them induces a forest of G. In this note, we prove this conjecture for graphs G with $\Delta(G) \geq |G|/2$.

1. Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of G, respectively. By $\alpha'(G)$ and G^c, we denote the largest size of a matching in the graph G and the completement graph of G. For other basic undefined concepts we refer the reader to [1].

The *vertex-arboricity* $a(G)$ of a graph G is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a forest. This notation was first introduced by Chartrand, Kronk and Wall [2] in 1968, who named it point-arboricity and proved that $a(G) \leq \lceil(\Delta(G)+1)/2\rceil$ for every graph G. Recently, Wu, Zhang and Li [4] introduced the equitable version of vertex arboricity. If the set of vertices of a graph G can be equitably partitioned into k subsets (i.e. the size of each subset is either $\lceil |G|/k\rceil$ or $\lfloor |G|/k\rfloor$) such that each subset of vertices induce a forest of G, then we call that G admits an *equitable k-tree-coloring*. The minimum integer k such that G has an equitable k-tree-coloring is the *equitable vertex arboricity* $a_{eq}(G)$ of G. As an extension of the result of Chartrand, Kronk and Wall on vertex arboricity, Wu, Zhang and Li [4] raised the following conjecture and they proved it for complete bipartite graphs, graphs with maximum average degree less than 3, and graphs with maximum average degree less than 10/3 and maximum degree at least 4.

Conjecture 1.1. $a_{eq}(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ *for every simple graph G.*

In this note, we establish this conjecture for graphs G with $\Delta(G) \geq |G|/2$.

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2. Main results and the proofs

For convenience, we set $\Gamma(G) = \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ throughout this section. To begin with, we introduce two useful lemmas of Chen, Lih and Wu.

Lemma 2.1. [3] *If G is a disconnected graph, then $\alpha'(G) \geq \delta(G)$.*

Lemma 2.2. [3] *If G is a connected graph such that $|G| > 2\delta(G)$, then $\alpha'(G) \geq \delta(G)$.*

Lemma 2.3. *If G is a connected graph with $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G) + 1$.*

Proof. Consider the longest path $P = [v_0v_1 \dots v_k]$ in G. We see immediately that $N(v_0) \subseteq V(P)$, because otherwise we would construct a longer path. Let v_i be a neighbor of v_0 so that i is maximum. Since $\delta(G) \geq 2$, $C = [v_0v_1 \dots v_iv_0]$ is a cycle of length $i + 1 \geq \delta(G) + 1$. □

In what follows, we prove three independent theorems, which together imply Conjecture 1.1 for graphs G with $\Delta(G) \geq |G|/2$.

Theorem 2.4. *If $\Delta(G) \geq \frac{2}{3}|G| - 1$, then $a_{eq}(G) \leq \Gamma(G)$.*

Proof. If $\Delta(G) = |G| - 1$, then $a_{eq}(G) \leq \Gamma(G)$ and this upper bound can be attained by the complete graphs, since we can arbitrarily partition $V(G)$ into $\Gamma(G)$ subsets so that each of them consists of one or two vertices, thus we assume $\Delta(G) \leq |G| - 2$. Since $\Delta(G) + \delta(G^c) = |G| - 1$ and $\Delta(G) \geq \frac{2}{3}|G| - 1$, $|G^c| \geq 3\delta(G^c)$ and $\delta(G^c) \geq |G^c| - 2\Gamma(G)$. By Lemmas 2.1 and 2.2, we have $\alpha'(G^c) \geq \delta(G^c)$, so there exists a matching $M = [x_1y_1, \dots, x_\delta y_\delta]$ of size $\delta := \delta(G^c)$ in G^c. Since $|G^c| \geq 3\delta(G^c)$, $|V(G^c) \setminus V(M)| \geq \delta$, thus we can select δ distinct vertices $z_1, \dots, z_\delta$ among $V(G^c) \setminus V(M)$. Denote $\beta = |G^c| - 2\Gamma(G)$ and $\mu = 3\Gamma(G) - |G^c|$. Since $|G| - 2 \geq \Delta(G) \geq \frac{2}{3}|G| - 1$, $\beta, \mu \geq 0$. We now use β colors to color 3β vertices of G so that the i-th color class consists of the three vertices x_i, y_i and z_i, and then use μ colors to color the remaining 2μ vertices of G so that each color class consists of two vertices. One can check that each color class of G induces a (linear) forest and the coloring of G is equitable. Therefore, $a_{eq}(G) \leq \beta + \mu = \Gamma(G)$. □

Theorem 2.5. *If $\frac{2}{3}|G| - 1 > \Delta(G) \geq \frac{2}{3}|G| - 2$, then $a_{eq}(G) \leq \Gamma(G)$.*

Proof. If $|G| \leq 3$, then the result is trivial, so we assume $|G| \geq 4$. If $|G| = 3k$, then $\Delta(G) = 2k - 2$ and $\delta(G^c) = k + 1$, since $\frac{2}{3}|G| - 1 > \Delta(G) \geq \frac{2}{3}|G| - 2$ and $\Delta(G) + \delta(G^c) = |G| - 1$. By Lemmas 2.1 and 2.2, we have $\alpha'(G^c) \geq \delta(G^c) > k$. Let $M_1 = [x_{11}y_{11}, \dots, x_{1k}y_{1k}]$ be a matching of G^c. We now partition the vertices of G into k subsets so that the i-th subset consists of the vertices x_{1i}, y_{1i} and one another vertex different from the vertices in $V(M_1)$. It is easy to check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \leq k = \Gamma(G)$. If $|G| = 3k + 2$, then $\Delta(G) = 2k$ and $\delta(G^c) = k + 1$. This also implies, by Lemma 2.1 and 2.2, that $\alpha'(G^c) \geq \delta(G^c) > k$. Let $M_2 = [x_{21}y_{21}, \dots, x_{2k}y_{2k}]$ be a matching of G^c. We now partition the vertices of G into $k + 1$ subsets so that the i-th subset with $i \leq k$ consists of the vertices x_{2i}, y_{2i} and one another vertex different from the vertices in $V(M_2)$ and the $(k+1)$-th subset consists of two vertices in $V(G) \setminus V(M_2)$. It is easy to check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \leq k + 1 = \Gamma(G)$. If $|G| = 3k + 1$, then $\Delta(G) = 2k - 1$ and $\delta(G^c) = k + 1$. By Lemmas 2.1 and 2.2, we have $\alpha'(G^c) \geq \delta(G^c)$. Let $M_3 = [x_{31}y_{31}, \dots, x_{3(k+1)}y_{3(k+1)}]$ be a matching of G^c. If x_{31} has a neighbor in G^c among $\{x_{32}, y_{32}, \dots, x_{3(k+1)}, y_{3(k+1)}\}$ (without loss of generality, assume that $x_{31}x_{32} \in E(G^c)$), then we can partition the vertices of G into k subsets so that the the first subset consists of the four vertices x_{31}, y_{31}, x_{32} and y_{32} and the i-th subset with $2 \leq i \leq k$ consists of the vertices $x_{3(i+1)}, y_{3(i+1)}$ and one another vertex different from the vertices in $V(M_2)$. One can check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \leq k = \Gamma(G)$. Hence, we shall assume that $x_{31}x_{3j}, x_{31}y_{3j} \notin E(G^c)$ for each $2 \leq j \leq k + 1$. Since $d_{G^c}(x_{31}) \geq \delta(G^c) = k + 1$ and $|G^c| = 3k + 1$, $x_{31}z \in E(G^c)$ for each $z \in V(G^c) \setminus V(M_3)$. Similarly, we shall assume that $y_{31}z \in E(G^c)$ for each $z \in V(G^c) \setminus V(M_3)$, because otherwise we return to a case we have considered before. We now partition the vertices of G into k subsets so that the the first subset consists of the two vertices x_{31}, y_{31} and two distinct vertices $z_1, z_2 \in V(G^c) \setminus V(M_3)$ and the i-th subset with

$2 \le i \le k$ consists of the vertices x_{3i}, y_{3i} and one another vertex different from the vertices in $V(M_2)$. One can again check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \le k = \Gamma(G)$. □

Theorem 2.6. *If $\frac{2}{3}|G| - 2 > \Delta(G) \ge \frac{1}{2}|G|$, then $a_{eq}(G) \le \Gamma(G)$.*

Proof. Since $\Delta(G) + \delta(G^c) = |G| - 1$ and $\Delta(G) \ge \frac{1}{2}|G|$, $|G^c| \ge 2\delta(G^c) + 2$. We split our proof into two cases.

Case 1: G^c is connected.

Since $|G^c| \ge 2\delta(G^c) + 2 > 2\delta(G^c)$, there exists a path $P = [x_0, x_1, \ldots, x_{2\delta}]$ of length $2\delta := 2\delta(G^c)$ in G^c (see [1, Exercise 4.2.9]). Denote $\beta = |G| - 3\Gamma(G)$ and $\mu = 4\Gamma(G) - |G|$. Since $\frac{2}{3}|G| - 2 > \Delta(G) \ge \frac{1}{2}|G|$, $\beta, \mu \ge 1$. Since $2\Gamma(G) > \Delta(G) = |G| - \delta(G^c) - 1$, $\delta(G^c) \ge |G| - 2\Gamma(G) = 2\beta + \mu$. Thus, the vertex sets $V_i = \{x_{4i-4}, x_{4i-3}, x_{4i-2}, x_{4i-1}\}$ with $1 \le i \le \beta$ and $U_i = \{x_{4\beta+2i-2}, x_{4\beta+2i-1}\}$ with $1 \le i \le \mu$ are well defined. Note that $V(P) \supseteq \bigcup_{i=1}^{\beta} V_i \cup \bigcup_{i=1}^{\mu} U_i$. Since $|G| - 4\beta - 3\mu = \mu$, $|G - \bigcup_{i=1}^{\beta} V_i - \bigcup_{i=1}^{\mu} U_i| = \mu$. Let $V(G) \setminus \left(\bigcup_{i=1}^{\beta} V_i \cup \bigcup_{i=1}^{\mu} U_i\right) = \{y_1, \ldots, y_\mu\}$ and let $W_i = U_i \cup \{y_i\}$ with $1 \le i \le \mu$. We now partition the vertices of G into $\beta + \mu$ subsets $V_1, \ldots, V_\beta, W_1, \ldots, W_\mu$. One can check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \le \beta + \mu = \Gamma(G)$.

Case 2: G^c is disconnected.

Let $G_1, \ldots, G_t$ be the components of G^c with $t \ge 2$. Since $\Delta(G) + \delta(G^c) = |G| - 1$ and $\Delta(G) < \frac{2}{3}|G| - 2$, $\min\{\delta(G_1), \ldots.\delta(G_t)\} \ge \delta(G^c) \ge 2$. This implies, by Lemma 2.3, that G_i contains a cycle $C_i = [x_0^i x_1^i \ldots x_{l(C_i)}^i x_0^i]$ of length $l(C_i) + 1 \ge \delta(G_i) + 1$ for each $1 \le i \le t$. Let $V_j^i = \{x_{4j-4}^i, x_{4j-3}^i, x_{4j-2}^i, x_{4j-1}^i\}$ with $1 \le i \le t$ and $1 \le j \le n_i$, in which $4n_i - 1 \le l(C_i)$ and $n_1 + \ldots + n_t = \beta$. Note that V_j^i is well defined by Claim 2.7.

Claim 2.7. $\sum_{i=1}^{t} \left\lfloor \frac{\delta(G_i)+1-4n_i}{2} \right\rfloor \ge \mu$.

Proof. Otherwise, $\delta(G^c) \le \frac{1}{2}t\delta(G^c) \le \frac{1}{2}\sum_{i=1}^{t} \delta(G_i) \le \sum_{i=1}^{t} \left\lfloor \frac{\delta(G_i)+1}{2} \right\rfloor < 2\beta + \mu = |G| - 2\Gamma(G) \le |G| - \Delta(G) - 1$, contradicting to $\Delta(G) + \delta(G^c) = |G| - 1$. □

We conclude, by Claim 2.7, that there exists a matching M of size at least μ in $G^c - \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} V_j^i$. Therefore, we can partition the vertices of G into $\beta + \mu$ subsets so that the i-th subset with $1 \le i \le \mu$ consists of a pair of vertices matched under M and one vertex in $V(G) \setminus \left(V(M) \cup \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} V_j^i\right)$ and the last β subsets are $V_1^1, \ldots, V_{n_1}^1, \ldots, V_1^t, \ldots, V_{n_t}^t$. One can check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \le \beta + \mu = \Gamma(G)$. □

From the proofs of the above three theorems, we can immediately deduce the following conclusion.

Conclusion 2.8. *If G is a simple graph with $\Delta(G) \ge \frac{1}{2}|G|$, then $V(G)$ can be equitably partitioned into $\Gamma(G)$ subsets so that each of them induces a linear forest of G, i.e., the equitable linear vertex arboricity of G is at most $\Gamma(G)$, and the upper bound $\Gamma(G)$ is sharp.*

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