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ON THE SUM OF POWERS OF THE DEGREES OF GRAPHS

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Abstract

For positive integers p and q, let $\mathcal{G}_{p,q}$ be a class of graphs such that $|E(G)| \leq p|V(G)| - q$ for every $G \in \mathcal{G}_{p,q}$. In this paper, we consider the sum of the kth powers of the degrees of the vertices of a graph $G \in \mathcal{G}_{p,q}$ with $\Delta(G) \geq 2p$. We obtain an upper bound for this sum that is linear in Δ^{k-1}. These graphs include the planar, 1-planar, t-degenerate, outerplanar, and series-parallel graphs.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. For a graph G, by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ we denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For convenience, we set $n = |V(G)|$, $m = |E(G)|$, $\Delta = \Delta(G)$ and $\delta = \delta(G)$ throughout this paper. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbours of v in G and let $d_G(v) = |N_G(v)|$ be the degree of v in G. For a positive integer k, the sum of the kth powers of the degrees of the vertices of G, denoted by $\sum_k(G)$, is the value of $\sum_{v \in V(G)} d_G^k(v)$. For other undefined notation and terminology we refer the reader to [4].

In this paper, we consider the sum of the kth powers of the degrees of the vertices of certain classes of graphs. First of all, it is trivial that $\sum_1(G) = 2m$ for every graph G. For $k \geq 2$, de Caen [2] proved that

$$\sum\nolimits_2(G) \leq m\Big(\frac{2m}{n-1} + n - 2\Big).$$

This bound was generalised to hypergraphs by Bey [1] and improved to

$$m\Big(\frac{2m}{n-1} + \frac{n-2}{n-1}\Delta + (\Delta - \delta)\Big(1 - \frac{\Delta}{n-1}\Big)\Big)$$

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by Das [8]. De Caen's inequality was also used by Li and Pan [7] to provide an upper bound on the largest eigenvalue of the Laplacian of a graph. In [9], Cioabă generalised Das' bound to

$$\sum\nolimits_{k+1}(G) \leq \frac{2m}{n}\Big(\sum\nolimits_{k}(G) + (n-1)(\Delta^k - \delta^k)\Big) - \frac{\Delta^k - \delta^k}{n}\sum\nolimits_{2}(G).$$

Now we restrict G to be a planar graph, that is, a graph that can be drawn in the plane so that there are no crossed edges. Harant *et al.* [5] proved that

$$\sum\nolimits_{k}(G) \leq \frac{(6-\delta)\Delta^k + (\Delta-6)\delta^k}{\Delta-\delta}\Big(n - \frac{12}{6-\delta}\Big) + \frac{12}{6-\delta}\delta^k \tag{1}$$

if $\Delta(G) \geq 6$.

The aim of this paper is to extend this inequality to a larger class. For our purpose, we define $\mathcal{G}_{p,q}$ to be a class of graphs such that $|E(G)| \leq p|V(G)| - q$ for every $G \in \mathcal{G}_{p,q}$, where p and q are positive integers. The following theorem is the main result.

THEOREM 1.1. *For every simple graph $G \in \mathcal{G}_{p,q}$ with $\Delta(G) \geq 2p$,*

$$\sum\nolimits_{k}(G) \leq \frac{(2p-\delta)\Delta^k + (\Delta-2p)\delta^k}{\Delta-\delta}\Big(n - \frac{2q}{2p-\delta}\Big) + \frac{2q}{2p-\delta}\delta^k.$$

It is easy to check that Theorem 1.1 (with $p = 3$ and $q = 6$) implies (1). Moreover, the implicit condition $\Delta(G) \geq 2p$ in Theorem 1.1 is necessary. This is because there exists a $(2p-1)$-regular graph G with order at least 4 such that $e \leq p(n-2)$, where $e = |E(G)|$; thus the kth powers of the degrees of the vertices of G are exactly $\Delta^k n$. However, the leading coefficient of n in Theorem 1.1 is at most $(2p-\delta)\Delta^{k-1} + o(\Delta^{k-1})$.

A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge. Pach and Tóth [6] proved that a simple 1-planar graph on n vertices has at most $4n - 8$ edges. This immediately yields a corollary of Theorem 1.1.

COROLLARY 1.2. *For every simple 1-planar graph G with $\Delta \geq 8$,*

$$\sum\nolimits_{k}(G) \leq \frac{(8-\delta)\Delta^k + (\Delta-8)\delta^k}{\Delta-\delta}\Big(n - \frac{16}{8-\delta}\Big) + \frac{16}{8-\delta}\delta^k.$$

Since every 7-regular 1-planar graph (for the existence of such a graph, see [3]) has $\sum_k(G) = \Delta^k n$, but the coefficient of n in Corollary 1.2 is at most $(8-\delta)\Delta^{k-1} + o(\Delta^{k-1})$, the lower bound 8 for Δ in Corollary 1.2 is necessary.

A graph G is t-degenerate if $\delta(H) \leq t$ for every $H \subseteq G$. If G is a t-degenerate graph, then $G_1 := G$ can be reduced to the null graph by the following steps.

Step i $(1 \leq i \leq n-t)$	Remove a vertex of degree at most t from G_i, and denote the resulting graph by G_{i+1}.
Step $n-t+1$	Remove all the vertices of G_{n-t+1}.

In each of the first $n - t$ steps, at most t edges are removed, and in the last step (note that G_{n-t+1}, which has t vertices, may be a complete graph), at most $t(t-1)/2$ edges are removed. Therefore,

$$|E(G)| \leq t(|V(G)| - t) + \frac{t(t-1)}{2} = t|V(G)| - \frac{t(t+1)}{2}.$$

Setting $p = t$ and $q = t(t+1)/2$ in Theorem 1.1, we immediately have the following corollary.

COROLLARY 1.3. *For every simple t-degenerate graph G with $\Delta \geq 2t$,*

$$\sum\nolimits_k(G) \leq \frac{(2t-\delta)\Delta^k + (\Delta - 2t)\delta^k}{\Delta - \delta}\Big(n - \frac{t^2+t}{2t-\delta}\Big) + \frac{t^2+t}{2t-\delta}\delta^k.$$

A graph is series-parallel if it may be turned into K_2 by a sequence of the following operations: (a) replacement of a pair of parallel edges with a single edge that connects their common endpoints, (b) replacement of a pair of edges incident to a vertex of degree two with a single edge. By this definition, one can see that every series-parallel graph is 2-degenerate and contains at least two vertices of degree at most 2. Let G be a series-parallel graph. If $\Delta = 3$, then it is easy to verify that $\sum_k(G) \leq 2^{k+1} + (n-2)3^k$. If $\Delta \geq 4$, then we can obtain an upper bound for the kth powers of the degrees of the vertices of G as in Corollary 1.3 by setting $t = 2$ there. Combining these two cases, we have the following corollary.

COROLLARY 1.4. *For every simple series-parallel graph G with $\Delta \geq 3$,*

$$\sum\nolimits_k(G) \leq \frac{(4-\delta)\Delta^k + (\Delta - 4)\delta^k}{\Delta - \delta}\Big(n - \frac{6}{4-\delta}\Big) + \frac{6}{4-\delta}\delta^k.$$

Since outerplanar graphs are 2-degenerate, the bound in Corollary 1.4 also applies to outerplanar graphs with $\Delta \geq 4$.

2. Proof of Theorem 1.1

Since $\Delta(G) \geq 2p$, and $G \in \mathcal{G}_{p,q}$ yields that $\delta \leq 2p - 1 < 2p$, we have $1 \leq \delta < \Delta$. It is easy to see that Theorem 1.1 holds for $k = 1$. Thus in the following we let $k \geq 2$.

By n_i we denote the number of vertices of degree i of a graph G. It holds trivially that $\sum_{\delta \leq i \leq \Delta} n_i = n$. Since G has at most $pn - q$ edges, $\sum_{\delta \leq i \leq \Delta} in_i \leq 2pn - 2q$. Consider the following program $\mathcal{P}$.

$$\begin{aligned} \max: \quad & f(x_\delta, \ldots, x_\Delta) = \sum_{\delta \leq i \leq \Delta} i^k x_i \\ \text{such that} \quad & \sum_{\delta \leq i \leq \Delta} x_i = n, \\ & \sum_{\delta \leq i \leq \Delta} i x_i \leq 2pn - 2q, \\ & x_i \geq 0 \ (x_i \text{ real}, i = \delta, \ldots, \Delta). \end{aligned}$$

Let $(x_\delta, \ldots, x_\Delta)$ be an optimal solution of $\mathcal{P}$. It follows that $\sum_k(G) \leq f(x_\delta, \ldots, x_\Delta)$.

CLAIM 2.1. *If $\Delta \geq 2p+2$, then $x_i = 0$ for $2p+1 \leq i \leq \Delta - 1$.*

To prove the claim, assume that $x_i > 0$ for some $i \in 2p+1, \ldots, \Delta - 1$. Let $y_j = x_j$ for $j \in \{\delta, \ldots, \Delta - 1\} \backslash \{i, 2p\}$, $y_i = 0$, $y_{2p} = x_{2p} + (1 - (2p-i)/(2p-\Delta))x_i$ and $y_\Delta = x_\Delta + ((2p-i)/(2p-\Delta))x_i$. Then $\sum_{\delta \leq j \leq \Delta} y_j = n$, $\sum_{\delta \leq j \leq \Delta} jy_j \leq 2pn - 2q$, $y_j \geq 0$ for $j = \delta, \ldots, \Delta$ and

$$\begin{aligned} f(y_\delta, \ldots, y_\Delta) - f(x_\delta, \ldots, x_\Delta) &= \Big(-i^k + \frac{2p-i}{2p-\Delta}\Delta^k + \Big(1 - \frac{2p-i}{2p-\Delta}\Big)(2p)^k\Big)x_i \\ &= \Big((2p)^k - i^k + \frac{i-2p}{\Delta-2p}(\Delta^k - (2p)^k)\Big)x_i \\ &= (i-2p)(((2p)^{k-1} + (2p)^{k-2}\Delta + \cdots + 2p\Delta^{k-2} + \Delta^{k-1}) \\ &\qquad - ((2p)^{k-1} + (2p)^{k-2}i + \cdots + 2pi^{k-2} + i^{k-1}))x_i \\ &> 0 \end{aligned}$$

for $k \geq 2$, a contradiction.

CLAIM 2.2. *If $\delta \leq 2p-2$, then $x_i = 0$ for $\delta + 1 \leq i \leq 2p-1$.*

Assume that $x_i > 0$ for an $i \in \{\delta+1, \ldots, 2p-1\}$. Let $y_j = x_j$ for $j \in \{\delta+1, \ldots, \Delta\} \backslash \{i, 2p\}$, $y_i = 0$, $y_\delta = x_\delta + ((2p-i)/(2p-\delta))x_i$ and

$$y_{2p} = x_{2p} + \Big(1 - \frac{2p-i}{2p-\delta}\Big)x_i.$$

Then $\sum_{\delta \leq j \leq \Delta} y_j = n$, $\sum_{\delta \leq j \leq \Delta} jy_j \leq 2pn - 2q$, $y_j \geq 0$ for $j = \delta, \ldots, \Delta$ and

$$\begin{aligned} f(y_\delta, \ldots, y_\Delta) - f(x_\delta, \ldots, x_\Delta) &= \Big(-i^k + \frac{2p-i}{2p-\delta}\delta^k + \Big(1 - \frac{2p-i}{2p-\delta}\Big)(2p)^k\Big)x_i \\ &= \Big((2p)^k - i^k + \frac{2p-i}{2p-\delta}(\delta^k - (2p)^k)\Big)x_i \\ &= (2p-i)(((2p)^{k-1} + (2p)^{k-2}i + \cdots + 2pi^{k-2} + i^{k-1}) \\ &\qquad - ((2p)^{k-1} + (2p)^{k-2}\delta + \cdots + 2p\delta^{k-2} + \delta^{k-1}))x_i \\ &> 0 \end{aligned}$$

for $k \geq 2$, a contradiction.

CLAIM 2.3. *If $\Delta \geq 2p+1$, then, among $x_\delta, \ldots, x_\Delta$, only x_δ, x_{2p} and x_Δ may be nonzero; if $\Delta = 2p$, then, among $x_\delta, \ldots, x_\Delta$, only x_δ and x_Δ may be nonzero.*

We only prove the first part of this claim, since the proof of the second part is similar. Recall that $\delta \leq 2p-1$. If $\Delta \geq 2p+2$ and $\delta \leq 2p-2$, then by Claims 2.1 and 2.2, we have $x_i = 0$ for $i \in \{\delta+1, \ldots, \Delta-1\} \backslash \{2p\}$, and this claim holds. If $2p \leq \Delta \leq 2p+1$ and $\delta \leq 2p-2$, then by Claim 2.2, $x_i = 0$ for $\delta+1 \leq i \leq 2p-1$, so only x_δ, x_{2p} and x_Δ may be nonzero. If $\Delta \geq 2p+2$ and $\delta = 2p-1$, then by Claim 2.1, $x_i = 0$ for $2p+1 \leq i \leq \Delta - 1$, so only x_δ, x_{2p} and x_Δ may be nonzero. If $2p \leq \Delta \leq 2p+1$ and $\delta = 2p-1$, then this claim follows trivially.

We come back to the proof of Theorem 1.1. If $\Delta \geq 2p+1$, then by Claim 2.3 and the restrictions of $\mathcal{P}$, we obtain that $x_\delta + x_{2p} + x_\Delta = n$ and $\delta x_\delta + 2p x_{2p} + \Delta x_\Delta \leq 2pn - 2q$, which imply that $(2p-\delta)x_\delta \geq 2q + (\Delta - 2p)x_\Delta$ and

$$x_{2p} = n - x_\delta - x_\Delta \leq n - \frac{2q}{2p-\delta} - \frac{\Delta-\delta}{2p-\delta}x_\Delta.$$

Furthermore, since

$$(2p-\delta)x_\Delta = (2p-\delta)n - (2p-\delta)x_\delta - (2p-\delta)x_{2p}$$

and

$$(\Delta - 2p)x_\Delta \leq (2p-\delta)x_\delta - 2q = (2p-\delta)n - (2p-\delta)x_\Delta - (2p-\delta)x_{2p} - 2q,$$

we have

$$(\Delta-\delta)x_\Delta \leq (2p-\delta)n - (2p-\delta)x_{2p} - 2q.$$

It follows that $x_\Delta \leq ((2p-\delta)n - (2p-\delta)x_{2p} - 2q)/(\Delta-\delta)$ and

$$\begin{aligned}
f(x_\delta, \ldots, x_\Delta) &= \delta^k x_\delta + (2p)^k x_{2p} + \Delta^k x_\Delta \\
&= \delta^k(n - x_{2p} - x_\Delta) + (2p)^k x_{2p} + \Delta^k x_\Delta \\
&= \delta^k n + ((2p)^k - \delta^k)x_{2p} + (\Delta^k - \delta^k)x_\Delta \\
&\leq \delta^k n + ((2p)^k - \delta^k)\Big(n - \frac{2q}{2p-\delta} - \frac{\Delta-\delta}{2p-\delta}x_\Delta\Big) + (\Delta^k - \delta^k)x_\Delta \\
&= (2p)^k\Big(n - \frac{2q}{2p-\delta}\Big) + \frac{2q}{2p-\delta}\delta^k + \Big(\Delta^k - \delta^k - \frac{(2p)^k - \delta^k}{2p-\delta}(\Delta-\delta)\Big)x_\Delta \\
&\leq (2p)^k\Big(n - \frac{2q}{2p-\delta}\Big) + \frac{2q}{2p-\delta}\delta^k \\
&\qquad + \Big(\Delta^k - \delta^k - \frac{(2p)^k - \delta^k}{2p-\delta}(\Delta-\delta)\Big)\frac{(2p-\delta)n - 2q}{\Delta-\delta} \\
&= \frac{(2p-\delta)\Delta^k + (\Delta-2p)\delta^k}{\Delta-\delta}\Big(n - \frac{2q}{2p-\delta}\Big) + \frac{2q}{2p-\delta}\delta^k.
\end{aligned}$$

If $\Delta = 2p$, then by Claim 2.3 and the restrictions of $\mathcal{P}$, we obtain that $x_\delta + x_{2p} = n$ and $\delta x_\delta + 2p x_{2p} \leq 2pn - 2q$. It follows that $(2p-\delta)x_\delta \geq 2q$ and $x_{2p} = n - x_\delta \leq n - 2q/(2p-\delta)$, which implies that

$$\begin{aligned}
f(x_\delta, \ldots, x_\Delta) &= \delta^k x_\delta + (2p)^k x_{2p} \\
&= \delta^k(n - x_{2p}) + (2p)^k x_{2p} \\
&= \delta^k n + ((2p)^k - \delta^k)x_{2p} \\
&\leq \delta^k n + ((2p)^k - \delta^k)\Big(n - \frac{2q}{2p-\delta}\Big) \\
&= (2p)^k\Big(n - \frac{2q}{2p-\delta}\Big) + \frac{2q}{2p-\delta}\delta^k.
\end{aligned}$$

This completes the proof of Theorem 1.1.

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