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The structure of plane graphs with independent crossings and its applications to coloring problems

Research Article

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Abstract: If a graph G has a drawing in the plane in such a way that every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short. In this paper, the structure of IC-planar graphs with minimum degree at least two or three is studied. By applying their structural results, we prove that the edge chromatic number of G is Δ if $\Delta \geq 8$, the list edge (resp. list total) chromatic number of G is Δ (resp. $\Delta + 1$) if $\Delta \geq 14$ and the linear arboricity of G is $\lceil \Delta/2 \rceil$ if $\Delta \geq 17$, where G is an IC-planar graph and Δ is the maximum degree of G.

MSC: 05C10, 05C15

Keywords: Independent crossing • IC-planar graph • Light edge • Coloring • Discharging

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, edge set, minimum degree and maximum degree of a graph G, respectively. The *crossing number* of G, denoted by $\mathrm{cr}(G)$, is the minimum possible number of crossings in a drawing of G in the plane. A *k-alternating cycle* in a graph G is a cycle of even length in which alternate vertices have degree k in G. Throughout this paper, a k-, k^+- and k^--*vertex* (resp. *face*) in a graph is a vertex (resp. face) of degree k, at least k and at most k, respectively. Any undefined notation follows that of Bondy and Murty [3].

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The famous Four Color Theorem states that every graph with crossing number zero (i.e. planar graph) is 4-colorable. By this fact one can easily see that $\chi(G) \leq 4 + \mathrm{cr}(G)$ for every graph G, see [8]. Thus, for a class $\mathcal{G}$ of graphs, if we can prove that the crossing number of any given graph $G \in \mathcal{G}$ is bounded by a constant k, then the chromatic number of the graph class $\mathcal{G}$ is also bounded by a constant. However, such a constant k may not exist for some classes of graphs. A graph is 1-*planar* if it can be drawn on the plane in such a way that each edge is crossed by at most one other edge. This notion was introduced by Ringel [14], who proved that the chromatic number of each 1-planar graph is at most 7, even though the crossing number of the class of 1-planar graphs is not bounded by a fixed constant. In the same paper, Ringel conjectured that six colors are sufficient to color any 1-planar graph. This conjecture was confirmed by Borodin [4, 5]. In 2008, Albertson [2] considered graphs with even more restricted structure of crossings. The *cluster* of a crossing is the set of the four end-vertices of its two crossed edges. Two crossings are *independent* if their clusters do not intersect. It is easy to see that if every two crossings in G are independent, then each edge of G is crossed by at most one other edge and thus G is 1-planar. Settling a conjecture of Albertson [2], Král and Stacho [11] showed that every graph that can be drawn in the plane with all its crossings independent (for brevity, such a graph is called an *IC-planar graph* throughout this paper) is 5-colorable.

From the above definitions, one can see that the class of IC-planar graphs lies between planar graphs and 1-planar graphs. The structures of planar graphs are well established in the literature and the structures of 1-planar graphs have first been investigated by Pach and Tóth [13] in 1997. However, to our knowledge it appears that no other work besides the ones of Albertson [2] and Král and Stacho [11] on IC-planar graphs has been done.

In this paper, we first remark that every IC-planar graph is 6-degenerate, and, moreover, prove that every IC-planar graph with minimum degree at least 3 (resp. 4) contains a light edge uv with $d(u) + d(v) \leq 18$ (resp. 12). In Section 2, some more detailed structural properties of IC-planar graphs are investigated. In Section 3, we apply these properties to obtain some results on edge coloring, edge choosability, total choosability and linear arboricity of IC-planar graphs.

2. Structure of IC-planar graphs

Throughout this paper, we always assume that any given IC-planar graph has already been drawn in the plane with all its crossings independent and with the minimum number of crossings. It is known that $|E(G)| \leq 3|V(G)| - 6$ for every planar graph [3] and $|E(G)| \leq 4|V(G)| - 8$ for every 1-planar graph [9]. Similarly, we can also obtain a linear upper bound for the size of an IC-planar graph.

Proposition 2.1.

Let G be an IC-planar graph on n vertices and m edges. Then $m \leq 13n/4 - 6$ and this bound is the best possible.

Proof. Since every two crossings in G are independent, $\mathrm{cr}(G) \leq n/4$. We delete one edge from each crossing and denote the resulting graph by G'. It is clear that G' is a planar graph. Since $|E(G')| = m - \mathrm{cr}(G)$ and $|V(G')| = n$, $m = |E(G')| + \mathrm{cr}(G) \leq 3|V(G')| - 6 + \mathrm{cr}(G) \leq 3n - 6 + n/4 = 13n/4 - 6$. To show the tightness of this upper bound, we consider the Ladder graph $L_{2t} = P_2 \square P_{2t}$. Note that L_{2t} has $4t$ vertices and there are t vertex-disjoint C_4 contained in L_{2t} (here we call these C_4's *operable*). Triangulate L_{2t} and then we get a planar graph with $4t$ vertices and $12t - 6$ edges. Now add one crossing edge inside each original operable C_4. One can check that the resulting graph is an IC-planar graph with $4t$ vertices and $13t - 6$ edges. The left graph in Figure 1 is a graph obtained from the Ladder graph L_4 by the above operation. □

Proposition 2.2.

Every IC-planar graph contains a vertex of degree at most 6; the bound 6 is the best possible.

Proof. Suppose, to the contrary, that G is an IC-planar graph with $\delta(G) \geq 7$. Then we have $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 7|V(G)|$. On the other hand, by Proposition 2.1 we have $2|E(G)| \leq 13|V(G)|/2 - 12 < 7|V(G)|$. This a contradiction. Therefore, every IC-planar graph contains a vertex of degree at most 6. The right graph in Figure 1 is a 6-regular IC-planar graph, hence, the bound 6 is sharp. □

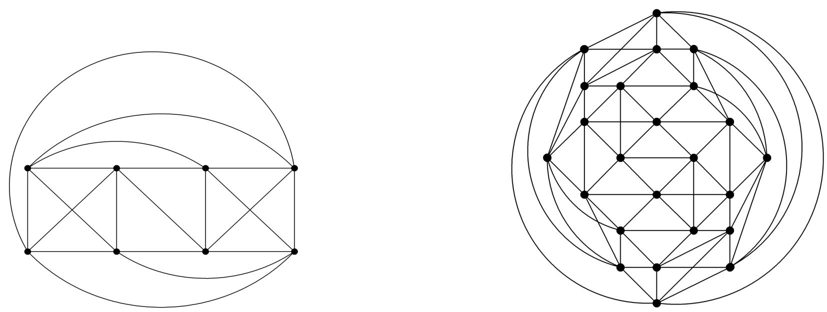

Figure 1. Two extremal IC-planar graphs with respect to Proposition 2.1 and Proposition 2.2

Actually, Proposition 2.2 also follows trivially by deleting one edge from each crossing (which forms a matching), resulting in a planar graph, which has minimum degree at most 5. In the following, we will use Proposition 2.2 to prove three stronger results. Before we proceed, we first introduce some definitions and notations. Let G be an IC-planar graph. The *associated plane graph* $G^\times$ of G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is called *false* if it is not a vertex of G and *true* otherwise. By *false face*, we mean a face f in $G^\times$ that is incident with at least one false vertex; and otherwise we call f a *true face*.

Let v be a vertex of an IC-planar graph G. By $N_2(v)$ we denote the number of 2-vertices that are adjacent to v in G. By $t_G(v)$ and $f_G(v)$ we denote the number of true and false 3-faces that are incident with v in $G^\times$, respectively. Since every two crossings in G are independent, $f_G(v) \leq 2$ for every v in G. Throughout this section, we call a vertex in G *small* if it is of degree at most 3. Denote by $s_G(v)$ the number of small vertices that are adjacent to v in G. In particular, if u is a 2-vertex in G with $N_G(u) = \{v, w\}$ such that uv crosses xy at a point z and $vxzv, vyzv, wxzuw, wyzuw$ are faces of $G^\times$ (see the left graph of Figure 2), then we call u a *special neighbor* of w in G. Denote by $n_G(w)$ the number of special neighbors of w. One can easily deduce the following inequality.

$$2n_G(w) + t_G(w) + f_G(w) \leq d_G(w). \tag{1}$$

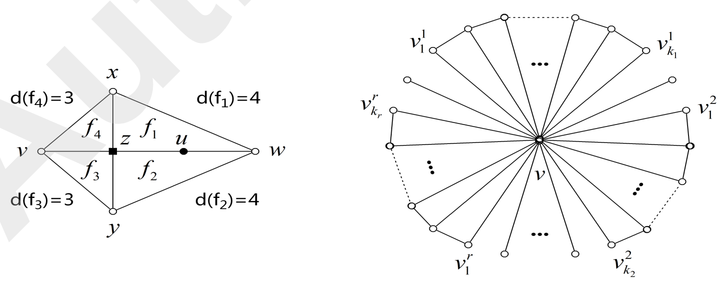

Figure 2. Some relative definitions to the proofs of the main theorems

Theorem 2.3.

Every IC-planar graph with minimum degree at least 2 contains one of the following configurations:

(a) *a 2-alternating cycle* $C = v_1v_2\cdots v_{2n}v_1$, $n \geq 2$, *such that* $d_G(v_1) = d_G(v_3) = \ldots = d_G(v_{2n-1}) = 2$ *and* $\max_{1\leq i\leq n} |N_2(v_{2i})| \geq 3$;

(b) *an edge* uv *such that* $d(u) \leq 6$ *and* $d(u) + d(v) \leq 18$.

Proof. Suppose, to the contrary, that G is a minimal counterexample to the theorem. Then one can see that G is connected. Let G_2 be the subgraph induced by the edges that are incident with the 2-vertices of G. Since G does not contain the configuration (b), no two 2-vertices are adjacent in G, which implies that G_2 contains no odd cycles. Since G does not contain the configuration (a) either, any component of G_2 is either an even cycle or a tree. Therefore, G_2 contains a matching M that covers all 2-vertices. If $uv \in E(M)$ and $d_G(u) = 2$, then v is called the 2-*master* of u and u is called the 2-*dependent* of v. It is easy to see that each 2-vertex has one 2-master and each vertex in G may have at most one 2-dependent.

Let v be a vertex in G and let $G^\times$ be the associated plane graph of G. Denote by $m_G(v)$ the number of false vertices that are adjacent to v in $G^\times$. Since G is IC-planar, $m_G(v) \leq 1$. Moreover, G also satisfies the following properties.

(P1) If $uv \in E(G)$ and $d_G(u) \leq 6$, then $d_G(v) \geq 19 - d_G(u)$.

(P2) If f is a false face in $G^\times$ that is incident with a 2-vertex, then $d_{G^\times}(f) \geq 4$.

(P3) If $m_G(v) = 0$, then $t_G(v) + s_G(v) \leq \left\lfloor \frac{3d_G(v)}{2} \right\rfloor$.

(P4) If $m_G(v) = 1$, $f_G(v) = 2$ and $t_G(v) = d_G(v) - 2$, then $t_G(v) + s_G(v) \leq \left\lfloor \frac{3d_G(v) - 3}{2} \right\rfloor$.

(P5) If $m_G(v) = 1$, $f_G(v) = 2$ and $t_G(v) \leq d_G(v) - 3$, then $t_G(v) + s_G(v) \leq \left\lfloor \frac{3d_G(v) - 4}{2} \right\rfloor$.

(P6) If $m_G(v) = 1$ and $f_G(v) \leq 1$, then $t_G(v) + s_G(v) \leq \left\lfloor \frac{3d_G(v) - 2}{2} \right\rfloor$.

(P1) follows directly from the absence of the configuration (b). Let $f = uvwu$ be a false 3-face in $G^\times$ that is incident with a 2-vertex v. Without loss of generality, assume that u is a false vertex and that vv', ww' are two mutually crossed edges in G. Since $N_G(v) = \{v', w\}$, we can redraw graph G by putting v into the face (with respect to the original drawing) that is incident with the path $v'uw$. By doing so, we reduce the number of crossings by one and then get a contradiction to our global assumption that the number of crossings is minimum. Therefore, f shall be a 4^+-face and we have proved (P2). Now we prove (P3). If $t_G(v) = d_G(v)$, then by (P1), it is easy to see that $s_G(v) \leq \lfloor d_G(v)/2 \rfloor$ and thus (P3) is satisfied. So we assume that $t_G(v) \leq d_G(v) - 1$. Let $v_1^1, \dots, v_{k_1}^1, v_1^2, \dots, v_{k_2}^2 \dots, v_1^r, \dots, v_{k_r}^r$ be the neighbors of v in a clockwise sequence with respect to the drawing of G such that F_i is a fan for every $1 \leq i \leq r$ and

$$t_G(v) = \sum_{i=1}^{r} k_i - r, \tag{2}$$

where F_i is the subgraph of G induced by the edge set $E_i = \bigcup_{j=1}^{k_i - 1} \{vv_j^i, v_j^i v_{j+1}^i\} \cup \{vv_{k_i}^i\}$ (see the right graph of Figure 2). Note that $r \geq 1$ and we do not necessarily have $\sum_{i=1}^{r} k_i = d_G(v)$ here. It is easy to see that

$$s_G(v) \leq \sum_{i=1}^{r} \left\lceil \frac{k_i}{2} \right\rceil + d_G(v) - \sum_{i=1}^{r} k_i \tag{3}$$

by (P1). Without loss of generality, assume that k_i is odd for every $i \leq a$ and is even for every $i > a$. Combine (2) and (3), we have

$$\begin{aligned} t_G(v) + s_G(v) &\leq \sum_{i=1}^{r} \left\lceil \frac{k_i}{2} \right\rceil + d_G(v) - r = \sum_{i=1}^{a} \frac{k_i + 1}{2} + \sum_{i=a+1}^{r} \frac{k_i}{2} + d_G(v) - r = \sum_{i=1}^{r} \frac{k_i}{2} + \frac{a - 2r}{2} + d_G(v) \\ &\leq \frac{\sum_{i=1}^{r} k_i + 2d_G(v) - r}{2} = \frac{t_G(v) + 2d_G(v)}{2} \qquad (4) \\ &\leq \frac{3d_G(v) - 1}{2} \qquad (5) \end{aligned}$$

since $a \leq r$ and $t_G(v) \leq d_G(v) - 1$. This completes the proof of (P3). Now we show that (P4) is a direct corollary of (P3). Suppose that uv crosses xy in G. Since $f(v) = 2$, $vx, vy \in E(G)$. With respect to the graph $G' = G - uv$, we have $m_{G'}(v) = 0$ and $t_{G'}(v) = d_{G'}(v) = d_G(v) - 1$. Therefore, by (P3) we have $t_{G'}(v) + s_{G'}(v) \le \lfloor 3d_{G'}(v)/2 \rfloor = \lfloor (3d_G(v) - 3)/2 \rfloor$. Note that $vxyv$ is a true 3-face in $G'^{\times}$ but is not a face in $G^{\times}$. So we have $t_G(v) = t_{G'}(v) - 1$. Meanwhile, since $u \notin V(G')$ and u may be a small vertex that is adjacent to v in G, $s_G(v) \le s_{G'}(v) + 1$. Thus we deduce that $t_G(v) + s_G(v) \le t_{G'}(v) + s_{G'}(v) \le \lfloor (3d_G(v) - 3)/2 \rfloor$, which completes the proof of (P4). Similarly, one can also prove (P5) by using the inequality (5). At last we prove (P6). Here we also suppose that uv crosses xy in G but consider the graph $G'' = G - xy$ instead. Note that we have now $m_{G''}(v) = 0$, $t_G(v) \le t_{G''}(v) \le t_G(v) + 1 \le d_G(v) - 1$ and $s_G(v) \le s_{G''}(v)$. If $t_{G''}(v) = t_G(v) + 1$, then

$$t_G(v) + s_G(v) \le t_{G''}(v) + s_{G''}(v) - 1 \le \frac{3d_{G''}(v) - 2}{2}$$

by (P3). If $t_{G''}(v) = t_G(v)$, then

$$t_G(v) + s_G(v) \le t_{G''}(v) + s_{G''}(v) \le \frac{t_{G''}(v) + 2d_{G''}(v)}{2} \le \frac{d_G(v) + 2d_{G''}(v) - 2}{2} = \frac{3d_{G''}(v) - 2}{2}$$

by (4). Hence the proof of (P6) completes.

Now we assign an initial charge c to $V(G) \cup F(G^{\times})$ by letting $c(v) = d_G(v) - 4$ for every $v \in V(G)$ and $c(f) = d_{G^{\times}}(f) - 4$ for every $f \in F(G^{\times})$. Since $G^{\times}$ is a plane graph, by Euler's formula, one can easily deduce that

$$\sum_{x \in V(G) \cup F(G^{\times})} c(x) = \sum_{v \in V(G)} (d_G(v) - 4) + \sum_{f \in F(G^{\times})} (d_{G^{\times}}(f) - 4) = \sum_{v \in V(G^{\times})} (d_{G^{\times}}(v) - 4) + \sum_{f \in F(G^{\times})} (d_{G^{\times}}(f) - 4) = -8. \qquad (6)$$

We redistribute the charges of vertices in G and faces in $G^{\times}$ according to the following rules (see Figure 3) and check that the final charge c' on each vertex and each face is nonnegative. Since our rules only move charge around and do not affect the total charges, this leads to a contradiction to (6) and completes our proof.

R1. Let $f = uvwu$ be a false 3-face in $G^{\times}$ and let u be a false vertex.

R1.1. If $d_G(v) \le 4$, then f receives 1 from w.

R1.2. If $\min\{d_G(v), d_G(w)\} \ge 5$, then f receives 1/2 from v and w, respectively.

R2. Let $f = uvwu$ be a true 3-face in $G^{\times}$.

R2.1. If $d_G(u) = 7$, then f receives 1/3 from u.

R2.2. If $d_G(u) \ge 8$, then f receives 1/2 from u.

R3. Let v be a 2-vertex in G, $N_G(v) = \{u, w\}$ and let u be a 2-master of v. Then v receives 5/3 from u and 1/3 from w.

R4. Let v be a 3-vertex in G. Then v receives 1/3 from each of the neighbors of v in G.

R5. Suppose that uv crosses xy in G at a point z in $G^{\times}$ and $vxzv$, $vyzv$ are 3-faces in $G^{\times}$. Let $N_G(u) = \{v, w\}$ and let f_1, f_2 be the face that is incident with the path $xzuw$ and $yzuw$ in $G^{\times}$, respectively.

R5.1. If $d_{G^{\times}}(f_1) \ge 5$, then v receives 1/3 from f_1 through the 2-vertex u.

R5.2. If $d_{G^{\times}}(f_1) = d_{G^{\times}}(f_2) = 4$, then v receives 1/3 from w through the 2-vertex u.

By R1, one can easily see that every false 3-face f in $G^{\times}$ receives exactly 1 from the vertices incident with f. Thus $c'(f) = -1 + 1 = 0$. For a true 3-face f in $G^{\times}$, we can conclude that f is incident with either three 7^+-vertices or two 8^+-vertices by (P1). So by R2 we have

$$c'(f) \ge -1 + \min\left\{3 \times \frac{1}{3}, 2 \times \frac{1}{2}\right\} = 0.$$

Since 4-faces are not involved in the discharging rules, $c'(f) = c(f) = 0$ for every 4-face $f \in F(G^{\times})$. Let f be a 5^+-face in $G^{\times}$. Then f may send out charges only by R5.1. However, no two 2-vertices are adjacent in G by (P1). This implies that $c'(f) \ge d_{G^{\times}}(f) - 4 - \lfloor d_{G^{\times}}(f)/2 \rfloor / 3 > 0$ for $d_{G^{\times}}(f) \ge 5$ by R5.1.

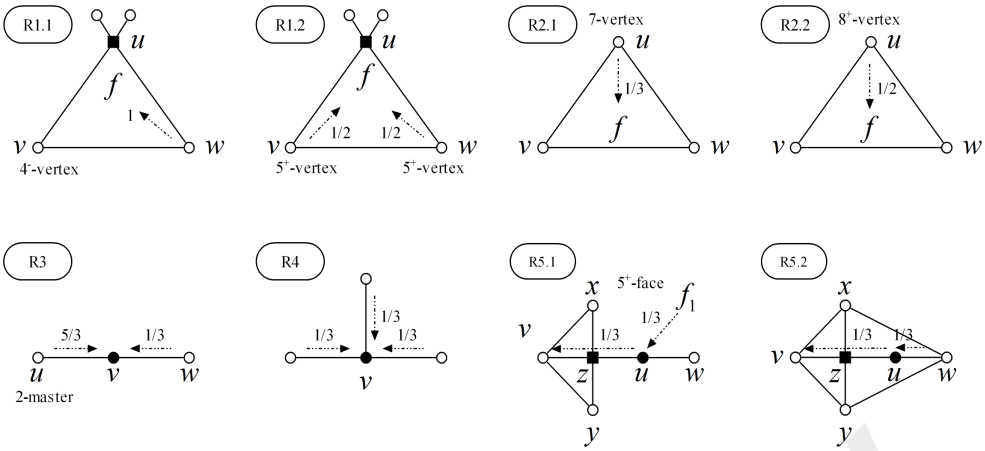

Figure 3. Discharging rules for the proof of Theorem 2.3

Let v be a vertex in G. If $d_G(v) = 2$, then $c'(v) = -2 + 5/3 + 1/3 = 0$ by R3, since 4^--vertices would not send out any charges by the above rules and one neighbor of v in G must be a 2-master while the other neighbor is not. If $d_G(v) = 3$, then by R4 one can also see that $c'(v) = -1 + 3 \times (1/3) = 0$. If $d_G(v) = 4$, then it is easy to see $c'(v) = c(v) = 0$ since 4-vertices are not involved in the discharging rules. If $5 \leq d_G(v) \leq 6$, then $c'(v) \geq d_G(v) - 4 - 2 \times (1/2) \geq 0$ by R1 and R2, since v is incident with at most two false 3-faces in $G^\times$ and v would not send out charges to the true 3-faces that are incident with it. If $d_G(v) = 7$, then by R1 and R2.1,

$$c'(v) \geq 3 - 2 \times \frac{1}{2} - 5 \times \frac{1}{3} > 0.$$

If $8 \leq d_G(v) \leq 14$, then v is adjacent to no 4^--vertices in G by (P1), which implies that $c'(v) \geq d_G(v) - 4 - d_G(v)/2 \geq 0$ by R1.2 and R2.2. If $d_G(v) = 15$, then v is adjacent to no 3^--vertices in G by (P1). Meanwhile, R1.1 can be applied to v only once, because otherwise v would be incident with two false 3-faces uvx and uvy such that u is a false vertex and x, y are both 4^--vertices, however, xy cannot be an edge in G by (P1), which is a contradiction. Thus,

$$c'(v) \geq 11 - 1 - \frac{1}{2} \times 14 > 0$$

by R1 and R2.2. If $d_G(v) = 16$, then v is adjacent to no 2-vertices in G by (P1). So by R1, R2 and R4,

$$c'(v) \geq 12 - 1 - \frac{1}{2} - \frac{1}{2} t_G(v) - \frac{1}{3} s_G(v) = \frac{21}{2} - \frac{1}{3}(t_G(v) + s_G(v)) - \frac{1}{6} t_G(v) \geq 0$$

if $f_G(v) \geq 1$, and

$$c'(v) \geq 12 - \frac{1}{2} t_G(v) - \frac{1}{3} s_G(v) \geq 12 - \frac{1}{2}(t_G(v) + s_G(v)) \geq 0$$

if $f_G(v) = 0$, since $t_G(v) + s_G(v) \leq 24$ by (P3)–(P6) and $t_G(v) \leq 16 - f_G(v)$.

The last cases are $d_G(v) \geq 17$. In the following we only prove $c'(v) \geq 0$ for every 17-vertex v in G, since the proofs of another cases are almost the same. Now consider that v may send out charges through its special neighbors by R5.2. First of all, we suppose that v is adjacent to no false vertices in $G^\times$, that is, $m_G(v) = 0$. So we have $f_G(v) = 0$. Note that there are $s_G(v)$ many 3^--vertices that are adjacent to v in G, at most one of which may be a 2-dependent of v. So by R2.2, R3, R4 and R5.2,

$$c'(v) \geq 13 - \frac{1}{2} t_G(v) - \frac{1}{3}(s_G(v) - 1) - \frac{5}{3} - \frac{1}{3} n_G(v) = \frac{35}{3} - \frac{1}{3}(t_G(v) + s_G(v)) - \frac{1}{6}(t_G(v) + 2n_G(v)) > 0,$$

since $t_G(v) + s_G(v) \leq 25$ by (P3) and $t_G(v) + 2n_G(v) \leq 17$ by (1). By the same argument, one can also prove that that $c'(v) \geq 0$ if $f_G(v) = 0$. Therefore, we shall now assume that $m_G(v) = 1$ and $f_G(v) \geq 1$.

If $t_G(v) \leq d_G(v) - 3$ and $f_G(v) = 1$, then by R1, R2.2, R3, R4 and R5.2,

$$c'(v) \geq 13 - 1 - \frac{1}{2}t_G(v) - \frac{1}{3}(s_G(v) - 1) - \frac{5}{3} - \frac{1}{3}n_G(v) = \frac{32}{3} - \frac{1}{3}(t_G(v) + s_G(v)) - \frac{1}{6}(t_G(v) + 2n_G(v)) \geq 0,$$

since $t_G(v) + s_G(v) \leq 24$ by (P6) and $t_G(v) + 2n_G(v) \leq 16$ by (1).

If $t_G(v) \leq d_G(v) - 3$ and $f_G(v) = 2$, then by R1, R2.2, R3, R4 and R5.2,

$$c'(v) \geq 13 - 1 - \frac{1}{2} - \frac{1}{2}t_G(v) - \frac{1}{3}(s_G(v) - 1) - \frac{5}{3} - \frac{1}{3}n_G(v) = \frac{61}{6} - \frac{1}{3}(t_G(v) + s_G(v)) - \frac{1}{6}(t_G(v) + 2n_G(v)) \geq 0,$$

since $t_G(v) + s_G(v) \leq 23$ by (P5) and $t_G(v) + 2n_G(v) \leq 15$ by (1).

If $t_G(v) = d_G(v) - 2$ and $f_G(v) = 1$, then by (1), we have $n_G(v) = 0$, that is to say, v has no special neighbors in G. Thus by R1, R2.2, R3 and R4,

$$c'(v) \geq 13 - 1 - \frac{1}{2}t_G(v) - \frac{1}{3}(s_G(v) - 1) - \frac{5}{3} = \frac{49}{6} - \frac{1}{3}(t_G(v) + s_G(v)) > 0,$$

since $t_G(v) + s_G(v) \leq 24$ by (P6).

If $t_G(v) = d_G(v) - 2$ and $f_G(v) = 2$, then we also have $n_G(v) = 0$. If v is adjacent to no 2-vertices in G, then by R1, R2.2 and R4,

$$c'(v) \geq 13 - 1 - \frac{1}{2} - \frac{1}{2}t_G(v) - \frac{1}{3}s_G(v) = 9 - \frac{1}{3}(t_G(v) + s_G(v)) > 0,$$

since $t_G(v) + s_G(v) \leq 24$ by (P4). So v has a neighbor u of degree 2 in G. Furthermore, since $t_G(v) = d_G(v) - 2$ and $f_G(v) = 2$, u is the only 2-vertex that is adjacent to v in G and uv must be a crossed edge. Let xy be the edge that crosses uv in G at a point z in $G^\times$ and w be the other neighbor of u in G. Now $vxzv$ and $vyzv$ are two false 3-faces that are incident with v. Let f_1, f_2 be the face that is incident with the path $xzuw$ and $yzuw$ in $G^\times$, respectively. Then we must have $\min\{d_{G^\times}(f_1), d_{G^\times}(f_2)\} \geq 4$ by (P2). So by R5, v shall receive 1/3 from some element in $V(G) \cup F(G^\times)$. Hence by R1, R2.2, R3, R4 and R5,

$$c'(v) \geq 13 - 1 - \frac{1}{2} - \frac{1}{2}t_G(v) - \frac{1}{3}(s_G(v) - 1) - \frac{5}{3} + \frac{1}{3} = 8 - \frac{1}{3}(t_G(v) + s_G(v)) \geq 0,$$

since $t_G(v) + s_G(v) \leq 24$ by (P4). This completes the proof of the theorem. □

A bipartite subgraph F, with two partite sets X and Y, of the graph G is called a 3-*alternating subgraph* if $d_F(x) = d_G(x) \leq 3$ for each $x \in X$ and $d_F(y) \geq d_G(y) + 3 - \Delta(G)$ for each $y \in Y$. This important notion was introduced by Borodin, Kostochka and Woodall in [6] and then used in some papers on graph colorings.

Theorem 2.4.

Every IC-planar graph with minimum degree at least 2 contains one of the following configurations:

(a) *a 2-alternating cycle;*

(b) *a 3-alternating subgraph;*

(c) *an edge uv such that $d(u) \leq 6$ and $d(u) + d(v) \leq 15$.*

Proof. Suppose that G is a minimal counterexample to the theorem. Then G is connected. Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. Then by a similar proof as in Theorem 2.3, one can conclude that G_2 is a forest. It follows that G_2 contains a matching M that covers all 2-vertices. If $uv \in E(M)$ and $d_G(u) = 2$, then v is called the 2-*master* of u and u is called the 2-*dependent* of v.

Let $X = \{x : 2 \le d_G(x) \le 3\}$ and $Y = \{y : y \in N_G(x), x \in X\}$. Since G contains no 3-alternating subgraph, we can prove the following result.

Claim. If X is not empty, then there exists a bipartite subgraph F of G, with partite sets X and Y, such that $d_F(x) = 1$ for each $x \in X$ and $d_F(y) \le 2$ for each $y \in Y$.

The proof idea of the claim is borrowed from the proof of [6, Theorem 8]. Actually, one can also find that the proof of it is just a part of the proof of [19, Lemma 2.4], so we omit the detailed proof of the claim here. Let F be the bipartite subgraph from the claim. If $xy \in F$ and $x \in X$, then y is called the 3-*master* of x and x is called the 3-*dependent* of y.

Combine these lines of discussions, we conclude that G has the following properties.

(P1) Every 2-vertex in G has a 2-master and a 3-master.

(P2) Every 3-vertex in G has a 3-master.

(P3) Every vertex in Y may have at most one 2-dependent and at most two 3-dependents.

Our proof of the theorem also uses the discharging method. First of all, we assign an initial charge c to $V(G) \cup F(G^\times)$ by letting $c(v) = d_G(v) - 4$ for every $v \in V(G)$ and $c(f) = d_{G^\times}(f) - 4$ for every $f \in F(G^\times)$. Then by Euler's formula one can similarly deduce that $\sum_{x \in V(G) \cup F(G^\times)} c(x) = -8$. Let us now discharge according to the following rules (see Figure 4).

R1. Let $f = uvwu$ be a false 3-face in $G^\times$ and let u be a false vertex.

- R1.1. If $d_G(v) \le 4$, then f receives 1 from w.
- R1.2. If $\min\{d_G(v), d_G(w)\} \ge 5$, then f receives 1/2 from v and w, respectively.

R2. Let $f = uvwu$ be a true 3-face in $G^\times$.

- R2.1. If $d_G(u) = 7$, then f receives 1/3 from u.
- R2.2. If $d_G(u) \ge 8$, then f receives 1/2 from u.

R3. Let v be a 2-vertex in G. Then v receives 1 from its 2-master and another 1 from its 3-master.

R4. Let v be a 3-vertex in G. Then v receives 1 from its 3-master.

R5. Suppose that uv crosses xy in G at a point z in $G^\times$ and $vxzv$, $vyzv$ are 3-faces in $G^\times$. Let $N_G(u) = \{v, w\}$ and let f_1, f_2 be the face that is incident with the path $xzuw$ and $yzuw$ in $G^\times$, respectively.

- R5.1. If $d_{G^\times}(f_1) \ge 5$, then v receives 1/2 from f_1 through the 2-vertex u.
- R5.2. If $d_{G^\times}(f_1) = d_{G^\times}(f_2) = 4$, then v receives 1/2 from w through the 2-vertex u.

In the following we show that the final charge c' for every vertex and face is nonnegative. This implies that

$$\sum_{x \in V(G) \cup F(G^\times)} c(x) \;=\; \sum_{x \in V(G) \cup F(G^\times)} c'(x) \;\ge\; 0,$$

a contradiction.

Note that the rules R1, R2 and R5 are highly similar to the corresponding ones in the proof of Theorem 2.3. So by some analogous arguments, one can check that $c'(f) \ge 0$ for every face in $G^\times$ and $c'(v) \ge 0$ for every vertex of degree between 4 and 12 in G. Since 2-vertices and 3-vertices participate only in R3 and R4, one can also conclude that $c'(v) = 0$ for every 2-vertex and every 3-vertex in G by (P1) and (P2). If v is a 13-vertex in G, then v is adjacent to

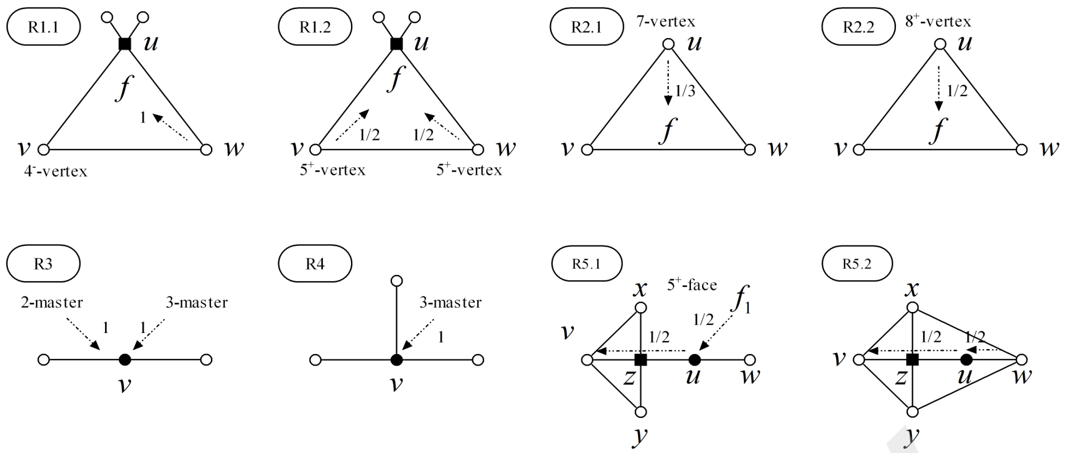

Figure 4. Discharging rules for the proof of Theorem 2.4

no 2-vertices since the configuration (c) is forbidden in G. This implies that R5.2 would not be applied to v. Moreover, v would take part in R1.1 at most once. It follows that $c'(v) \geq 9 - 1 - 12 \times (1/2) - 2 \times 1 = 0$ by R1, R2, R4 and (P3). At last we work with 14^+-vertices, which may have special neighbors (recall the definition stated before (1)).

Let v be a vertex in G with $d_G(v) \geq 14$. First of all, suppose that v has at least one special neighbor, that is, $n_G(v) \geq 1$. Then by R1, R2.2, R3, R4, R5.2, (P3) and (1),

$$
\begin{aligned}
c'(v) &\geq d_G(v) - 4 - 1 - \frac{1}{2} - \frac{1}{2} t_G(v) - \frac{1}{2} n_G(v) - 1 - 2 \times 1 = d_G(v) - \frac{1}{2}(t_G(v) + 2n_G(v)) + \frac{1}{2} n_G(v) - \frac{17}{2} \\
&\geq d_G(v) - \frac{1}{2}(d_G(v) - 2) - 8 = \frac{1}{2} d_G(v) - 7 \geq 0
\end{aligned}
$$

if $f_G(v) = 2$, and

$$
\begin{aligned}
c'(v) &\geq d_G(v) - 4 - 1 - \frac{1}{2} t_G(v) - \frac{1}{2} n_G(v) - 1 - 2 \times 1 = d_G(v) - \frac{1}{2}(t_G(v) + 2n_G(v)) + \frac{1}{2} n_G(v) - 8 \\
&\geq d_G(v) - \frac{1}{2}(d_G(v) - 1) - \frac{15}{2} = \frac{1}{2} d_G(v) - 7 \geq 0
\end{aligned}
$$

if $f_G(v) = 1$, and

$$
c'(v) \geq d_G(v) - 4 - \frac{1}{2} t_G(v) - \frac{1}{2} n_G(v) - 1 - 2 \times 1 = d_G(v) - \frac{1}{2}(t_G(v) + 2n_G(v)) + \frac{1}{2} n_G(v) - 7 \geq d_G(v) - \frac{1}{2} d_G(v) - \frac{13}{2} > 0
$$

if $f_G(v) = 0$. Thus we assume that v has no special neighbors in G. If v has no 2-dependents or v is incident with at most $d_G(v) - 1$ many 3-faces, then

$$
\begin{aligned}
&c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 1) - 2 \times 1 = \frac{1}{2} d_G(v) - \frac{13}{2} > 0 \qquad \text{or} \\
&c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 2) - 1 - 2 \times 1 = \frac{1}{2} d_G(v) - 7 \geq 0
\end{aligned}
$$

by R1, R2, R3, R4 and (P3), respectively. Hence we shall assume that v has one 2-dependent u and all faces that are incident with v in $G^\times$ are of degree 3. Moreover, one can consequently deduce that uv is a crossed edge in G. Let xy

cross uv in G at a point z in $G^\times$. Now $vxzv$ and $vyzv$ are two false 3-faces in $G^\times$. Let f_1 and f_2 be the face that is incident with the path $xzuw$ and $yzuw$ in $G^\times$, respectively. Then we must have $\min\{d_{G^\times}(f_1), d_{G^\times}(f_2)\} \geq 4$ since 2-vertex is incident with no false 3-faces in $G^\times$. This implies that v would get 1/2 by R5 from either f_i or w. Hence by R1, R2, R3, R4 and (P3),

$$c'(v) \geq d_G(v) - 4 - 1 - \frac{1}{2}(d_G(v) - 1) - 1 - 2 \times 1 + \frac{1}{2} = \frac{1}{2} d_G(v) - 7 \geq 0$$

for $d_G(v) \geq 14$. □

Theorem 2.5.

Every IC-planar graph with minimum degree at least 3 contains one of the following configurations:

(a) *a 3-alternating cycle;*

(b) *an edge uv such that $d(u) + d(v) \leq 12$.*

Proof. Suppose that G is a minimal counterexample to the theorem. Then it is easy to see that G is connected. Let V_3 be the set of 3-vertices and let V_{10}^+ be the set of 10^+-vertices in G. Let F be the set of edges in G having one end-vertex in V_3 and let H be the bipartite subgraph with vertex set $V_3 \cup V_{10}^+$ and edge set F. Since G does not contain 3-alternating cycles, H is a forest and thus $|V(H)| = |V_3| + |V_{10}^+| > |E(H)|$. Moreover, the neighbors of every vertex in V_3 belong to the vertex set V_{10}^+ due to the absence of the configuration (b) in G. This implies that $|E(H)| = 3|V_3|$. Hence we conclude that

$$|V_{10}^+| > 2|V_3|. \tag{7}$$

Let us assign an initial charge c to $V(G) \cup F(G^\times)$ by letting $c(v) = d_G(v) - 4$ for every $v \in V(G)$ and $c(f) = d_{G^\times}(f) - 4$ for every $f \in F(G^\times)$. It follows easily from Euler's formula that $\sum_{x \in V(G) \cup F(G^\times)} c(x) = -8$. Now we redistribute the charges of vertices and faces according to the following rules (see Figure 5).

R1. Every 10^+-vertex gives 1/2 to a common pot from which each 3-vertex receives 1.

R2. Let $f = uvwu$ be a false 3-face in $G^\times$ and let u be a false vertex.

R2.1. If $d_G(v) \leq 4$, then f receives 1 from w.

R2.2. If $\min\{d_G(v), d_G(w)\} \geq 5$, then f receives 1/2 from v and w, respectively.

R3. Let $f = uvwu$ be a true 3-face in $G^\times$ such that $d_G(u) \leq d_G(v) \leq d_G(w)$.

R3.1. If $d_G(u) \leq 5$, then f receives 1/2 from v and w, respectively.

R3.2. If $d_G(u) = 6$, then f receives 1/4 from u and receives 3/8 from v and w, respectively.

R3.3. If $d_G(u) \geq 7$, then f receives 1/3 from u, v and w, respectively.

We check that the final charge c' on each vertex and each face is nonnegative. And we also show that the final charge of the common pot is nonnegative. Since our rules only move charge around and do not affect the sum, this leads to a contradiction that the total final charge is nonnegative and completes the proof.

Since $|V_{10}^+| > 2|V_3|$ by (7), the charge of the common pot is positive. By the above rules, one can easily check that every 3-face in $G^\times$ would receive exactly 1 from the vertices that are incident with it and every 4^+-face does not take part in the discharging rules. So we conclude that $c'(f) \geq 0$ for every $f \in F(G^\times)$.

Let v be a vertex in G. If $d_G(v) = 3$, then v does not participate in R2 and R3. So by R1, $c'(v) = -1 + 1 = 0$. If $d_G(v) = 4$, then it is trivial that $c'(v) = c(v) = 0$. If $d_G(v) = 5$, then v would send out no charges to its incident true 3-faces by R3. Meanwhile, v may be incident with at most two false 3-faces since every two crossings in G are independent. So by R2.2, $c'(v) \geq 1 - 2 \times (1/2) = 0$. If $d_G(v) = 6$, then v is adjacent to no 6^--vertices since the configuration (b) is forbidden in G. So to each true 3-face v shall send 1/4 by R3.2 and to each false 3-face v shall send 1/2 by R2.2. This implies that

$$c'(v) \geq 2 - 2 \times \frac{1}{2} - 4 \times \frac{1}{4} = 0$$

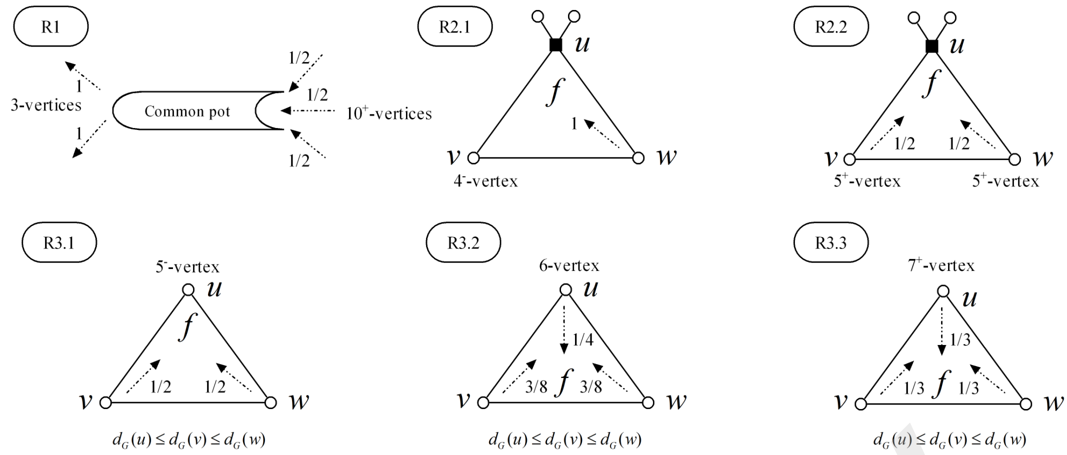

Figure 5. Discharging rules for the proof of Theorem 2.5

because v can be incident with at most two false 3-faces. Similarly one can also check that

$$c'(v) \geq 3 - 2 \times \frac{1}{2} - 5 \times \frac{3}{8} > 0$$

by R2.2 and R3.2 if $d_G(v) = 7$ and $c'(v) \geq 4 - 8 \times \frac{1}{2} = 0$ by R2.2 and R3.1 if $d_G(v) = 8$. If $d_G(v) \geq 9$, then by the same argument as in the proof of Theorem 2.3 one can conclude that R2.1 may be applied to v at most once. It follows that $c'(v) \geq 5 - 1 - 8 \times (1/2) = 0$ if $d_G(v) = 9$ by R2 and R3 and $c'(v) \geq d_G(v) - 4 - 1 - (d_G(v) - 1)/2 - 1/2 \geq 0$ if $d_G(v) \geq 10$ by R1, R2 and R3. □

From Theorem 2.3 and Theorem 2.5, we can deduce the following two corollaries, which prove the existence of light edges in IC-planar graphs with prescribed minimum degree.

Corollary 2.6.
Every IC-planar graph with minimum degree at least 3 *contains an edge* uv *with* $d(u) + d(v) \leq 18$.

Corollary 2.7.
Every IC-planar graph with minimum degree at least 4 *contains an edge* uv *with* $d(u) + d(v) \leq 12$.

To end this section, we would like to point out that the upper bound 12 mentioned in Theorem 2.5 and Corollary 2.7 is the best possible due to the existence of a 6-regular IC-planar graph (see the right graph of Figure 1). Meanwhile, the condition $\delta(G) \geq 3$ in Corollary 2.6 is essential because the weight of any edge of the complete bipartite graph $K_{2,n}$ is $2 + n$, not bounded by a constant. But we still do not know whether Theorems 2.3 and 2.4 are tight or not. We leave this as an open problem for further research.

3. Applications to coloring problems

A proper edge (resp. total) coloring of a graph is an assignment of colors to the edges (resp. to the vertices and edges) of a graph so that no two adjacent edges (resp. elements) receive the same color. The smallest number of colors needed in a proper edge (resp. total) coloring of a graph G is the edge (resp. total) chromatic number, denoted by $\chi'(G)$ (resp. $\chi''(G)$).

For edge coloring, the well-known Vizing's theorem states that for any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. This theorem divides all graphs into two classes: class one graphs have $\chi'(G) = \Delta(G)$ and class two graphs have $\chi'(G) = \Delta(G)+1$. Consequently, a major question in the area of edge colorings is that of determining to which of these two classes a given graph belongs. It is known that every planar graph with maximum degree at least 7 [16] and every 1-planar graph with maximum degree at least 10 [22] is of class one. What can we say about IC-planar graph? The following answer can be seen as a corollary of Proposition 2.1.

Proposition 3.1.

Every IC-planar graph with maximum degree at least 8 *is of class one.*

Proof. Let G be a minimum counterexample to the theorem with maximum degree $\Delta \geq 8$. Then G is a connected graph of class two and $\chi'(G-e) < \chi'(G)$ for every edge e of G. In fact, such a graph G is also called Δ-*critical graph* in the literature. In [12], Li showed that the average degree of every Δ-critical graph with $\Delta \geq 8$ is at least 13/2. This implies that $|E(G)| \geq 13|V(G)|/4$. However, on the other hand we have $|E(G)| \leq 13|V(G)|/4 - 6$ by Proposition 2.1. This is a contradiction. $\square$

Proposition 3.2.

There are IC-planar graphs of class two with maximum degree Δ for each $\Delta \leq 6$.

Proof. Since Vizing [17] presented examples of planar graphs of class two with maximum degree no more than five, the theorem holds for $\Delta \leq 5$. Now we construct an IC-planar graph G' with maximum degree 6 by replacing an edge uv of the 6-regular IC-planar graph G shown in Figure 1 with a path uwv satisfying $d_G(w) = 2$. Note that $\Delta(G') = 6$, $|V(G')| = 25$ and $|E(G')| = 73$. Clearly the largest matching in G' has size at most $\lfloor |V(G')|/2 \rfloor = 12$. Since $|E(G')| = 73 > 12\Delta(G')$, the graph G' is of class two. $\square$

In view of the above two propositions, we leave a conjecture here.

Conjecture 3.3.

Every IC-planar graph with maximum degree 7 *is of class one.*

Let f be a function into positive integers. We say that G is *edge-f-choosable* (resp. *totally-f-choosable*) if, whenever we are given a list of $f(x)$ colors to each element $x \in E(G)$ (resp. $x \in E(G) \cup V(G)$), there exists a proper edge (resp. total) coloring of G such that each element is colored with a color from its own list. In particular, if $f(x) = k$ for every element, then we say that G is *edge-k-choosable* (resp. *totally-k-choosable*). The *list edge* (resp. *list total*) *chromatic number* of G, denoted by $\chi'_l(G)$ (resp. $\chi''_l(G)$), is the smallest integer k such that G is edge-k-choosable (resp. totally-k-choosable). The following is the well-known List Coloring Conjecture, see [6, 10].

Conjecture 3.4.

For any graph G, $\chi'_l(G) = \chi'(G)$ and $\chi''_l(G) = \chi''(G)$.

As far as we know, this conjecture was confirmed for several classes of graphs including planar graphs with maximum degree at least 12 [6] and 1-planar graphs with maximum degree at least 21 [23]. We now focus on IC-planar graphs. Actually, based on Theorem 2.4 and Theorem 2.5, one can respectively prove the following two theorems.

Theorem 3.5.

Let G be an IC-planar graph. If $\Delta(G) \geq 14$, then $\chi'_l(G) = \Delta(G)$ and $\chi''_l(G) = \Delta(G)+1$.

Theorem 3.6.

Let G be an IC-planar graph. If $\Delta(G) \geq 10$, then $\chi'_l(G) \leq \Delta(G)+1$; and if $\Delta(G) \geq 11$, then $\chi''_l(G) \leq \Delta(G)+2$.

The proofs of the above two theorems are omitted here since highly similar proofs constructed by Borodin et al. can be found in [6]. The interested readers can refer to Theorem 9 of their paper.

We have now confirmed the List Coloring Conjecture for IC-planar graphs with maximum degree at least 14. By the way, let us recall the well-known Total Coloring Conjecture, which asserts that every graph of maximum degree Δ admits a $(\Delta+2)$-total coloring. This conjecture has already been confirmed for planar graphs with maximum degree at least 7 [15] and 1-planar graphs with maximum degree at least 13 [21]. By Theorem 3.6, we can additionally conclude that the Total Coloring Conjecture also holds for IC-planar graphs with maximum degree at least 11.

A linear forest is a forest in which every connected component is a path. A mapping ϕ from $E(G)$ to $\{1, 2, \dots, t\}$ is called a t-*linear coloring* if the subgraph induced by $\phi^{-1}(s)$ is a linear forest for every $1 \le s \le t$. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum number t for which G has a t-linear coloring. Akiyama, Exoo and Harary [1] conjectured that $\mathrm{la}(G) = \lceil(\Delta(G)+1)/2\rceil$ for any regular graph G. It is obviously that $\mathrm{la}(G) \ge \lceil\Delta(G)/2\rceil$ for every graph G and $\mathrm{la}(G) \ge \lceil(\Delta(G)+1)/2\rceil$ for every regular graph G. So this conjecture is equivalent to the following conjecture, which is known as Linear Arboricity Conjecture.

Conjecture 3.7.

For any graph G, $\left\lceil \frac{\Delta(G)}{2} \right\rceil \le \mathrm{la}(G) \le \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

This conjecture has been completely confirmed for planar graphs by Wu [18, 20]. Moreover, Cygan et al. [7] proved that $\mathrm{la}(G) = \lceil\Delta(G)/2\rceil$ for every planar graph with maximum degree at least 9. For an IC-planar graph, we can also determine its linear arboricity provided that its maximum degree is large enough. The proof of the following theorem is based on Theorem 2.3. We omit it here since a highly similar proof can be found in [18]. The interested readers can refer to [18, Theorem 2.2].

Theorem 3.8.

Let G be an IC-planar graph. If $\Delta(G) \ge 17$, then $\mathrm{la}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

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References

[1] Akiyama J., Exoo G., Harary F., Covering and packing in graphs. III: Cyclic and acyclic invariants, Math. Slovaca, 1980, 30(4), 405–417

[2] Albertson M.O., Chromatic number, independence ratio, and crossing number, Ars Math. Contemp., 2008, 1(1), 1–6

[3] Bondy J.A., Murty U.S.R., Graph Theory with Applications, Elsevier, New York, 1976

[4] Borodin O.V., Solution of the Ringel problem on the vertex-face coloring of plane graphs and on the coloring of 1-planar graphs, Metody Diskret. Analiz., 1984, 41, 12–26

[5] Borodin O.V., A new proof of the 6-color theorem, J. Graph Theory, 1995, 19(4), 507–521

[6] Borodin O.V., Kostochka A.V., Woodall D.R., List edge and list total colorings of multigraphs, J. Combin. Theory Ser. B, 1997, 71(2), 184–204

[7] Cygan M., Hou J.-F., Kowalik Ł., Lužar B., Wu J.-L., A planar linear arboricity conjecture, J. Graph Theory, 2012, 69(4), 403–425
[8] Erman R., Havet F., Lidický B., Pangrác O., 5-coloring graphs with 4 crossings, SIAM J. Discrete Math., 2011, 25(1), 401–422
[9] Fabrici I., Madaras T., The structure of 1-planar graphs, Discrete Math., 2007, 307(7-8), 854–865
[10] Jensen T.R., Toft B., Graph Coloring Problems, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, New York, 1995
[11] Král D., Stacho L., Coloring plane graphs with independent crossings, J. Graph Theory, 2010, 64(3), 184–205
[12] Li X., Average degrees of critical graphs, Ars Combin., 2005, 74, 303–322
[13] Pach J., Tóth G., Graphs drawn with few crossings per edge, Combinatorica, 1997, 17(3), 427–439
[14] Ringel G., Ein Sechsfarbenproblem auf der Kugel, Abh. Math. Sem. Univ. Hamburg, 1965, 29, 107–117
[15] Sanders D.P., Zhao Y., On total 9-coloring planar graphs of maximum degree seven, J. Graph Theory, 1999, 31(1), 67–73
[16] Sanders D.P., Zhao Y., Planar graphs of maximum degree seven are class I, J. Combin. Theory Ser. B, 2001, 83(2), 201–212
[17] Vizing V.G., Critical graphs with given chromatic class, Diskret. Analiz., 1965, 5, 9–17
[18] Wu J.-L., On the linear arboricity of planar graphs, J. Graph Theory, 1999, 31(2), 129–134
[19] Wu J., Wang P., List-edge and list-total colorings of graphs embedded on hyperbolic surfaces, Discrete Math., 2008, 308(4), 6210–6215
[20] Wu J.-L., Wu Y.-W., The linear arboricity of planar graphs of maximum degree seven is four, J. Graph Theory, 2008, 58(3), 210–220
[21] Zhang X., Hou J., Liu G., On total colorings of 1-planar graphs, preprint available at http://xinzhang.hpage.com/get_file.php?id=1513981&vnr=342077
[22] Zhang X., Wu J.-L., On edge colorings of 1-planar graphs, Inform. Process. Lett., 2011, 111(3), 124–128
[23] Zhang X., Wu J., Liu G., List edge and list total coloring of 1-planar graphs, Front. Math. China (in press), DOI: 10.1007/s11464-012-0184-7