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List total coloring of pseudo-outerplanar graphs☆

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ABSTRACT

A graph is pseudo-outerplanar if each of its blocks has an embedding in the plane so that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. It is proved that every pseudo-outerplanar graph with maximum degree $\Delta \geq 5$ is totally $(\Delta + 1)$-choosable.

1. Introduction

In this paper, all graphs are finite, simple and undirected. By $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$, we denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G, respectively. By $VE(G)$, we denote the set $V(G) \cup E(G)$. For undefined concepts we refer the readers to [1].

A total coloring of a graph G is an assignment of colors to the vertices and edges of G such that every pair of adjacent/incident elements receives different colors. A *total k-coloring* of a graph G is a total coloring of G from a set of k colors. The minimum positive integer k for which G has a total k-coloring, denoted by $\chi''(G)$, is the *total chromatic number* of G.

Suppose that a set $L(x)$ of colors, called a *list* of x, is assigned to each element $x \in VE(G)$. A total coloring φ is called a *list total coloring* of G for L, or a *total L-coloring*, if $\varphi(x) \in VE(G)$ for each element $x \in VE(G)$. If $|L(x)| \equiv k$ for every $x \in VE(G)$, then a total L-coloring is called a *list total k-coloring* and we say that G is *totally k-choosable*. The minimum integer k for which G has a list total k-coloring, denoted by $\chi_l''(G)$, is the *total choosability* of G. It is obvious that $\chi_l''(G) \geq \chi''(G) \geq \Delta(G) + 1$.

In 1997, Borodin, Kostochka and Woodall [2] raised the following conjecture, which is known as list total conjecture (LTC). In the same paper, they gave an affirmative answer to LTC for planar graphs with maximum degree at least 12.

Conjecture 1. *For any graph G, $\chi_l''(G) = \chi''(G)$.*

Recently, LTC was investigated by many authors including [3–6,8,7,9,11,14]. In particular, Wang and Lih [9] confirmed LTC for outerplanar graphs with maximum degree at least 4, and this result was generalized to series–parallel graphs by Zhou, Matsuo and Nishizeki [14] in 2005. However, this "list total conjecture" is still very much open.

In this paper, we investigate the list total colorings of pseudo-outerplanar graphs, another class of graphs (different from series–parallel graphs) between outerplanar graphs and planar graphs. A graph is *pseudo-outerplanar* if each of its blocks has an embedding in the plane so that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with

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each of them crossing at most one another. For example, $K_{2,3}$ and K_4 are both pseudo-outerplanar graphs. The concept of pseudo-outerplanar graph was first introduced by Zhang, Liu and Wu [13] in 2012. They proved that the class of outerplanar graphs is the intersection of the classes of pseudo-outerplanar graphs and series–parallel graphs.

The purpose of this paper is to show that LTC holds for pseudo-outerplanar graphs with maximum degree at least 5, and thus extending one result of [12], where is proved that every pseudo-outerplanar graphs with maximum degree at least 5 is totally $(\Delta + 1)$-colorable.

2. Structural properties of PO-graphs

In what follows, we always assume that every pseudo-outerplanar graph H considered in this paper has been drawn on the plane so that its pseudo-outerplanarity is satisfied and call such a drawing a *pseudo-outerplanar diagram*. Let H be a pseudo-outerplanar diagram and let G be a block of H. Denote by $v_1, v_2, \ldots, v_{|G|}$ the vertices of G that lie in a clockwise sequence. Let $\mathcal{V}[v_i, v_j] = \{v_i, v_{i+1}, \ldots, v_j\}$ and $\mathcal{V}(v_i, v_j) = \mathcal{V}[v_i, v_j] \setminus \{v_i, v_j\}$, where the subscripts are taken modular $|G|$. A vertex set $\mathcal{V}[v_i, v_j]$ is a *non-edge* if $j = i + 1$ and $v_iv_j \notin E(G)$, is a *path* if $v_kv_{k+1} \in E(G)$ for all $i \le k < j$, and is a *subpath* if $j > i + 1$ and some edges in the form v_kv_{k+1} for $i \le k < j$ are missing. An edge v_iv_j in G is a *chord* if $j - i \neq 1$ or $1 - |G|$. By $\mathcal{C}[v_i, v_j]$, we denote the set of chords xy with $x, y \in \mathcal{V}[v_i, v_j]$. We say that a chord v_kv_l is contained in a chord v_iv_j if $i \le k \le l \le j$.

Lemma 2. *Let v_i and v_j be vertices of a 2-connected pseudo-outerplanar diagram G. If there are no crossed chords in $\mathcal{C}[v_i, v_j]$ and no edges between $\mathcal{V}(v_i, v_j)$ and $\mathcal{V}(v_j, v_i)$, then $\mathcal{V}[v_i, v_j]$ is either non-edge or path.*

Proof. The proof is same to the one of Claim 1 in [13], we refer the readers to [13, p. 2794]. □

Lemma 3 (*[10]*)**.** *Each outerplanar graph G with minimum degree at least 2 contains a 2-vertex that is adjacent to a 4^--vertex.*

Theorem 4. *Each pseudo-outerplanar graph G with minimum degree at least 2 contains at least one of the following configurations:*

(a) *a 2-vertex u adjacent to a 4^--vertex v;*
(b) *a path $v_1u_1v_2u_2v_3u_3v_4$ with $v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4 \in E(G)$, $d(u_1) = d(u_2) = d(u_3) = 2$ and $d(v_2) = d(v_3) = 5$;*
(c) *a cycle $u_1u_2u_3u_4$ with $d(u_2) = d(u_4) = 2$;*
(d) *a cycle $u_1u_2u_3u_4$ with $u_2u_4 \in E(G)$, $d(u_2) = d(u_4) = 3$ and $d(u_3) \le 4$;*
(e) *a cycle $u_1u_2u_3u_4$ with $u_2u_4 \in E(G)$, $d(u_2) = d(u_4) = 3$ and u_3 being adjacent to a 2-vertex v;*
(f) *a cycle $u_1u_2u_3u_4$ with $u_2u_4 \in E(G)$, $d(u_2) = d(u_4) = 3$ and u_3 being adjacent to a 3-vertex v and a vertex x with $vx \in E(G)$;*
(g) *a cycle $u_1u_2u_3u_4$ with $u_1u_3, u_2u_4 \in E(G)$, $d(u_2) = d(u_4) = 3$ and u_3 being adjacent to a vertex v with $u_1v \in E(G)$.*

Proof. We first assume that G is a 2-connected pseudo-outerplanar diagram with $v_1, \ldots, v_{|G|}$ being the vertices of this diagram that lie in a clockwise sequence. If G contains no crossings, then G is outerplanar, which implies that G contains (a) by Lemma 3. If G contains a crossing, then we can choose one pair of crossed chords v_iv_j and v_kv_j such that

(1) v_iv_j crosses v_kv_l in G;
(2) v_i, v_k, v_j and v_l lie in a clockwise sequence;
(3) besides v_iv_j and v_kv_l, there are no crossed chords in $\mathcal{C}[v_i, v_l]$.

Suppose that this theorem is false. By a same proof of Theorem 4.2 in [13], we can prove that

$$l - j = j - k = k - i = 1 \quad \text{and} \quad v_iv_k, v_kv_j, v_jv_l \in E(G), \tag{1}$$

since G does not contain (a), (b) or (c). This pair of crossed chords v_iv_j and v_kv_j satisfying (1) are called *co-crossed chords*.

Since the configuration (d) is absent from G, $\min\{d(v_i), d(v_l)\} \ge 5$. This implies that there are at least one chord v_lv_s with $s \neq i, k$ and at least one chord v_mv_i with $m \neq j, l$. We now choose s and m so that there is no chord v_lv_t contained in v_lv_s and no chord v_iv_n contained in v_iv_m. In the following, we call the graph induced by $v_iv_j, v_kv_l, v_iv_k, v_kv_j, v_jv_l, v_lv_s$ or by $v_iv_j, v_kv_l, v_iv_k, v_kv_j, v_jv_l, v_iv_m$ an *inner cluster* of G, denoted by $IC(i, l, s)$ or $IC(m, i, l)$, respectively. The *width* of the two inner clusters defined above is $|\mathcal{V}[v_i, v_s]|$ and $|\mathcal{V}[v_m, v_l]|$, respectively.

Claim 1. *If $IC(i, l, s)$ is an inner cluster with the shortest width among all the inner clusters that contained in the graphs induced by $\mathcal{V}[v_i, v_s]$, then the chord v_lv_s is crossed.*

Proof. Without loss of generality, assume that $i = 1$ and $l = 4$. If v_4v_s is a non-crossed chord, then there are no edges between $\mathcal{V}(v_4, v_s)$ and $\mathcal{V}(v_s, v_4)$. If there are no chords contained in v_4v_s, then (a) or (e) would appear in G. If there are chords contained in v_4v_s, then we consider two cases.

Case 1.1. *Every chord contained in v_4v_s is non-crossed.*

If every chord contained in v_4v_s is non-crossed, then by Lemma 2, $\mathcal{V}[v_4, v_s]$ is a path. We now claim that there exists a chord in $S := \mathcal{C}[v_4, v_s] \setminus \{v_4v_s\}$ that contains at least one other chord. If this proposition does not hold, then we choose one chord v_iv_j with $4 < i < j \le s$ so that v_iv_j contains no other chords. If $|j - i| \ge 3$, then we can find two adjacent 2-vertices in

$\mathcal{V}[v_i, v_j]$, a contradiction. If $|j-i| = 2$, then $d(v_{i+1}) = 2$ and $d(v_i) \geq 5$. This implies that there are at least two non-crossed chords besides v_iv_j that are incident with v_i. Therefore, we would find two chords in S so that one contains the other in G, a contradiction. Hence, we can choose a chord v_iv_j in S so that v_iv_j contains at least one other chord, say v_av_b, and moreover, every chord contained in v_iv_j contains no other chords. Without loss of generality, assume that $b \neq j$. If $|b-a| \geq 3$, then we can find two adjacent 2-vertices in $\mathcal{V}[v_a, v_b]$, a contradiction. If $|b-a| = 2$, then $d(v_{a+1}) = 2$ and $d(v_b) \geq 5$. This implies that besides v_av_b, there are at least two non-crossed chords that are incident with v_b, therefore, we would find two chords in $\mathcal{C}[v_i, v_j] \setminus \{v_iv_j\}$ so that one contains the other, a contradiction to our assumption.

Case 1.2. *There is at least one pair of crossed chords that are contained in* $\mathcal{C}[v_4, v_s]$.

If there is at least one pair of crossed chords that are contained in v_4v_s, then we choose one pair of co-crossed chords v_av_b and v_cv_d with $c-a = b-c = d-b = 1$ and $v_av_c, v_cv_b, v_bv_d \in E(G)$. Since the configuration (d) is absent from G, $\min\{d(v_a), d(v_d)\} \geq 5$. If $a = 4$, then it is easy to see that (f) occurs in G. If $d = s$, then there exists an inner cluster $IC(x, a, d)$ with $4 \leq x < a$ and width $|\mathcal{V}[v_x, v_d]| < s$, a contradiction. Therefore, we assume that $a \neq 4$ and $d \neq s$. Since $IC(1, 4, s)$ is an inner cluster with the shortest width in G, there is no chord in the form v_av_i with $4 \leq i < a$ or in the form v_bv_j with $d < j \leq s$. Since the configuration (d) is absent from G, $\min\{d(v_a), d(v_d)\} \geq 5$. The above two facts imply that $v_av_d \in E(G)$ and there are a chord v_iv_d with $4 \leq i < a$ and a chord v_av_j with $d < j \leq s$. We call the graph induced by $v_av_b, v_av_c, v_av_d, v_bv_c, v_bv_d.v_cv_d, v_av_i$ and v_dv_j a K_4-*cluster* derived from v_av_b and v_cv_d, and by $|\mathcal{V}[v_i, v_j]|$, we denote the *width* of this K_4-cluster. Without loss of generality, we assume that the width of the above K_4-cluster is the shortest among all the K_4-clusters contained in the graph induced by $\mathcal{V}[v_4, v_s]$. If there are no crossed chords in $\mathcal{C}[v_d, v_j]$, then by Lemma 2, $\mathcal{V}[v_d, v_j]$ is either a non-edge or a path, because there are no edges between $\mathcal{V}(v_d, v_j)$ and $\mathcal{V}(v_j, v_d)$. Since $d(v_d) \geq 5$, $\mathcal{V}[v_d, v_j]$ cannot be a non-edge, thus it is a path. If there are no chords that are contained in $\mathcal{V}[v_d, v_j]$, then either (a), (e) or (g) would occur in G. If there are chords contained in $\mathcal{V}[v_d, v_j]$, then by similar arguments as in Case 1.1, one can prove that there are no non-crossed chords contained in $\mathcal{V}[v_d, v_j]$. If there is at least one pair of co-crossed chords $v_{a'}v_{b'}$ and $v_{c'}v_{d'}$ with $a' < c' < b' < d'$ that are contained in $\mathcal{V}[v_d, v_j]$, then $a' \neq d$, because otherwise $IC(a, d, b')$ would be an inner cluster shorter than $IC(1, 4, s)$, a contradiction. This implies, by similar arguments as above, that either there is an inner cluster $IC(x, a', d')$ with $d \leq x < a'$ and width $|\mathcal{V}[v_x, v_{d'}]| < s$, or $d' \neq j$ and there is an inner cluster $IC(a', d', y)$ with $d' < y \leq j$ and width $|\mathcal{V}[v_{a'}, v_y]| < s$, or $d' \neq j$ and there is a K_4-cluster derived from $v_{a'}v_{b'}$ and $v_{c'}v_{d'}$ with width no more than $|\mathcal{V}[v_d, v_j]| < |\mathcal{V}[v_i, v_j]|$. In either case, we would obtain a contradiction to our assumption.

Hence, the chord v_4v_s is crossed. □

Claim 2. *If* $IC(i, l, s)$ *is an inner cluster with the shortest width among all the inner clusters that contained in the graphs induced by* $\mathcal{V}[v_i, v_s]$, *then the chords* v_lv_s *cannot be crossed.*

Proof. Without loss of generality, assume that $i = 1$ and $l = 4$. Suppose, to the contrary, that v_lv_s is crossed by one other chord v_av_b with $4 < a < s$. If there is at least one pair of crossed chords that are contained in $\mathcal{C}[v_4, v_a]$ or $\mathcal{C}[v_a, v_s]$, then by similar arguments as in Case 1.2, one can obtain contradictions. Therefore, every chord contained in $\mathcal{C}[v_4, v_a]$ or $\mathcal{C}[v_a, v_s]$ is non-crossed. Since there are no edges between $\mathcal{V}(v_4, v_a)$ and $\mathcal{V}(v_a, v_4)$, or between $\mathcal{V}(v_a, v_s)$ and $\mathcal{V}(v_s, v_a)$, by Lemma 2, $\mathcal{V}[v_4, v_a]$ or $\mathcal{V}[v_a, v_s]$ is either non-edge or path. If $\mathcal{V}[v_4, v_a]$ and $\mathcal{V}[v_a, v_s]$ are non-edges, then $d(v_a) = 1$, a contradiction. If $\mathcal{V}[v_4, v_a]$ and $\mathcal{V}[v_a, v_s]$ are paths, then by similar arguments as in Case 1.1, (a) or (f) would appear in G. If $\mathcal{V}[v_4, v_a]$ is path and $\mathcal{V}[v_a, v_s]$ is non-edge, then by similar arguments as in Case 1.1, (a) would appear in G unless $a = 5$, in which case (e) occurs in G. Hence, we assume that $\mathcal{V}[v_4, v_a]$ is non-edge and $\mathcal{C}[v_a, v_s]$ is path in the following. By similar arguments as in Case 1.1, one can obtain contradictions if $s-a \geq 2$, so assume that $s-a = 1$, that is, $a = 5$ and $s = 6$. Since $d(v_a) = 2$, $b \neq 1$, because otherwise we would find (e). In the following, the graph induced by $v_1v_2, v_2v_3, v_3v_4, v_1v_3, v_2v_4.v_4v_6, v_5v_6$ and v_5v_b (or a graph isomorphic to this graph) is called a $\rtimes$-*cluster*, and the *width* of this $\rtimes$-cluster is $|\mathcal{V}[v_1, v_b]|$. Without loss of generality, we can assume the width of the above $\rtimes$-cluster is the shortest among all the $\rtimes$-clusters that are contained in the graph induced by $\mathcal{V}[v_1, v_b]$.

If there are no chords contained in v_6v_b, then (a) appears in G, so we assume that there are chords contained in v_6v_b. If every chord contained in v_6v_b is non-crossed, then by Lemma 2, $\mathcal{V}[v_6, v_b]$ is either non-edge or path. If $\mathcal{V}[v_6, v_b]$ is a non-edge, then v_5 and v_6 are two adjacent 2-vertices, a contradiction, so we assume that $\mathcal{V}[v_6, v_b]$ is a path. In this case, we can use similar arguments as in Case 1.1 to obtain contradictions. Therefore, we shall assume that there is at least one pair of crossed chords that are contained in $\mathcal{C}[v_6, v_b]$.

We arbitrarily choose one pair of co-crossed chords $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$ with $i' < k' < j' < l'$ that are contained in $\mathcal{C}[v_6, v_b]$. Since both v_6 and v_b are adjacent to a 2-vertex v_5, $i' \neq 6$ and $l' \neq b$, because otherwise we would find (e) in G. Due to the absence of (d), we have $\min\{d(v_{i'}), d(v_{l'})\} \geq 5$, which implies that there exist $s' \neq i', k'$ and $m' \neq j', l'$ so that $v_{l'}v_{s'}$ and $v_{i'}v_{m'}$ are chords in G. If $l' < s' \leq b$, then we can assume, without loss of generality, that $IC(i', l', s')$ is an inner cluster with the shortest width among all the inner clusters contained in the graph induced by $\mathcal{V}[v_{i'}, v_{s'}]$. If $v_{l'}v_{s'}$ is a non-crossed chord, then we use similar arguments as in the proof of Claim 1 to obtain contradictions. If $v_{l'}v_{s'}$ is a chord crossed by one other chord $v_{a'}v_{b'}$ with $l' < a' < s'$, then by similar arguments as in the first part of this proof, one can deduce that $s'-a' = a'-l' = 1$, $v_{l'}v_{a'} \notin E(G)$ and $v_{a'}v_{s'} \in E(G)$. This implies that $6 \leq b' < i'$, because otherwise we would find a shorter $\rtimes$-cluster, a contradiction to our assumption. Since $v_{a'}v_{b'}$ has already crossed $v_{l'}v_{s'}$ in G, $b' \leq m' < i'$. Without loss of generality, assume that $IC(m', i', l')$ is an inner cluster with the shortest width among all the inner clusters contained in the graph induced by $\mathcal{V}[v_{m'}, v_{l'}]$. If $v_{i'}v_{m'}$ is a non-crossed chord, then we use similar arguments as in the proof of Claim 1 to obtain contradictions. If $v_{i'}v_{m'}$ is crossed by one other chord $v_{c'}v_{d'}$ with $m' < c' < i'$, then by similar arguments as in the first part of this proof, one can deduce that $m' - c' = c' - i' = 1$, $v_{c'}v_{i'} \notin E(G)$ and $v_{c'}v_{m'} \in E(G)$. Since $v_{a'}v_{b'}$ has already crossed $v_{l'}v_{s'}$ in G, $b' \leq d' < m'$, which implies a shorter $\rtimes$-cluster that is contained in the graph induced by $\mathcal{V}[v_1, v_b]$, a contradiction to our assumption. Therefore, for any chord $v_{l'}v_{s'}$ with $s' \neq i', k'$, we have $6 \leq s' < i'$. Similarly, we can prove, for any chord $v_{i'}v_{m'}$ with $m' \neq j', l'$, that $l' < m' \leq b$. Since $\min\{d(v_{i'}), d(v_{l'})\} \geq 5$, $v_{i'}v_{l'} \in E(G)$, which implies a K_4-cluster derived from $v_{i'}v_{j'}$ and $v_{k'}v_{l'}$. Without loss of generality, assume that the width of this K_4-cluster is the shortest among all the K_4-clusters contained in the graph induced by $\mathcal{V}[v_{m'}, v_{s'}]$, then by similar arguments as in the proof of Claim 1, we can obtain contradictions. □

It is easy to see that the above two claims are conflicting. Hence, every 2-connected pseudo-outerplanar graphs contains one of the configurations among (a)–(g). We now assume that G has cut vertices and choose one of its end-blocks. Denote the chosen end-block by B and the vertices of this end-block that lie in a clockwise sequence by $v_1, \ldots, v_{|B|}$. Without loss of generality, assume that v_1 is the unique cut-vertex of B.

First, assume that there are no crossings in the end-block B. Since B is a 2-connected outerplanar graph, B is Hamiltonian, which implies that $\mathcal{V}[v_1, v_{|B|}]$ is a path. If there is at most one chord in B, then it is easy to see that G contains (a). If there are two chords v_iv_j and v_sv_t in B, then without loss of generality, we can assume that $1 \leq j < t < s < i$, therefore, by similar arguments as in Subcase 1.1, one can prove that G contains (a).

At last, assume that there is at least one pair of crossed chords v_iv_j and v_kv_l in B. Without loss of generality, assume that $1 < i < k < j < l \leq |B|$ and that v_iv_j and v_kv_l are a pair of co-crossed chords (so they satisfy (1)). Since (d) is absent from G, $\min\{d_B(v_i), d_B(v_l)\} \geq 5$.

If there is a vertex v_s with $l < s \leq |B|$ or $s = 1$ so that v_lv_s is a non-crossed chord, then by the proof of Claim 1, one can find one of the configurations (a)–(g) in the graph induced by $\mathcal{V}[v_i, v_s]$, and moreover, v_1 is not the vertex with bounded degree in the configuration. If there is a vertex v_m with $1 \leq m < i$ so that v_iv_m is a non-crossed chord, then we can prove the theorem similarly. Therefore, we have the following.

Claim 3. *There do not exist vertex v_s with $l < s \leq |B|$ or $s = 1$ so that v_lv_s is a non-crossed chord or vertex v_m with $1 \leq m < i$ so that v_iv_m is a non-crossed chord.* □

Suppose that there is a vertex v_s with $l < s \leq |B|$ or $s = 1$ so that v_lv_s is a chord crossed by one other chord v_av_b with $l < a < s$. If the graph induced by $v_iv_k, v_kv_j, v_jv_l, v_iv_j, v_kv_l, v_lv_s, v_sv_a$ and v_av_b is not a $\rtimes$-cluster, then by the proof of Claim 2, one can find one of the configurations (a)–(g) in the graph induced by $\mathcal{V}[v_i, v_s]$, and thus in G. If $s < b \leq |B|$ or $b = 1$, then by the proof of Claim 2, one can also find one of the configurations (a)–(g) in the graph induced by $\mathcal{V}[v_i, v_b]$, and moreover, v_1 is not the vertex with bounded degree in the configuration. Thus, we have $a - l = s - a = 1$, $v_lv_a \notin E(G)$, $v_av_s \in E(G)$ and $1 < b \leq i$. If $b = i$, then it is easy to prove that $d_B(v_l) \leq 4$, a contradiction. If $b \neq i$, then there is a chord v_mv_i with $b \leq m < i$, since $d_B(v_i) \geq 5$ and v_av_b is crossed by v_lv_s. By Claim 3, v_mv_i is a crossed chord, and we assume that it is crossed by v_nv_t with $m < n < i$. Similarly as above, we shall also assume that $i - n = n - m = 1$, $v_nv_i \notin E(G)$ and $v_mv_n \in E(G)$. If $b \leq t < m$, then by similar arguments as in the proof of Claim 2, one can find one of the configurations (a)–(g) in the graph induced by $\mathcal{V}[v_t, v_l]$, and thus in G. If $t = l$, then $d_B(v_i) \leq 4$, a contradiction. Therefore, we immediately deduce the following claim.

Claim 4. *There do not exist vertex v_s with $l < s \leq |B|$ or $s = 1$ so that v_lv_s is a crossed chord, and similarly, there do not exist v_m with $1 \leq m < i$ so that v_mv_i is a crossed chord.* □

Since $\min\{d_B(v_i), d_B(v_l)\} \geq 5$, by Claims 3 and 4, there exist vertices v_s with $1 < s < i$ and v_m with $l < m \leq |B|$ so that v_lv_s and v_iv_m are two chords that cross each other. If there is at least one pair of crossed chords that are contained in $\mathcal{C}[v_l, v_m]$, then by similar arguments as in Case 1.2, one can obtain contradictions. If every chord contained in $\mathcal{C}[v_l, v_m]$ is non-crossed, then by Lemma 2, $\mathcal{V}[v_l, v_m]$ is either a path or a non-edge. However, if $\mathcal{V}[v_l, v_m]$ is a path with $m - l \geq 2$, then by similar arguments as in Case 1.1, one can find (a) or (e) in G; if $\mathcal{V}[v_l, v_m]$ is a path with $m - l = 1$, then (g) occurs in G, since $d_B(v_l) \geq 5$ implies $v_iv_l \in E(G)$ and if $\mathcal{V}[v_l, v_m]$ is a non-edge, then $d_B(v_l) \leq 4$, a contradiction. □

3. List total coloring of PO-graphs

In this section, we present a sufficient condition for a pseudo-outerplanar graph to have a list total coloring and prove the following theorem.

Theorem 5. *Let G be a pseudo-outerplanar graph, and let L be a list of G. If*

$$|L(x)| \geq \max\{6, \Delta(G) + 1\}$$

for each $x \in VE(G)$, then G has a total L-coloring.

Before proving Theorem 5, we introduce some necessary notations. Let L be a list of a graph G and let L' be a list of a graph $G' \subset G$ with $L'(x) = L(x)$ for each element $x \in VE(G)$. Suppose that we have already obtained a total L'-coloring φ' of G', and that we are to extend φ' to a total L-coloring φ of G without altering the colors in G'. For each $x \in VE(G)$, let $L_{av}(x, \varphi')$ be the *available list* (the set of all colors in $L(x)$ that are available) for x when φ' is extended to a total L-coloring φ of G.

Lemma 6. *Suppose that G contains a path* $P = v_1u_1v_2u_2v_3u_3v_4$ *with* $v_1v_2, v_2v_3, v_3v_4 \in E(G)$ *and* $d(u_1) = d(u_2) = d(u_3) = 2$. *Let* φ' *be a partial total L-coloring of G so that the uncolored elements under* φ' *are* $u_1, u_2, u_3, v_2, v_3, v_1v_2, v_2v_3, v_3v_4$ *and the edges of the path P, where L is a list assignment of G. If*

$$\min\{|L_{\text{av}}(u_1v_1, \varphi')|, |L_{\text{av}}(u_3v_4, \varphi')|, |L_{\text{av}}(v_1v_2, \varphi')|, |L_{\text{av}}(v_3v_4, \varphi')|\} \geq 2,$$
$$\min\{|L_{\text{av}}(v_2, \varphi')|, |L_{\text{av}}(v_3, \varphi')|\} \geq 3,$$
$$|L_{\text{av}}(v_2v_3, \varphi')| \geq 4,$$

and

$$\min\{|L_{\text{av}}(u_1v_2, \varphi')|, |L_{\text{av}}(u_2v_2, \varphi')|, |L_{\text{av}}(u_2v_3, \varphi')|, |L_{\text{av}}(u_3v_3, \varphi')|\} \geq 5,$$

then φ' *can be extended to a total L-coloring* φ *of G without altering the colors in* G'.

Proof. Without loss of generality, we assume that $|L_{\text{av}}(u_1v_1, \varphi')| = |L_{\text{av}}(u_3v_4, \varphi')| = |L_{\text{av}}(v_1v_2, \varphi')| = |L_{\text{av}}(v_3v_4, \varphi')| = 2$, $|L_{\text{av}}(v_2, \varphi')| = |L_{\text{av}}(v_3, \varphi')| = 3$, $|L_{\text{av}}(v_2v_3, \varphi')| = 4$, and $|L_{\text{av}}(u_1v_2, \varphi')| = |L_{\text{av}}(u_2v_2, \varphi')| = |L_{\text{av}}(u_2v_3, \varphi')| = |L_{\text{av}}(u_3v_3, \varphi')| = 5$. (otherwise we can shorten some lists that assigned to the elements of $VE(G)$ so that those conditions are satisfied). We extend φ' to a total L-coloring φ of G by two stages.

Stage 1. Color u_3v_3, u_3v_4, v_3v_4 and v_3 so that the resulted partial coloring φ^1 satisfies one of the following conditions:

(1) $|L_{\text{av}}(v_2v_3, \varphi^1)| \geq 3$,
(2) $|L_{\text{av}}(v_2v_3, \varphi^1)| = 2$ and $L_{\text{av}}(v_2v_3, \varphi^1) \neq L_{\text{av}}(v_2, \varphi^1)$ if $|L_{\text{av}}(v_2, \varphi^1)| = 2$,
(3) $|L_{\text{av}}(v_2v_3, \varphi^1)| = 2$ and $L_{\text{av}}(v_1v_2, \varphi^1) \neq L_{\text{av}}(v_2, \varphi^1)$ if $|L_{\text{av}}(v_2, \varphi^1)| = 2$.

We now prove that the coloring φ^1 constructed in stage 1 exists. Assume that $L_{\text{av}}(v_3v_4, \varphi') = \{1, 2\}$. We now color v_3 with a color, say 3, from $L_{\text{av}}(v_3, \varphi') \setminus \{1, 2\}$.

Case 1. $2 \in L_{\text{av}}(u_3v_4, \varphi')$ (the case when $1 \in L_{\text{av}}(u_3v_4, \varphi')$ is similar).

Color v_3v_4 and u_3v_4 with 1 and 2, and then discuss two subcases.

Case 1.1. $\{1, 2, 3\} \subseteq L_{\text{av}}(u_3v_3, \varphi')$.

Assume that $L_{\text{av}}(u_3v_3, \varphi') = \{1, 2, 3, 4, 5\}$. Denote the current partial coloring by ϕ_0.

If $4 \notin L_{\text{av}}(v_2v_3, \varphi') \setminus \{1, 3\}$, then color u_3v_3 with 4, and we have $\{1, 3\} \subseteq L_{\text{av}}(v_2v_3, \varphi')$, otherwise ϕ_0 satisfies (1) and we let $\varphi^1 := \phi_0$. Let $L_{\text{av}}(v_2v_3, \varphi') = \{1, 3, n_1, n_2\}$, where $\{1, 3\} \cap \{n_1, n_2\} = \emptyset$. If $L_{\text{av}}(v_2, \varphi') \neq \{3, n_1, n_2\}$, then ϕ_0 satisfies (2) and we let $\varphi^1 := \phi_0$, so we assume that $L_{\text{av}}(v_2, \varphi') = \{3, n_1, n_2\}$. Now we erase the color on u_3v_3 and recolor v_3v_4 and u_3v_4 with 2 and a color $\phi_1(u_3v_4) \in L_{\text{av}}(u_3v_4, \varphi') \setminus \{2\}$, respectively. If $\phi_1(u_3v_4) \neq 4$, then color u_3v_3 with 4. Since the current coloring ϕ_1 satisfies (1) or (2), we let $\varphi^1 := \phi_1$. If $\phi_1(u_3v_4) = 4$, then $L_{\text{av}}(u_3v_4, \varphi') = \{2, 4\}$, and color u_3v_3 with 5. If $\{n_1, n_2\} \neq \{2, 5\}$, then the current coloring ϕ_1 satisfies (1) or (2), so let $\varphi^1 := \phi_1$. If $\{n_1, n_2\} = \{2, 5\}$, then recolor u_3v_3 with 3. If $1 \notin L_{\text{av}}(v_3, \varphi') \setminus \{2, 3\}$, then $L_{\text{av}}(v_3, \varphi') = \{2, 3, 5\}$, otherwise we can recolor v_3 with a color from $L_{\text{av}}(v_3, \varphi') \setminus \{2, 3\}$, and the resulted partial coloring satisfies (2). In this case, we recolor v_3v_4, u_3v_4 and u_3v_3 with 1, 2 and 4. If the current coloring does not satisfy (3), then $L_{\text{av}}(v_1v_2, \varphi') = \{2, 5\}$, thus we can construct a partial coloring satisfying (3) by recolor v_3 with 5. Therefore, we assume that $1 \in L_{\text{av}}(v_3, \varphi') \setminus \{2, 3\}$. If $5 \notin L_{\text{av}}(v_3, \varphi') \setminus \{2, 3\}$, then $L_{\text{av}}(v_3, \varphi') = \{1, 2, 3\}$, otherwise we can recolor v_3 with a color from $L_{\text{av}}(v_3, \varphi') \setminus \{1, 2, 3\}$, and the resulted partial coloring satisfies (2). In this case, we recolor v_3v_4, u_3v_4 and u_3v_3 with 1, 2 and 4, and color v_3 with 2. If the current coloring does not satisfy (3), then recolor v_3 with 3 and one can check that the new coloring satisfies (3). Therefore, $5 \in L_{\text{av}}(v_3, \varphi') \setminus \{2, 3\}$, which implies that $L_{\text{av}}(v_3, \varphi') = \{1, 3, 5\}$. In this case, we recolor v_3v_4, u_3v_4 and u_3v_3 with 1, 2 and 4, and color v_3 with 3. If the current coloring does not satisfy (3), then recolor v_3 with 5 and the new coloring satisfies (3).

If $5 \notin L_{\text{av}}(v_2v_3, \varphi') \setminus \{1, 3\}$, then we can do the similar arguments as above by symmetry, so we assume that $\{4, 5\} \subseteq L_{\text{av}}(v_2v_3, \varphi')$. Assume that $L_{\text{av}}(v_2v_3, \varphi') = \{4, 5, n_1, n_2\}$, where $\{n_1, n_2\} \cap \{4, 5\} = \emptyset$. If $\{n_1, n_2\} \neq \{1, 3\}$, then color u_3v_3 with 4. If $5 \notin L_{\text{av}}(v_2, \varphi')$, then it is easy to see that the current partial coloring satisfies (1) or (2). If $5 \in L_{\text{av}}(v_2, \varphi')$, then recolor u_3v_3 with 5 and the resulted partial coloring also satisfies (1) or (2). Therefore, we assume that $L_{\text{av}}(v_2v_3, \varphi') = \{1, 3, 4, 5\}$. Now we recolor v_3v_4 and u_3v_4 with 2 and a color $\phi_2(u_3v_4) \in L_{\text{av}}(u_3v_4, \varphi') \setminus \{2\}$, then color u_3v_3 with a color $\phi_2(u_3v_3) \in \{4, 5\} \setminus \{\phi_2(u_3v_4)\}$. Without loss of generality, assume that $\phi_2(u_3v_3) = 4$. We now have $L_{\text{av}}(v_2, \varphi') = \{1, 3, 5\}$, otherwise ϕ_2 satisfies (2) and let $\varphi^1 := \phi_2$. If $\phi_2(u_3v_4) \neq 5$, then recolor u_3v_3 with 5 and the resulted coloring satisfies (2). If $\phi_2(u_3v_4) = 5$, then recolor u_3v_3 with 1 and the resulted coloring also satisfies (2).

Case 1.2. $\{1, 2, 3\} \nsubseteq L_{\text{av}}(u_3v_3, \varphi')$.

Since $|L_{\text{av}}(u_3v_3, \varphi')| = 5$, we can assume that $\{4, 5, 6\} \subseteq L_{\text{av}}(u_3v_3, \varphi')$. If $\{4, 5\} \subseteq L_{\text{av}}(v_2v_3, \varphi')$, then $L_{\text{av}}(v_2v_3, \varphi') = \{1, 3, 4, 5\}$, otherwise we color u_3v_3 with 6 and get a partial coloring satisfying (1). We now color u_3v_3 with 6, and deduce that $L_{\text{av}}(v_2, \varphi') = \{3, 4, 5\}$, otherwise the current partial coloring satisfies (2). In this case, we recolor v_3v_4, u_3v_4 and u_3v_3 with 2, $\phi_3(u_3v_4) \in L_{\text{av}}(u_3v_4, \varphi') \setminus \{2\}$ and $\phi_3(u_3v_3) \in \{4, 5, 6\} \setminus \{\phi_3(u_3v_4)\}$. It is easy to check that the partial coloring ϕ_3 satisfies (2), so we let $\varphi' := \phi_3$. By symmetry, one can prove the same result if $\{4, 6\} \subseteq L_{\text{av}}(v_2v_3, \varphi')$ or $\{5, 6\} \subseteq L_{\text{av}}(v_2v_3, \varphi')$. Therefore, we assume, without loss of generality, that $5, 6 \notin L_{\text{av}}(v_2v_3, \varphi')$. We now color u_3v_3 with 5, and deduce that $\{1, 3\} \subseteq L_{\text{av}}(v_2v_3, \varphi')$, otherwise the current partial coloring satisfies (1). Assume that $L_{\text{av}}(v_2v_3, \varphi') = \{1, 3, n_1, n_2\}$, where $\{n_1, n_2\} \cap \{1, 3\} = \emptyset$. We then have $L_{\text{av}}(v_2, \varphi') = \{3, n_1, n_2\}$, because otherwise the current coloring satisfies (2). In this case, we recolor v_3v_4, u_3v_4 and u_3v_3 with 2, $\phi_4(u_3v_4) \in L_{av}(u_3v_4, \varphi') \setminus \{2\}$ and $\phi_4(u_3v_3) \in \{5, 6\} \setminus \{\phi_4(u_3v_4)\}$. One can check that the resulted partial coloring ϕ_4 satisfies (2), thus we let $\varphi' := \phi_4$.

Case 2. $3 \in L_{av}(u_3v_4, \varphi')$.

We first color u_3v_4 with 3. Assume that $\{4, 5\} \subseteq L_{av}(u_3v_3, \varphi')$.

If $\{4, 5\} \subseteq L_{av}(v_2v_3, \varphi')$, then $1 \in L_{av}(v_2v_3, \varphi')$. Otherwise, we color v_3v_4 and u_3v_3 with 1 and 4. If the current coloring does not satisfy (2), then recolor u_3v_3 with 5 and get a partial coloring satisfying (2). Similarly, $3 \in L_{av}(v_2v_3, \varphi')$, so $L_{av}(v_2v_3, \varphi') = \{1, 3, 4, 5\}$. In this case, we color v_3v_4 and u_3v_3 with 2 and 4. If the current does not satisfy (2), then recolor u_3v_3 with 5 and the resulted coloring satisfies (2). If $\{4, 5\} \not\subseteq L_{av}(v_2v_3, \varphi')$, then we assume, without loss of generality, that $4 \notin L_{av}(v_2v_3, \varphi')$. We now color u_3v_3 and v_3v_4 with 4 and 1. If $\{1, 3\} \not\subseteq L_{av}(v_2v_3, \varphi')$, then it is easy to see that the current partial coloring satisfies (1). If $L_{av}(v_2v_3, \varphi') = \{1, 3, n_1, n_2\}$, where $\{n_1, n_2\} \cap \{1, 3\} = \emptyset$, then $L_{av}(v_2, \varphi') = \{3, n_1, n_2\}$, otherwise the current partial coloring satisfies (2). In this case, we can also get a partial coloring satisfying (2) by recoloring v_3v_4 with 2.

Case 3. $L_{av}(u_3v_4, \varphi') \cap \{1, 2, 3\} = \emptyset$.

Without loss of generality, assume that $L_{av}(u_3v_4, \varphi') = \{4, 5\}$. We claim that $1 \in L_{av}(v_2v_3, \varphi')$. Otherwise, color v_3v_4, u_3v_4 and u_3v_3 with 1, 4 and $\phi_5(u_3v_3) \in L_{av}(u_3v_3, \varphi') \setminus \{1, 3, 4\}$. If the current coloring ϕ_5 does not satisfy (2), then recolor u_3v_3 with a color $\phi_6(u_3v_3) \in L_{av}(u_3v_3, \varphi') \setminus \{1, 3, 4, \phi_5(u_3v_3)\}$. It is easy to see that ϕ_6 satisfies (2), thus we let $\varphi' := \phi_6$. Similarly, $2, 3 \in L_{av}(v_2v_3, \varphi')$. Assume that $L_{av}(v_2v_3, \varphi') = \{1, 2, 3, n_1\}$, where $n_1 \notin \{1, 2, 3\}$. We now color v_3v_4, u_3v_4 and u_3v_3 with 1, 4 and a color $\phi_7(u_3v_3) \in L_{av}(u_3v_3, \varphi') \setminus \{1, 2, 3, 4\}$. If $n_1 \neq \phi_7(u_3v_3)$, then we recolor v_3v_4 with 2 if ϕ_7 does not satisfy (2), and the resulted coloring satisfies (2), so $n_1 = \phi_7(u_3v_3)$. If $L_{av}(u_3v_3, \varphi') \neq \{1, 2, 3, 4, n_1\}$, then recolor u_3v_3 with a color from $L_{av}(u_3v_3, \varphi') \setminus \{1, 2, 3, 4, n_1\}$ and the resulted coloring can be dealt with as above, so $L_{av}(u_3v_3, \varphi') = \{1, 2, 3, 4, n_1\}$ and $n_1 \neq 4$. In this case, we color v_3v_4, u_3v_4 and u_3v_3 with 1, 5 and 4. If the current partial coloring does not satisfy (2), then we can construct another partial coloring that satisfies (2) by recoloring v_3v_4 with 2.

Stage 2. Extend φ^1 to a total L-coloring φ of G without altering the assigned colors.

Note that φ^1 satisfies $|L_{av}(u_1v_1, \varphi^1)| = |L_{av}(v_1v_2, \varphi^1)| = 2$, $|L_{av}(u_1v_2, \varphi^1)| = |L_{av}(u_2v_2, \varphi^1)| = 5$ by the choice of φ', and $|L_{av}(u_2v_2, \varphi^1)|, |L_{av}(v_2v_3, \varphi^1)|, |L_{av}(v_2, \varphi^1)| \geq 2$, and moreover, either $L_{av}(v_2v_3, \varphi^1) \neq L_{av}(v_2, \varphi^1)$ or $L_{av}(v_1v_2, \varphi^1) \neq L_{av}(v_2, \varphi^1)$ if $|L_{av}(v_2, \varphi^1)| = 2$, by the choice of φ^1 in Stage 1.

Without loss of generality, we assume that $|L_{av}(u_2v_2, \varphi^1)| = |L_{av}(v_2v_3, \varphi^1)| = |L_{av}(v_2, \varphi^1)| = 2$ and $L_{av}(v_2v_3, \varphi^1) \neq L_{av}(v_2, \varphi^1) = \{1, 2\}$ in the following arguments.

If $L_{av}(v_1v_2, \varphi^1) \cap L_{av}(u_2v_3, \varphi^1) \neq \emptyset$, then color v_1v_2 and u_2v_3 with $\mu(v_1v_2) = \mu(u_2v_3) \in L_{av}(v_1v_2, \varphi^1) \cap L_{av}(u_2v_3, \varphi^1)$. Since $L_{av}(v_2v_3, \varphi^1) \neq L_{av}(v_2, \varphi^1)$, we can color v_2 and v_2v_3 with $\mu(v_2) \in L_{av}(v_2, \varphi^1) \setminus \{\mu(v_1v_2)\}$ and $\mu(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{\mu(u_2v_3)\}$ so that $\mu(v_2) \neq \mu(u_2v_3)$. We then color u_1v_1, u_1v_2 and u_2v_2 with $\mu(u_1v_1) \in L_{av}(u_1v_1, \varphi^1) \setminus \{\mu(v_1v_2)\}$, $\mu(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{\mu(u_1v_1), \mu(v_1v_2), \mu(v_2), \mu(v_2v_3)\}$ and $\mu(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{\mu(u_2v_3), \mu(v_2v_3), \mu(v_2), \mu(u_1v_2)\}$, respectively. Since u_1, u_2 and u_3 are 2-vertices, they can be easily colored at the last stage. Therefore, we have a total L-coloring μ of G. In what follows, we assume that $L_{av}(v_1v_2, \varphi^1) \cap L_{av}(u_2v_3, \varphi^1) = \emptyset$.

Case 1′. $L_{av}(v_2v_3, \varphi^1) = \{1, 3\}$.

If $1 \in L_{av}(u_1v_1, \varphi^1)$ and $L_{av}(v_1v_2, \varphi^1) \neq \{1, 2\}$, then color u_1v_1 and v_2v_3 with 1, and color $v_2, v_1v_2, u_2v_3, u_2v_2$ and u_1v_2 with 2, $\mu_1(v_1v_2) \in L_{av}(v_1v_2, \varphi^1) \setminus \{1, 2\}$, $\mu_1(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{1\}$, $\mu_1(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{1, 2, \mu_1(v_1v_2), \mu_1(u_2v_3)\}$ and $\mu_1(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{1, 2, \mu_1(v_1v_2), \mu_1(u_2v_2)\}$, respectively. If $1 \in L_{av}(u_1v_1, \varphi^1)$ and $L_{av}(v_1v_2, \varphi^1) = \{1, 2\}$, then color u_1v_1 and v_2 with 1, and color $v_2v_3, v_1v_2, u_2v_3, u_2v_2$ and u_1v_2 with 3, 2, $\mu_1(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{3\}$, $\mu_1(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{1, 2, 3, \mu_1(u_2v_3)\}$ and $\mu_1(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{1, 2, 3, \mu_1(u_2v_2)\}$, respectively. In each case, we can extend μ_1 to the 2-vertices u_1, u_2 and u_3 and get a total L-coloring of G. Therefore, $1 \notin L_{av}(u_1v_1, \varphi^1)$. Similarly, $2, 3 \notin L_{av}(u_1v_1, \varphi^1)$. We assume, without loss of generality, that $L_{av}(u_1v_1, \varphi^1) = \{4, 5\}$.

If $4 \notin L_{av}(u_1v_2, \varphi^1)$, then color u_1v_1 and v_1v_2 with 4 and $\mu_2(v_1v_2) \in L_{av}(v_1v_2, \varphi^1) \setminus \{4\}$. If $\mu_2(v_1v_2) \neq 2$, then color $v_2, v_2v_3, u_2v_3, u_2v_2$ and u_1v_2 with 2, $\mu_2(v_2v_3) \in \{1, 3\} \setminus \{\mu_2(v_1v_2)\}$, $\mu_2(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{\mu_2(v_2v_3)\}$, $\mu_2(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{2, \mu_2(v_1v_2), \mu_2(v_2v_3), \mu_2(u_2v_3)\}$ and $\mu_2(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{2, \mu_2(v_1v_2), \mu_2(v_2v_3), \mu_2(u_2v_2)\}$. If $\mu_2(v_1v_2) = 2$, then color $v_2, v_2v_3, u_2v_3, u_2v_2$ and u_1v_2 with 1, 3, $\mu_2(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{3\}$, $\mu_2(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{1, 2, 3, \mu_2(u_2v_3)\}$ and $\mu_2(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{1, 2, 3, \mu_2(u_2v_2)\}$. In each case, we can extend μ_2 to the 2-vertices u_1, u_2 and u_3 and get a total L-coloring of G. Therefore, $4 \in L_{av}(u_1v_2, \varphi^1)$. Similarly, we have $1, 2, 3, 5 \in L_{av}(u_1v_2, \varphi^1)$, so $L_{av}(u_1v_2, \varphi^1) = \{1, 2, 3, 4, 5\}$. By similar arguments as above, we can also prove that $L_{av}(u_2v_2, \varphi^1) = \{1, 2, 3, 4, 5\}$, $L_{av}(v_1v_2, \varphi^1) \subseteq \{1, 2, 3, 4, 5\}$ and $L_{av}(u_2v_3, \varphi^1) \subseteq \{1, 2, 3, 4, 5\}$.

If $1 \in L_{av}(v_1v_2, \varphi^1)$, then color v_1v_2, v_2, v_2v_3, u_1v_1, u_1v_2 and u_2v_2 with 1, 2, 3, 4, 5 and 4. If $L_{av}(u_2v_3, \varphi^1) \neq \{3, 4\}$, then color u_2v_3 with a color in $L_{av}(u_2v_3, \varphi^1) \setminus \{3, 4\}$. If $L_{av}(u_2v_3, \varphi^1) = \{3, 4\}$, then recolor u_1v_1, u_1v_2 and u_2v_2 with 5, 4 and 5, and color u_2v_3 with 4. In each case, we can extend the current coloring to the 2-vertices u_1, u_2 and u_3 and get a total L-coloring of G. By similar arguments as above, we can complete the proof of this lemma if $2 \in L_{av}(v_1v_2, \varphi^1)$ or $3 \in L_{av}(v_1v_2, \varphi^1)$. Therefore, we assume that $L_{av}(v_1v_2, \varphi^1) = \{4, 5\}$. In this case, we color v_1v_2, v_2, v_2v_3, u_1v_1, u_1v_2 and u_2v_2 with 4, 1, 3, 5, 2 and 5. Since $L_{av}(v_1v_2, \varphi^1) \cap L_{av}(u_2v_3, \varphi^1) = \emptyset$, $5 \notin L_{av}(u_2v_3, \varphi^1)$. Hence, we color u_2v_3 with a color from $L_{av}(u_2v_3, \varphi^1) \setminus \{3\}$ and then extend the coloring at this stage to u_1, u_2 and u_3 to obtain a total L-coloring of G.

Case 2′. $L_{av}(v_2v_3, \varphi^1) = \{3, 4\}$.

By similar arguments as in the first part of Case 1′, one can prove that $L_{av}(u_1v_1, \varphi^1) \cap \{1, 2, 3, 4\} = \emptyset$, so we assume that $L_{av}(u_1v_1, \varphi^1) = \{5, 6\}$. Since $|L_{av}(u_1v_2, \varphi^1)| = 5$, $\{1, 2, 3, 4, 5, 6\} \setminus L_{av}(u_1v_2, \varphi^1) \neq \emptyset$. Without loss of generality, assume that $1 \notin L_{av}(u_1v_2, \varphi^1)$ and color v_2 with 1. If $L_{av}(v_1v_2, \varphi^1) = \{1, 3\}$, then color $v_2v_3, v_1v_2, u_1v_1, u_2v_3, u_2v_2$ and u_1v_2 with 4, 3, 5, $\mu_3(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{4\}$, $\mu_3(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{1, 3, 4, \mu_3(u_2v_3)\}$ and $\mu_3(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{3, 4, 5, \mu_3(u_2v_2)\}$. If $L_{av}(v_1v_2, \varphi^1) \neq \{1, 3\}$, then color $v_2v_3, v_1v_2, u_1v_1, u_2v_3, u_2v_2$ and u_1v_2 with 3, $\mu_3(v_1v_2) \in L_{av}(v_1v_2, \varphi^1) \setminus \{1, 3\}$, 5, $\mu_3(u_2v_3) \in L_{av}(u_2v_3, \varphi^1) \setminus \{3\}$, $\mu_3(u_2v_2) \in L_{av}(u_2v_2, \varphi^1) \setminus \{1, 3, \mu_3(v_1v_2), \mu_3(u_2v_3)\}$ and $\mu_3(u_1v_2) \in L_{av}(u_1v_2, \varphi^1) \setminus \{3, 5, \mu_3(v_1v_2), \mu_3(u_2v_2)\}$, respectively. In each case, we can extend the partial coloring μ_3 to the 2-vertices u_1, u_2 and u_3 and get a total L-coloring of G. □

Lemma 7. *Suppose that G contains a cycle $u_1u_2u_3u_4$ with $u_2u_4 \in E(G)$ and $d(u_2) = d(u_4) = 3$ and that $G' = G - \{u_2, u_4\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$, where L is a list assignment of G. If*

$$|L_{av}(u_2u_4, \varphi')| \geq 6,$$
$$\min\{|L_{av}(u_2, \varphi')|, |L_{av}(u_4, \varphi')|\} \geq 4,$$
$$\min\{|L_{av}(u_1u_2, \varphi')|, |L_{av}(u_2u_3, \varphi')|, |L_{av}(u_3u_4, \varphi')|, |L_{av}(u_1u_4, \varphi')|\} \geq 2$$

and

$$L_{av}(u_1u_2, \varphi') \neq L_{av}(u_2u_3, \varphi') \quad \text{when } |L_{av}(u_1u_2, \varphi')| = |L_{av}(u_2u_3, \varphi')| = 2$$

then φ' can be extended to a total L-coloring φ of G without altering the colors in G'.

Proof. Without loss of generality, we assume that $|L_{av}(u_2u_4, \varphi')| = 6$, $|L_{av}(u_2, \varphi')| = |L_{av}(u_4, \varphi')| = 4$ and $|L_{av}(u_1u_2, \varphi')| = |L_{av}(u_2u_3, \varphi')| = |L_{av}(u_3u_4, \varphi')| = |L_{av}(u_1u_4, \varphi')| = 2$ (otherwise we can shorten some lists that assigned to the elements of $VE(G)$ so that those conditions are satisfied), and then split the proofs into the following two cases.

Case 1. $L_{av}(u_1u_2, \varphi') = \{1, 2\}$ and $L_{av}(u_2u_3, \varphi') = \{3, 4\}$.

If $L_{av}(u_1u_4, \varphi') = \{1, 2\}$, then color u_1u_2 and u_1u_4 with 1 and 2, u_3u_4 with $\varphi''(u_3u_4) \in L_{av}(u_3u_4, \varphi') \setminus \{2\}$ and u_2u_3 with $\varphi''(u_2u_3) \in \{3, 4\} \setminus \{\varphi''(u_3u_4)\}$. Denote the extended partial coloring by φ''. One can see that $|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')|, |L_{av}(u_2u_4, \varphi'')| \geq 2$. If $L_{av}(u_2, \varphi'') = L_{av}(u_4, \varphi'') = L_{av}(u_2u_4, \varphi'')$, then recolor u_1u_2 and u_1u_4 with 2 and 1 when $\varphi''(u_3u_4) \neq 1$, or recolor u_2u_3 with the color in $\{3, 4\} \setminus \{\varphi''(u_2u_3)\}$ when $\varphi''(u_3u_4) = 1$. We still denote current coloring by φ'' but now we do not have $L_{av}(u_2, \varphi'') = L_{av}(u_4, \varphi'') = L_{av}(u_2u_4, \varphi'')$. Therefore, we can easily extend φ'' to a total L-coloring φ of G by coloring u_2, u_4 and u_2u_4 properly.

If $L_{av}(u_1u_4, \varphi') \cap L_{av}(u_2u_3, \varphi') \neq \emptyset$, then color u_2u_3 and u_1u_4 with $\varphi''(u_2u_3) = \varphi''(u_1u_4) \in L_{av}(u_1u_4, \varphi') \cap L_{av}(u_2u_3, \varphi')$, u_1u_2 with $\varphi''(u_1u_2) \in L_{av}(u_1u_2, \varphi') \setminus \{\varphi''(u_2u_3)\}$, u_3u_4 with $\varphi''(u_3u_4) \in L_{av}(u_3u_4, \varphi') \setminus \{\varphi''(u_2u_3)\}$ and denote the extended coloring by φ''. One can see that $|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')| \geq 2$ and $|L_{av}(u_2u_4, \varphi'')| \geq 3$. Therefore, φ'' can be easily extended to a total L-coloring φ of G by coloring u_1u_2, u_2, u_2u_4 and u_4 properly.

If $L_{av}(u_1u_4, \varphi') \cap L_{av}(u_2u_3, \varphi') = \emptyset$ and $L_{av}(u_1u_4, \varphi') \neq \{1, 2\}$, then color u_1u_4 with $\varphi''(u_1u_4) \in L(u_1u_4, \varphi') \setminus \{1, 2\}$, u_3u_4 with $\varphi''(u_3u_4) \in L(u_3u_4, \varphi') \setminus \{\varphi''(u_1u_4)\}$, u_2u_3 with $\varphi''(u_2u_3) \in L(u_2u_3, \varphi') \setminus \{\varphi''(u_3u_4)\}$ and denote the extended coloring by φ''. One can see that $|L_{av}(u_1u_2, \varphi'')|, |L_{av}(u_4, \varphi'')| \geq 2$ and $|L_{av}(u_2, \varphi'')|, |L_{av}(u_2u_4, \varphi'')| \geq 3$. Therefore, φ'' can be easily extended to a total L-coloring φ of G by coloring u_1u_2, u_2, u_2u_4 and u_4 properly.

Case 2. $L_{av}(u_1u_2, \varphi') = \{1, 2\}$ and $L_{av}(u_2u_3, \varphi') = \{1, 3\}$.

By similar arguments as in the second part of Case 1, one can show that $L_{av}(u_1u_4, \varphi') \cap L_{av}(u_2u_3, \varphi') = \emptyset$ and $L_{av}(u_1u_2, \varphi') \cap L_{av}(u_3u_4, \varphi') = \emptyset$.

If $2 \in L_{av}(u_1u_4, \varphi')$, then we assume that $L_{av}(u_1u_4, \varphi') = \{2, 4\}$. If $L_{av}(u_3u_4, \varphi') \neq \{3, 4\}$, then color u_2u_3 and u_1u_4 with 3 and 4, u_3u_4 with $\varphi''(u_3u_4) \in L_{av}(u_3u_4, \varphi') \setminus \{3, 4\}$ and denote the extended coloring by φ''. One can see that $|L_{av}(u_1u_2, \varphi'')|, |L_{av}(u_4, \varphi'')| \geq 2$ and $|L_{av}(u_2, \varphi'')|, |L_{av}(u_2u_4, \varphi'')| \geq 3$. Therefore, φ'' can be easily extended to a total L-coloring φ of G by coloring u_1u_2, u_2, u_2u_4 and u_4 properly. If $L_{av}(u_3u_4, \varphi') = \{3, 4\}$, then we first color u_1u_2, u_2u_3, u_3u_4 and u_1u_4 with 1, 3, 4 and 2, respectively, and denote the extended coloring by φ''. It is easy to see that $|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')|, |L_{av}(u_2u_4, \varphi'')| \geq 2$. If the three sets $L_{av}(u_2, \varphi'')$, $L_{av}(u_4, \varphi'')$, $L_{av}(u_2u_4, \varphi'')$ are not the same or $L_{av}(u_2, \varphi'') = L_{av}(u_4, \varphi'') = L_{av}(u_2u_4, \varphi'')$ and $|L_{av}(u_2, \varphi'')| \geq 3$, then φ'' can be easily extended to a total L-coloring φ of G. If $L_{av}(u_2, \varphi'') = L_{av}(u_4, \varphi'') = L_{av}(u_2u_4, \varphi'') = \{5, 6\}$, then we revise the coloring φ'' by recoloring u_1u_2, u_2u_3, u_3u_4 and u_4u_1 by 2, 1, 3 and 4. We then have $L_{av}(u_2, \varphi'') = \{3, 5, 6\}$, $L_{av}(u_4, \varphi'') = \{2, 5, 6\}$ and $L_{av}(u_2u_4, \varphi'') = \{5, 6\}$, so we extend φ'' to a total L-coloring of G by coloring u_2, u_4 and u_2u_4 with 3, 2 and 5.

If $2 \notin L_{av}(u_1u_4, \varphi')$, then we assume that $L_{av}(u_1u_4, \varphi') = \{4, 5\}$. We now color u_2u_3 with 3, u_3u_4 with $\varphi''(u_3u_4) \in L_{av}(u_3u_4, \varphi'') \setminus \{3\}$, u_1u_4 with $\varphi''(u_1u_4) \in L_{av}(u_1u_4, \varphi'') \setminus \{\varphi''(u_3u_4)\}$ and denote the extended coloring by φ''. It is easy to see that $|L_{av}(u_1u_2, \varphi'')|, |L_{av}(u_4, \varphi'')| \geq 2$ and $|L_{av}(u_2, \varphi'')|, |L_{av}(u_2u_4, \varphi'')| \geq 3$. Therefore, φ'' can be easily extended to a total L-coloring φ of G by coloring u_1u_2, u_2, u_2u_4 and u_4 properly. □

Lemma 8. *Suppose that G contains a cycle $u_1u_2u_3u_4$ with $u_2u_4 \in E(G)$, $d(u_2) = d(u_4) = 3$ and u_3 being adjacent to a vertex v. Let φ' be a partial total L-coloring of G so that the uncolored elements under φ' are u_1u_2, u_2u_3, u_3u_4, u_1u_4, u_2u_4, u_3v, u_2, u_3, u_4 and v, where L is a list assignment of G. If*

$$|L_{av}(u_2u_4, \varphi')| \geq 6,$$
$$\min\{|L_{av}(u_2, \varphi')|, |L_{av}(u_4, \varphi')|\} \geq 5,$$

$$\min\{|L_{av}(u_2u_3, \varphi')|, |L_{av}(u_3u_4, \varphi')|\} \geq 4$$
$$\min\{|L_{av}(u_1u_2, \varphi')|, |L_{av}(u_1u_4, \varphi')|, |L_{av}(u_3v, \varphi')|, |L_{av}(u_3, \varphi')|, |L_{av}(v, \varphi')|\} \geq 2$$

and

at least two of $L_{av}(u_3v, \varphi')$, $L_{av}(u_3, \varphi')$, $L_{av}(v, \varphi')$ *are distinct when*
$|L_{av}(u_3v, \varphi')| = |L_{av}(u_3, \varphi')| = |L_{av}(v, \varphi')| = 2$,

then φ' *can be extended to a total L-coloring* φ *of G without altering the assigned colors.*

Proof. Without loss of generality, we assume that $|L_{av}(u_2u_4, \varphi')| = 6$, $|L_{av}(u_2, \varphi')| = |L_{av}(u_4, \varphi')| = 5$, $|L_{av}(u_2u_3, \varphi')| = |L_{av}(u_3u_4, \varphi')| = 4$ and $|L_{av}(u_1u_2, \varphi')| = |L_{av}(u_1u_4, \varphi')| = |L_{av}(u_3v, \varphi')| = |L_{av}(u_3, \varphi')| = |L_{av}(v, \varphi')| = 2$.

If $L_{av}(u_3v, \varphi') = L_{av}(u_3, \varphi') = \{1, 2\}$, then we color u_3 and u_3v with 1 and 2, and then color v with a color from $L_{av}(v, \varphi')$ that is different from 1 and 2. Denote the current partial coloring still by φ'. We then have $|L_{av}(u_2u_4, \varphi')| = 6$, $|L_{av}(u_1u_2, \varphi')| = |L_{av}(u_1u_4, \varphi')| = 2$, $|L_{av}(u_2, \varphi')|, |L_{av}(u_4, \varphi')| \geq 4$ and $|L_{av}(u_2u_3, \varphi')|, |L_{av}(u_3u_4, \varphi')| \geq 2$. Without loss of generality, assume that $|L_{av}(u_2, \varphi')| = |L_{av}(u_4, \varphi')| = 4$ and $|L_{av}(u_2u_3, \varphi')| = |L_{av}(u_3u_4, \varphi')| = 2$. Since every 4-cycle is 2-choosable, we color each edge of the cycle $u_1u_2u_3u_4$ from its available list and denote the coloring at this stage by φ''. It is easy to see that $|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')|, |L_{av}(u_2u_4, \varphi'')| \geq 2$. If at least two of $L_{av}(u_2, \varphi'')$, $L_{av}(u_4, \varphi'')$ and $L_{av}(u_2u_4, \varphi'')$ are distinct or $\max\{|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')|, |L_{av}(u_2u_4, \varphi'')|\} \geq 3$, then φ'' can be easily extended to a total L-coloring of G. If $L_{av}(u_2, \varphi'') = L_{av}(u_4, \varphi'') = L_{av}(u_2u_4, \varphi'') = 2$, then exchange the colors on u_3 and u_3v and denote this coloring by φ'''. This operation does not disturb the properness of the colors on the edges of the cycle $u_1u_2u_3u_4$, but implies that at least two of $L_{av}(u_2, \varphi''')$, $L_{av}(u_4, \varphi''')$ and $L_{av}(u_2u_4, \varphi''')$ are distinct if $|L_{av}(u_2, \varphi''')| = |L_{av}(u_4, \varphi''')| = |L_{av}(u_2u_4, \varphi''')| = 2$. Therefore, φ''' can be extended to a total L-coloring φ of G.

If $L_{av}(u_3v, \varphi') = \{1, 2\}$ and $L_{av}(u_3, \varphi') = \{1, 3\}$, then there are two ways to color u_3 and u_3v so that v can be colored from its available list so that the color assigned to v is different with the colors assigned to u_3 and u_3v. Without loss of generality, assume the above two ways of coloring are as follows: color u_3 and u_3v with 1 and 3, or with 2 and 1. We now color u_3 and u_3v with 1 and 3, and then color v properly. Denote the current coloring by φ''. Suppose that $L_{av}(u_1u_2, \varphi') = \{a, b\}$. If $L_{av}(u_2u_3, \varphi') \neq \{1, 3, a, b\}$ or $\{a, b\} \cap \{1, 3\} \neq \emptyset$, then $L_{av}(u_1u_2, \varphi'') \neq L_{av}(u_2u_3, \varphi'')$, therefore, by Lemma 7, φ'' can be extended to a total L-coloring of G. If $L_{av}(u_2u_3, \varphi') = \{1, 3, a, b\}$ and $\{a, b\} \cap \{1, 3\} = \emptyset$, then recolor v_3 and v by 2 and 1, and recolor v properly. Denote this coloring by φ'''. We then have $L_{av}(u_2u_3, \varphi''') = \{3, a, b\} \setminus \{2\} \neq \{a, b\} = L_{av}(u_1u_2, \varphi''')$, so by Lemma 7, φ''' can be extended to a total L-coloring of G.

If $L_{av}(u_3v, \varphi') = \{1, 2\}$ and $L_{av}(u_3, \varphi') = \{3, 4\}$, then there are two ways to color u_3 and u_3v so that v can be colored from its available list so that the color assigned to v is different with the colors assigned to u_3 and u_3v. Therefore, we can do similar arguments as above to complete the proof. □

Lemma 9. *Suppose that G contains a cycle* $u_1u_2u_3u_4$ *with* $u_1u_3, u_2u_4 \in E(G)$ *and* $d(u_2) = d(u_4) = 3$*. Let* φ' *be a partial total L-coloring of G so that the uncolored elements under* φ' *are* u_1u_2, u_2u_3, u_3u_4, u_1u_4, u_2u_4, u_1u_3, u_1, u_2, u_3 *and* u_4*, where L is a list assignment of G. If*

$$\min\{|L_{av}(u_2, \varphi')|, |L_{av}(u_4, \varphi')|, |L_{av}(u_2u_4, \varphi')|\} \geq 6,$$
$$\min\{|L_{av}(u_1u_2, \varphi')|, |L_{av}(u_2u_3, \varphi')|, |L_{av}(u_3u_4, \varphi')|, |L_{av}(u_1u_4, \varphi')|\} \geq 4$$
$$\min\{|L_{av}(u_1, \varphi')|, |L_{av}(u_3, \varphi')|, |L_{av}(u_1u_3, \varphi')|\} \geq 2$$

and

at least two of $L_{av}(u_1, \varphi')$, $L_{av}(u_3, \varphi')$, $L_{av}(u_1u_3, \varphi')$ *are distinct when*
$|L_{av}(u_1, \varphi')| = |L_{av}(u_3, \varphi')| = |L_{av}(u_1u_3, \varphi')| = 2$,

then φ' *can be extended to a total L-coloring* φ *of G without altering the assigned colors.*

Proof. Without loss of generality, we assume that $|L_{av}(u_2, \varphi')| = |L_{av}(u_4, \varphi')| = |L_{av}(u_2u_4, \varphi')| = 6$, $|L_{av}(u_1u_2, \varphi')| = |L_{av}(u_2u_3, \varphi')| = |L_{av}(u_3u_4, \varphi')| = |L_{av}(u_1u_4, \varphi')| = 4$ and $|L_{av}(u_1, \varphi')| = |L_{av}(u_3, \varphi')| = |L_{av}(u_1u_3, \varphi')| = 2$.

If $L_{av}(u_1u_3, \varphi') = L_{av}(u_3, \varphi') = \{1, 2\}$, then color u_3 and u_1u_3 with 1 and 2, u_1 with $\varphi''(u_1) \in L_{av}(u_1, \varphi') \setminus \{1, 2\} \neq \emptyset$ and denote the current coloring by φ''. Without loss of generality, assume that $\varphi''(u_1) = 3$. It is easy to see that $|L_{av}(u_2u_4, \varphi'')| = 6$, $|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')| \geq 4$ and $|L_{av}(u_1u_2, \varphi'')|, |L_{av}(u_2u_3, \varphi'')|, |L_{av}(u_3u_4, \varphi'')|, |L_{av}(u_1u_4, \varphi'')| \geq 2$. If $\{1, 2\} \not\subseteq L_{av}(u_2u_3, \varphi')$, or $\{2, 3\} \not\subseteq L_{av}(u_1u_2, \varphi')$, or $L_{av}(u_2u_3, \varphi') \setminus \{1, 2\} \neq L_{av}(u_1u_2, \varphi') \setminus \{2, 3\}$, then by Lemma 7, φ'' can be extended to a total L-coloring of G. If $L_{av}(u_1u_2, \varphi') = \{2, 3, a, b\}$, $L_{av}(u_2u_3, \varphi') = \{1, 2, a, b\}$ and $\{a, b\} \cap \{1, 2, 3\} \neq \emptyset$, then exchange the colors on u_3 and u_1u_3, and denote this coloring by φ'''. Since $|L_{av}(u_2u_4, \varphi''')| = 6$, $|L_{av}(u_2, \varphi''')|, |L_{av}(u_4, \varphi''')| \geq 4$, $|L_{av}(u_1u_2, \varphi''')| \geq 3$ and $|L_{av}(u_3u_4, \varphi''')|, |L_{av}(u_1u_4, \varphi''')|, |L_{av}(u_2u_3, \varphi''')| \geq 2$, by Lemma 7, φ''' can be extended to a total L-coloring φ of G.

If $L_{av}(u_1u_3, \varphi') = \{1, 3\}$ and $L_{av}(u_3, \varphi') = \{1, 2\}$, then we shall assume that $L_{av}(u_1, \varphi') \neq \{1, 3\}$ (otherwise we come back to the above case). If $L_{av}(u_1, \varphi') \setminus \{1, 2, 3\} \neq \emptyset$, then color u_1 with a color in $L_{av}(u_1, \varphi') \setminus \{1, 2, 3\}$, say 4, and color u_3 and u_1u_3 with 1 and 3. Denote the current coloring by φ''. If $\{1, 3\} \not\subseteq L_{av}(u_2u_3, \varphi')$, or $\{3, 4\} \not\subseteq L_{av}(u_1u_2, \varphi')$, or

$L_{av}(u_2u_3, \varphi') \setminus \{1, 3\} \neq L_{av}(u_1u_2, \varphi') \setminus \{3, 4\}$, then by Lemma 7 and similar arguments as before, φ'' can be extended to a total L-coloring of G. If $L_{av}(u_1u_2, \varphi') = \{3, 4, a, b\}$, $L_{av}(u_2u_3, \varphi') = \{1, 3, a, b\}$ and $\{a, b\} \cap \{1, 3, 4\} \neq \emptyset$, then recolor u_3 with 2 and denote the current coloring by φ'''. By Lemma 7 and similar arguments as before, one can prove that φ''' can be extended to a total L-coloring of G. If $L_{av}(u_1, \varphi') = \{1, 2\}$, then color u_3, u_1u_3 and u_1 with 1, 3 and 2. Denote this partial coloring of G by φ''. By similar arguments as above, one can prove that either φ'' can be extended to a total L-coloring of G by Lemma 7, or we can construct a new partial coloring φ''' by exchanging the colors on u_1 and u_3 that can be extended to a total L-coloring of G by Lemma 7. If $L_{av}(u_1, \varphi') = \{2, 3\}$, then color u_3, u_1u_3 and u_1 with 1, 3 and 2 and denote this partial coloring of G by φ''. One can also prove that either φ'' can be extended to a total L-coloring of G by Lemma 7, or we can construct a new partial coloring φ''' that can be extended to a total L-coloring of G by Lemma 7 via recoloring u_3, u_1u_3 and u_1 with 2, 1 and 3.

If $L_{av}(u_1u_3, \varphi') = \{3, 4\}$ and $L_{av}(u_3, \varphi') = \{1, 2\}$, then we shall assume that $L_{av}(u_1, \varphi') \cap \{3, 4\} = \emptyset$. (otherwise we come back to the above case). We now color u_1 with $\varphi''(u_1) \in L_{av}(u_1, \varphi')$, u_1u_3 with $\varphi''(u_1u_3) \in L_{av}(u_1u_3, \varphi') \setminus \{\varphi''(u_1)\}$, and u_3 with $\varphi''(u_3) \in L_{av}(u_3, \varphi') \setminus \{\varphi''(u_1)\}$. Denote this coloring by φ''. By Lemma 7 and similar arguments as before, φ'' can be extended to a total L-coloring of G if $\{\varphi''(u_1u_3), \varphi''(u_3)\} \not\subseteq L_{av}(u_2u_3, \varphi')$, or $\{\varphi''(u_1u_3), \varphi''(u_1)\} \not\subseteq L_{av}(u_1u_2, \varphi')$, or $L_{av}(u_2u_3, \varphi') \setminus \{\varphi''(u_1u_3), \varphi''(u_3)\} \neq L_{av}(u_1u_2, \varphi') \setminus \{\varphi''(u_1u_3), \varphi''(u_1)\}$. Therefore, we assume that $L_{av}(u_2u_3, \varphi') = \{\varphi''(u_1u_3), \varphi''(u_3), a, b\}$ and $L_{av}(u_1u_2, \varphi') = \{\varphi''(u_1u_3), \varphi''(u_1), a, b\}$, where $\{a, b\} \cap \{\varphi''(u_1u_3), \varphi''(u_1), \varphi''(u_3)\} = \emptyset$. If $\varphi''(u_1) \notin \{1, 2\}$, then we can construct a new partial coloring φ''' that can be extended to a total L-coloring of G by Lemma 7 via recoloring u_3 with a color from $L_{av}(u_3, \varphi') \setminus \{\varphi''(u_3)\}$. Thus, we assume, without loss of generality, that $\varphi''(u_1) = 1$, $\varphi''(u_3) = 2$ and $\varphi''(u_1u_3) = 3$. If $L_{av}(u_1, \varphi') \neq \{1, 2\}$, then recolor u_1 with a color from $L_{av}(u_1, \varphi') \setminus \{1, 2, 3, 4\}$, which is a non-empty set since $L_{av}(u_1, \varphi') \cap \{3, 4\} = \emptyset$. If $L_{av}(u_1, \varphi') = \{1, 2\}$, then exchange the colors on u_1 and u_3. In each case, the resulted coloring can be extended to a total L-coloring of G by Lemma 7. □

We are now ready to give a proof of Theorem 5.

Proof of Theorem 5. Suppose that G is a counterexample to this theorem with the smallest value of $|V(G)| + |E(G)|$. It is easy to see that $\delta(G) \geq 2$ and every 2-vertex in G is adjacent only to $\Delta(G)$-vertices, so by Lemma 3, G contains one of the configurations among (b)–(g).

If G contains (b), then $G' = G - \{u_1, u_2, u_3, v_2, v_3\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$. One can see that $|L_{av}(v_1v_2, \varphi')|, |L_{av}(v_1v_3, \varphi')|, |L_{av}(v_2v_4, \varphi')|, |L_{av}(v_3v_4, \varphi')| \geq 3$. If $L_{av}(v_1v_3, \varphi') \cap L_{av}(v_2v_4, \varphi') \neq \emptyset$, then color v_1v_3 and v_2v_4 with a same color from $L_{av}(v_1v_3, \varphi') \cap L_{av}(v_2v_4, \varphi')$. If $L_{av}(v_1v_3, \varphi') \cap L_{av}(v_2v_4, \varphi') = \emptyset$, then we can extend φ' to another partial coloring φ'' of G by coloring v_1v_3 and v_2v_4 with $\varphi''(v_1v_3) \in L_{av}(v_1v_3, \varphi')$ and $\varphi''(v_2v_4) \in L_{av}(v_2v_4, \varphi')$ so that $|L_{av}(v_1v_2, \varphi') \setminus \{\varphi''(v_1v_3), \varphi''(v_2v_4)\}| \geq 2$ and $|L_{av}(v_3v_4, \varphi') \setminus \{\varphi''(v_1v_3), \varphi''(v_2v_4)\}| \geq 2$. In either case, we can extend φ' to another partial coloring φ'' of G by coloring v_1v_3 and v_2v_4 from their available lists, and moreover, φ'' satisfies

$$|L_{av}(u_1v_1, \varphi'')|, |L_{av}(u_3v_4, \varphi'')|, |L_{av}(v_1v_2, \varphi'')|, |L_{av}(v_3v_4, \varphi'')| \geq 2,$$
$$|L_{av}(v_2, \varphi'')|, |L_{av}(v_3, \varphi')| \geq 3,$$
$$|L_{av}(v_2v_3, \varphi'')| \geq 4,$$

and

$$|L_{av}(u_1v_2, \varphi'')|, |L_{av}(u_2v_2, \varphi'')|, |L_{av}(u_2v_3, \varphi'')|, |L_{av}(u_3v_3, \varphi'')| \geq 5.$$

Therefore, by Lemma 6, φ'' can be extended to a total L-coloring φ of G without altering the assigned colors.

If G contains (c), then $G' = G - \{u_2, u_4\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$. Since there are at least two available colors for each edge of the cycle $u_1u_2u_3u_4$ and every 4-cycle is 2-edge-choosable, we can color u_1u_2, u_2u_3, u_3u_4 and u_4u_1 from their available lists so that the extended coloring is proper. At last, we color u_2 and u_4 from their available lists to obtain a total L-coloring φ of G. This can be easily done since u_2 or u_4 is incident with four colored elements right row.

If G contains (d), then $G' = G - \{u_2, u_4\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$. One can check that $|L_{av}(u_2u_4, \varphi')| \geq 6$, $|L_{av}(u_1u_2, \varphi')|, |L_{av}(u_1u_4, \varphi')| \geq 2$, $|L_{av}(u_2u_3, \varphi')|, |L_{av}(u_3u_4, \varphi')| \geq 3$ and $|L_{av}(u_2, \varphi')|, |L_{av}(u_4, \varphi')| \geq 4$. By Lemma 7, φ' can be extended to a total L-coloring φ of G without altering the colors in G'.

If G contains (e), then $G' = G - \{u_2, u_4, u_3v\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$. We now erase the color on v and still denote the current coloring by φ'. It is easy to check that $|L_{av}(u_2u_4, \varphi')| \geq 6$, $|L_{av}(u_1u_2, \varphi')|, |L_{av}(u_1u_4, \varphi')| \geq 2$, $|L_{av}(u_2u_3, \varphi')|, |L_{av}(u_3u_4, \varphi')| \geq 3$, $|L_{av}(u_2, \varphi')|, |L_{av}(u_4, \varphi')| \geq 4$, $|L_{av}(u_3v, \varphi')| \geq 2$ and $|L_{av}(v, \varphi')| \geq 3$. We now color u_3v with $\varphi''(u_3v) \in L_{av}(u_3v, \varphi')$ and v with $\varphi''(v) \in L_{av}(v, \varphi') \setminus \{\varphi''(u_3v)\}$ so that the extended partial coloring φ'' satisfies

(1) $|L_{av}(u_2u_4, \varphi'')| \geq 6$, $|L_{av}(u_2, \varphi'')|, |L_{av}(u_4, \varphi'')| \geq 4$, $|L_{av}(u_1u_2, \varphi'')|, |L_{av}(u_2u_3, \varphi'')|, |L_{av}(u_3u_4, \varphi'')|, |L_{av}(u_1u_4, \varphi'')| \geq 2$
(2) $L_{av}(u_1u_2, \varphi'') \neq L_{av}(u_2u_3, \varphi'')$ if $|L_{av}(u_1u_2, \varphi'')| = |L_{av}(u_2u_3, \varphi'')| = 2$.

This can be done since $|L_{av}(u_3v, \varphi')| \geq 2$. Therefore, φ' can be extended to a total L-coloring φ of G without altering the colors in G' by Lemma 7.

If G contains (f), then $G' = G - \{u_2, u_4\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$. We now erase the colors on u_3, u_3v and v and denote the current coloring by φ''. It is easy to see that

$$|L_{\text{av}}(u_2u_4, \varphi'')| \geq 6,$$
$$|L_{\text{av}}(u_2, \varphi'')|, |L_{\text{av}}(u_4, \varphi'')| \geq 5,$$
$$|L_{\text{av}}(u_2u_3, \varphi'')|, |L_{\text{av}}(u_3u_4, \varphi'')| \geq 4,$$

and

$$|L_{\text{av}}(u_1u_2, \varphi'')|, |L_{\text{av}}(u_1u_4, \varphi'')|, |L_{\text{av}}(u_3v, \varphi'')|, |L_{\text{av}}(u_3, \varphi'')|, |L_{\text{av}}(v, \varphi'')| \geq 2.$$

Since φ'' is a partial coloring of φ', at least two of $L_{\text{av}}(u_3v, \varphi''), L_{\text{av}}(u_3, \varphi''), L_{\text{av}}(v, \varphi'')$ are distinct if $|L_{\text{av}}(u_3v, \varphi'')| = |L_{\text{av}}(u_3, \varphi'')| = |L_{\text{av}}(v, \varphi'')| = 2$. Therefore, by Lemma 8, φ'' can be extended to a total L-coloring φ of G without altering the colors in G'.

If G contains (g), then $G' = G - \{u_2, u_4\}$ has a total L'-coloring φ' so that $L'(x) = L(x)$ for each $x \in VE(G')$. We now erase the colors on u_1, u_3 and u_1u_3 and denote the current coloring by φ''. One can see that

$$|L_{\text{av}}(u_2, \varphi'')|, |L_{\text{av}}(u_4, \varphi'')|, |L_{\text{av}}(u_2u_4, \varphi'')| \geq 6,$$
$$|L_{\text{av}}(u_1u_2, \varphi'')|, |L_{\text{av}}(u_2u_3, \varphi'')|, |L_{\text{av}}(u_3u_4, \varphi'')|, |L_{\text{av}}(u_1u_4, \varphi'')| \geq 4,$$

and

$$|L_{\text{av}}(u_1, \varphi'')|, |L_{\text{av}}(u_3, \varphi'')|, |L_{\text{av}}(u_1u_3, \varphi'')| \geq 2.$$

Since φ'' is a partial coloring of φ', at least two of $L_{\text{av}}(u_1u_3, \varphi''), L_{\text{av}}(u_1, \varphi''), L_{\text{av}}(u_3, \varphi'')$ are distinct if $|L_{\text{av}}(u_1, \varphi'')| = |L_{\text{av}}(u_3, \varphi'')| = |L_{\text{av}}(u_1u_3, \varphi'')| = 2$. Therefore, φ'' can be extended to a total L-coloring φ of G without altering the colors in G' by Lemma 9. □

Remark. Theorem 5 implies that every pseudo-outerplanar graphs with maximum degree at least 5 is totally $(\Delta + 1)$-choosable. In fact, the bound 5 for the maximum degree in this result cannot be improved to 3 since there exists a pseudo-outerplanar graph with maximum degree 3 and total chromatic number 5 (see Fig. 1 of the Ref. [12]). In another coming paper, we have proved that every pseudo-outerplanar graph with maximum degree 4 is totally 5-colorable. However, we still do not know whether every pseudo-outerplanar graph with maximum degree 4 is totally 5-choosable, thus we leave it as an open problem here.

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