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Equitable vertex arboricity of graphs[☆]

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ABSTRACT

An equitable (t, k)-tree-coloring of a graph G is a coloring of vertices of G such that the sizes of any two color classes differ by at most one and the subgraph induced by each color class is a forest of maximum degree at most k. The minimum t such that G has an equitable (t', k)-tree-coloring for every $t' \geq t$, denoted by $va_k^{\equiv}(G)$, is the strong equitable vertex k-arboricity. In this paper, we give sharp upper bounds for $va_1^{\equiv}(K_{n,n})$ and $va_k^{\equiv}(K_{n,n})$, and prove that $va_\infty^{\equiv}(G) \leq 3$ for every planar graph G with girth at least 5 and $va_\infty^{\equiv}(G) \leq 2$ for every planar graph G with girth at least 6 and for every outerplanar graph.

1. Introduction

All graphs considered in the paper are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the set of vertices, the set of edges, the minimum degree and the maximum degree of G, respectively. $N_G(v)$ denotes the set of neighbors of a vertex v in G and $d_G(v) = |N_G(v)|$ denotes the degree of v. Sometimes we use $d(v)$ instead of $d_G(v)$ for brevity. A k-, k^+- and k^--*vertex* in G is a vertex of degree k, at least k and at most k, respectively. If $uv \in E(G)$ and $d(u) = k$, then we say that u is a k-*neighbor* of v; k^--*neighbor* and k^+-*neighbor* can be similarly defined. For other undefined concepts we refer the reader to [1].

We associate positive integers $1, 2, \ldots, t$ with colors and call f a t-*coloring* of G if f is a mapping from $V(G)$ to $\{1, 2, \ldots, t\}$. For $1 \leq i \leq t$, let $V_i = \{v \mid f(v) = i\}$. A t-coloring f of G is *equitable* if $||V_i| - |V_j|| \leq 1$ for all i and j; that is, every color class has size $\lfloor |V(G)|/t \rfloor$ or $\lceil |V(G)|/t \rceil$. A t-coloring of G is *proper* if every two adjacent vertices have the different colors. The smallest number t such that G has a proper equitable t-coloring, denoted by $\chi^=(G)$, is the *equitable chromatic number*.

Note that a proper equitable t-colorable graph may admit no proper equitable t'-colorings for some $t' > t$. For example, the complete bipartite graph $H := K_{2m+1,2m+1}$ has no proper equitable $(2m+1)$-colorings, although it satisfies $\chi^=(H) = 2$. This fact motivates another interesting parameter for proper equitable coloring. The *equitable chromatic threshold* of G, denoted by $\chi^{\equiv}(G)$, is the smallest integer t such that G has proper equitable colorings for any number of colors greater than or equal to t. In 1970, Hajnal and Szemerédi [7] answered a question of Erdős by proving that every graph G with $\Delta(G) \leq r$ has a proper equitable $(r+1)$-coloring. In fact, Hajnal–Szemerédi Theorem implies $\chi^{\equiv}(G) \leq \Delta(G) + 1$ for every graph G. In 2008, Kierstead and Kostochka [8] simplified the proof of Hajnal–Szemerédi Theorem, and moreover, they [9]

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strengthened Hajnal–Szemerédi Theorem by proving that G has a proper equitable $(r+1)$-coloring if G is a graph such that $d(x)+d(y) \le 2r+1$ for every edge xy.

Regarding equitable colorings, there are two well-known conjectures. Note that Conjecture 1.2 is stronger than Conjecture 1.1.

Conjecture 1.1 ([11]). *For any connected graph G, except the complete graph and the odd cycle, $\chi^=(G) \le \Delta(G)$.*

Conjecture 1.2 ([4]). *For any connected graph G, except the complete graph, the odd cycle and the complete bipartite graph $K_{2m+1,2m+1}$, $\chi^{\equiv}(G) \le \Delta(G)$.*

The above two conjectures have been confirmed for many classes of graphs, such as graphs with $\Delta \le 3$ [4,5] or $\Delta \ge |V(G)|/3+1$ [4,5,14], bipartite graphs [10], outerplanar graphs [14], series–parallel graphs [16] and planar graphs with $\Delta \ge 9$ [12,15]. There are other related results; see [13,17].

In [6], Fan, Kierstead, Liu, Molla, Wu and Zhang first considered relaxed equitable coloring of graphs. They proved that every graph has an equitable Δ-coloring such that each color class induces a forest with maximum degree at most one. On the basis of this research, we aim to introduce the notion of equitable (t,k)-tree-coloring. A t-coloring f of a graph G is a (t,k)-*tree-coloring* of G if each component of $G[V_i]$ is a tree of maximum degree at most k. A (t,∞)-tree-coloring is called a t-tree-coloring for short. The vertex k-arboricity of G, denoted by $va_k(G)$, is the minimum t such that G has a (t,k)-tree-coloring. Indeed, the notion of (t,k)-tree-coloring is a uniform form of some familiar kinds of vertex coloring. For example, $va_0(G)=\chi(G)$, $va_2(G)=vla(G)$ and $va_\infty(G)=va(G)$, where $\chi(G)$ is the standard chromatic number, $vla(G)$ is the vertex linear arboricity and $va(G)$ is the vertex arboricity of G. It is also trivial that $va_k(K_{m,n})=2$ for the complete bipartite graph $K_{m,n}$ and an integer $k \ge 0$. In [3], it was proved that the set of vertices of every planar graph can be partitioned into three subsets such that each subset induces a forest. This implies $va_\infty(G) \le 3$ for every planar graph G.

An *equitable* (t,k)-*coloring* is a (t,k)-coloring that is equitable. The *equitable vertex k-arboricity* of a graph G, denoted by $va_k^=(G)$, is the smallest t such that G has an equitable (t,k)-tree-coloring. The *strong equitable vertex k-arboricity* of G, denoted by $va_k^{\equiv}(G)$, is the smallest t such that G has an equitable (t',k)-coloring for every $t' \ge t$. It is clear that $va_0^=(G)=\chi^=(G)$ and $va_0^{\equiv}(G)=\chi^{\equiv}(G)$ for every graph G, and $va_k^=(G)$ and $va_k^{\equiv}(G)$ may vary a lot. In this paper, we mainly investigate the strong equitable vertex k-arboricity of complete bipartite graphs and planar graphs. As one of the first results on this topic, we prove that $va_1^{\equiv}(K_{n,n}) \le 2\lfloor (n+1)/3 \rfloor$ and $va_\infty^{\equiv}(K_{n,n}) \le 2\lfloor (\sqrt{8n+9}-1)/4 \rfloor$ with those upper bounds being sharp, and then we prove that $va_\infty^{\equiv}(G) \le 3$ for every planar graph G with girth at least 5 and $va_\infty^{\equiv}(G) \le 2$ for every planar graph G with girth at least 6 and for every outerplanar graph.

2. Complete bipartite graphs

Lemma 2.1. *The complete bipartite graph $K_{n,n}$ has an equitable (t,k)-tree-coloring for every even integer $t \ge 2$.*

Proof. One can easily construct an equitable (t,k)-tree-coloring of $K_{n,n}$ by dividing each partite set into $t/2$ classes equitably and coloring the vertices of each class with one color. □

Theorem 2.2. $va_1^{\equiv}(K_{n,n}) \le 2\lfloor \frac{n+1}{3} \rfloor$.

Proof. By Lemma 2.1, in order to show $va_1^{\equiv}(K_{n,n}) \le 2\lfloor \frac{n+1}{3} \rfloor$, we only need to prove that $K_{n,n}$ has an equitable $(q,1)$-tree-coloring for every odd $q \ge 2\lfloor \frac{n+1}{3} \rfloor + 1$. Note that $3q-2n \ge 6\lfloor \frac{n+1}{3} \rfloor + 3 - 2n \ge 6 \times \frac{n-1}{3} + 3 - 2n \ge 1$. Let X and Y be the partite sets of $K_{n,n}$ and let $e=xy$ be an edge of $K_{n,n}$ with $x \in X$ and $y \in Y$. If $q \ge n$, then color x and y with 1, divide each of $X \setminus \{x\}$ and $Y \setminus \{y\}$ into $\frac{q-1}{2}$ classes equitably and color the vertices of each class with a color in $\{2,\dots,q\}$. One can easily check that the resulting coloring is an equitable $(q,1)$-tree-coloring of $K_{n,n}$ with the size of each color class being at most 2. Thus, we assume $q<n$. Suppose $2n = aq + r$, where $0 \le r \le a-1$. Since $a = \frac{2n-r}{q} \le \frac{2n}{q} \le \frac{2n}{2\lfloor \frac{n+1}{3} \rfloor + 1} < 3$, $a \le 2$. Now arbitrarily choose $3q-2n$ vertex-disjoint edges from $K_{n,n}$ and color the two end-vertices of each edge with a color in $\{1,\dots,3q-2n\}$. Let X' and Y' be the uncolored vertices in X and Y, respectively. One can see that $|X'|=|Y'|=n-(3q-2n)=3(n-q)>0$. Thus, we can divide each of X' and Y' into $n-q$ classes equitably and color the vertices of each class with a color in $\{3q-2n+1,\dots,q\}$. It is also easy to check that the resulting coloring of $K_{n,n}$ is an equitable $(q,1)$-tree-coloring with the size of each color class being either 2 or 3. Hence $va_1^{\equiv}(K_{n,n}) \le 2\lfloor \frac{n+1}{3} \rfloor$. □

Theorem 2.3. *If $n \equiv 2 \pmod 3$, then $va_1^{\equiv}(K_{n,n}) = 2\lfloor \frac{n+1}{3} \rfloor$.*

Proof. Let $n=3t+2$. If $G=K_{n,n}$ has an equitable $(2t+1,1)$-tree-coloring c, then the size of every color class in c is at least 3 because $\lceil \frac{2n}{2t+1} \rceil = \lceil \frac{6t+4}{2t+1} \rceil \ge 4$. This implies that there is no edge in G with its two end-vertices colored with the same color. Thus the vertices of every color class form an independent set. Without loss of generality, suppose that there are at least $t+1$ colors appearing in X. We then have $|X| \ge 3(t+1) = (3t+2)+1 = |X|+1$, a contradiction. This implies $va_1^{\equiv}(G) \ge 2t+2 = 2\lfloor \frac{n+1}{3} \rfloor$ and thus $va_1^{\equiv}(G) = 2\lfloor \frac{n+1}{3} \rfloor$ by Theorem 2.2. □

In the following we investigate the strong equitable vertex ∞-arboricity of $K_{n,n}$.

Let $K_{n,n}$ be a complete bipartite graph with two partite sets X and Y. For a partial q-coloring c (not needed to be proper) of $K_{n,n}$, let $V_1, \ldots, V_q$ be its color classes, $a = \lfloor \frac{2n}{q} \rfloor$ and let

$$\begin{aligned}
&c(X_1) = \{V_i \mid |V_i \cap X| = a+1, |V_i \cap Y| = 0\}, \qquad c(X_2) = \{V_i \mid |V_i \cap X| = a, |V_i \cap Y| = 0\},\\
&c(X_1') = \{V_i \mid |V_i \cap X| = a, |V_i \cap Y| = 1\}, \qquad c(X_2') = \{V_i \mid |V_i \cap X| = a-1, |V_i \cap Y| = 1\},\\
&c(Y_1) = \{V_i \mid |V_i \cap Y| = a+1, |V_i \cap X| = 0\}, \qquad c(Y_2) = \{V_i \mid |V_i \cap Y| = a, |V_i \cap X| = 0\},\\
&c(Y_1') = \{V_i \mid |V_i \cap Y| = a, |V_i \cap X| = 1\}, \qquad c(Y_2') = \{V_i \mid |V_i \cap Y| = a-1, |V_i \cap X| = 1\}.
\end{aligned}$$

We have the following lemma.

Lemma 2.4. *If $K_{n,n}$ is a complete bipartite graph with partite sets X and Y, where $2n = aq + r$ and $0 \le r \le a-1$, and c is a partial q-coloring of $K_{n,n}$, then c is an equitable (q, ∞)-tree-coloring of $K_{n,n}$ if and only if*

$$(a+1)|c(X_1)| + a|c(X_2)| + a|c(X_1')| + (a-1)|c(X_2')| + |c(Y_1')| + |c(Y_2')| = n, \tag{2.1}$$

$$(a+1)|c(Y_1)| + a|c(Y_2)| + a|c(Y_1')| + (a-1)|c(Y_2')| + |c(X_1')| + |c(X_2')| = n. \tag{2.2}$$

Proof. Let $V_1, \ldots, V_q$ be the color classes of c. First suppose that c is an equitable (q, ∞)-tree-coloring of $K_{n,n}$. Since $2n = aq + r$, the size of each color class of c is either a or $a+1$. It is easy to see that $\min\{|V_i \cap X|, |V_i \cap Y|\} \le 1$ for every $1 \le i \le q$, because otherwise we would find a 4-cycle in some color class V_i, a contradiction. Thus

$$c(X_1) \cup c(X_2) \cup c(X_1') \cup c(X_2') \cup c(Y_1) \cup c(Y_2) \cup c(Y_1') \cup c(Y_2') = \bigcup_{i=1}^{q} V_i \tag{2.3}$$

and Eqs. (2.1) and (2.2) hold accordingly. On the other hand, if Eqs. (2.1) and (2.2) hold, then c is a q-coloring of $K_{n,n}$ and the size of each color class of c is either a or $a+1$. Furthermore, we also have $\min\{|V_i \cap X|, |V_i \cap Y|\} \le 1$ for every $1 \le i \le q$. Hence c is an equitable (q, ∞)-tree-coloring of $K_{n,n}$. $\square$

Lemma 2.5. *The complete bipartite graph $K_{n,n}$ with $t(t+3) \le 2n < (t+1)(t+4)$ has an equitable (q, ∞)-tree-coloring for every integer $q \ge 2\lfloor \frac{t+1}{2} \rfloor$.*

Proof. By Lemma 2.1, we assume that q is an odd integer. This implies $q \ge t+1$. If $2n = aq + r$, where $0 \le r \le a-1$, then the two integers a and r would have the same parity. Note that $a = \frac{2n-r}{q} \le \frac{2n}{q} < \frac{(t+1)(t+4)}{q} \le t+4$ and $q \ge t+1$. We have

$$r \le a-2 \quad \text{and} \quad a \le t+3. \tag{2.4}$$

Now we prove the following two useful inequations:

$$2q \ge a + r, \tag{2.5}$$

$$q + r \ge a - 1. \tag{2.6}$$

First, if $a \le t+2$, then $q + r \ge q \ge t+1 \ge a-1$ and $2q \ge a + (a-2) \ge a + r$ by (2.4). Similarly, if $q \ge a-1$, then we would get the same results. Thus we assume that $a = t+3$ and $q \le a-2$. Since $q \ge t+1 = a-2$, $aq = (t+3)(t+1)$. This implies that $r = 2n - aq < (t+1)(t+4) - (t+1)(t+3) = t+1 = a-2$, so $r \le a-4$ and $2q = a + (a-4) \ge a + r$. On the other hand, q and a are both odd since $q = a-2$. It follows that $r = 2n - aq > 0$. Thus we have $q + r \ge q + 1 = a - 1$.

The proof of this lemma is constructive. Let X and Y be two partite sets of $K_{n,n}$ as described in Lemma 2.4. We are going to construct an equitable (q, ∞)-tree-coloring of $K_{n,n}$ by distinguishing three cases.

Case 1. $q \le 2r + 1$.

We construct a coloring c of $K_{n,n}$ by letting

$$|c(X_1)| = \frac{q-1}{2}, \qquad |c(Y_2)| = \frac{2q-a-r}{2}, \qquad |c(Y_1')| = \frac{2r+1-q}{2}, \qquad |c(Y_2')| = \frac{a-r}{2}$$

and $|c(X_2)| = |c(X_1')| = |c(X_2')| = |c(Y_2)| = 0$. Since $q \ge 1$, $2q \ge a + r$ by (2.5), $2r + 1 \ge q$, $a - 2 \ge r$, q is odd and a, r have the same parity, the four values $|c(X_1)|$, $|c(Y_2)|$, $|c(Y_1')|$ and $|c(Y_2')|$ must be nonnegative integers. Moreover, one can easily check that the two equations (2.1) and (2.2) in Lemma 2.4 would hold by our choice. Thus c is an equitable (q, ∞)-tree-coloring of $K_{n,n}$.

Case 2. $2r + 3 \le q \le a + r - 1$.

In this case we can construct a coloring c of $K_{n,n}$ by letting

$$|c(X_2')| = \frac{q+1}{2}, \qquad |c(Y_1)| = \frac{a+r-1-q}{2}, \qquad |c(Y_2)| = \frac{q-2r-1}{2}, \qquad |c(Y_1')| = \frac{q+r-a+1}{2}$$

and $|c(X_1)| = |c(X_2)| = |c(X_1')| = |c(Y_2')| = 0$. One can easily see that $|c(X_2')|$, $|c(Y_1)|$, $|c(Y_2)|$ and $|c(Y_1')|$ are all nonnegative integers, since $2r + 3 \le q \le a + r - 1$ and $q + r \ge a - 1$ by (2.6). On the other hand, the two equations (2.1) and (2.2) in Lemma 2.4 would also hold. Thus c is an equitable (q, ∞)-tree-coloring of $K_{n,n}$.

Case 3. $q \geq a + r + 1$.
Now we construct a coloring c of $K_{n,n}$ by setting

$$|c(X_2)| = \frac{q-1}{2}, \qquad |c(Y_2)| = \frac{q-a-r+1}{2}, \qquad |c(Y_1')| = r, \qquad |c(Y_2')| = \frac{a-r}{2}$$

and $|c(X_1)| = |c(X_1')| = |c(X_2')| = |c(Y_1)| = 0$. One can easily check that $|c(X_2)|$, $|c(Y_2)|$, $|c(Y_1')|$ and $|c(Y_2')|$ are all nonnegative integers and the two equations (2.1) and (2.2) in Lemma 2.4 hold. Hence, c is an equitable (q, ∞)-tree-coloring of $K_{n,n}$. □

Theorem 2.6. $va_\infty^{\equiv}(K_{n,n}) \leq 2\lfloor \frac{\sqrt{8n+9}-1}{4} \rfloor$.

Proof. Let $t = \lfloor \frac{\sqrt{8n+9}-3}{2} \rfloor$. One can easily check that $t(t+3) \leq 2n < (t+1)(t+4)$. Hence by Lemma 2.5, we have $va_\infty^{\equiv}(K_{n,n}) \leq 2\lfloor \frac{t+1}{2} \rfloor = 2\lfloor \frac{\sqrt{8n+9}-1}{4} \rfloor$. □

Lemma 2.7. *The complete bipartite graph $K_{n,n}$ with $2n = t(t+i)$, $i \geq 2$ and t being odd has no equitable (t, ∞)-tree-colorings.*

Proof. Suppose, to the contrary, that $K_{n,n}$ admits an equitable (t, ∞)-tree-coloring c. Since $2n = t(t+i)$, the size of every color class of c is exactly $t+i$. By Lemma 2.4, without loss of generation, we can assume that $|c(X_1)| + |c(X_2)| + |c(X_1')| + |c(X_2')| \geq \frac{t+1}{2}$. Here one should note that t had been supposed to be odd. Thus we have $2n = 2|X| \geq 2(t+i-1)(|c(X_1)| + |c(X_2)| + |c(X_1')| + |c(X_2')|) \geq (t+i-1)(t+1) = t(t+i) + i - 1 > t(t+i) = 2n$, a contradiction. □

Theorem 2.8. *If $2n = t(t+3)$ and t is odd, then $va_\infty^{\equiv}(K_{n,n}) = 2\lfloor \frac{\sqrt{8n+9}-1}{4} \rfloor$.*

Proof. By Lemma 2.7, G has no equitable (t, ∞)-tree-colorings. So we have $va_{\infty,2}^{\equiv}(G) = t + 1 = 2\lfloor \frac{\sqrt{8n+9}-1}{4} \rfloor$ by Theorem 2.6. □

Note that the upper bounds for $va_1^{\equiv}(K_{n,n})$ and $va_\infty^{\equiv}(K_{n,n})$ given in Theorems 2.2 and 2.6 are sharp in general case by Theorems 2.3 and 2.8, since there are infinity graphs with the strong vertex 1-arboricity and the strong ∞-arboricity reaching this bound, respectively. However, they are not very tight for some special graphs. For example, one can easily check that $va_1^{\equiv}(K_{43,43}) = 22 < 28$ and $va_\infty^{\equiv}(K_{65,65}) = 8 < 10$. Thus, to determine the strong equitable 1-arboricity or the strong equitable ∞-arboricity of the complete bipartite graphs seems not to be an easy task.

3. Planar graphs

Lemma 3.1. *Let $S = \{v_1, \ldots, v_t\}$, where $v_1, \ldots, v_t$ are distinct vertices in G. If $G - S$ has an equitable t-tree-coloring and $|N_G(v_i) \setminus S| \leq 2i - 1$ for every $1 \leq i \leq t$, then G has an equitable t-tree-coloring.*

Proof. Let $G_i = G \setminus \{v_1, \ldots, v_i\}$. It follows that $G = G_0$ and $G - S = G_t$. Let c_t be an equitable t-tree-coloring of G_t. For every $t \geq i \geq 1$, we extend the equitable t-tree-coloring c_i of G_i to an equitable t-tree-coloring c_{i-1} of G_{i-1} by giving v_i a color that is different from the colors in $\{c_i(v_{i+1}), \ldots, c_i(v_t)\}$ and that has been used on the neighbors of v_i at most once. This is possible since $|N_G(v_i) \setminus S| \leq 2i - 1$ for every $1 \leq i \leq t$. After t iterative extensions, one can check that the vertices in S receive different colors under the final coloring c_0. Hence, c_0 is an equitable t-tree-coloring of G. □

The maximum average degree of a graph is the maximum average degree over all its subgraphs. It is well-known that a planar graph with girth at least g has maximum average degree less than $2g/(g-2)$.

Lemma 3.2. *Every graph with maximum average degree less than $\frac{10}{3}$ contains at least one of the following configurations.*

(C1.1) *a vertex x of degree 1;*
(C1.2) *a 2-vertex x adjacent to a 6-vertex y;*
(C1.3) *a 3-vertex x adjacent to a 4^--vertex y and a 6^--vertex z;*
(C1.4) *an i-vertex x adjacent to at least $i-1$ 2-vertices, where $i = 7, 8, 9$.*

Proof. Suppose, to the contrary, that G contains none of the four configurations. It follows that $\delta(G) \geq 2$. Assign initial charge $c(v) = d(v)$ to every vertex $v \in V(G)$. We now redistribute the charges of vertices in G according to Rules 1 and 2 below.

Rule 1. A 7^+-vertex gives $\frac{2}{3}$ to each of its 2-neighbors.
Rule 2. A 4^+-vertex gives $\frac{1}{6}$ to each of its 3-neighbors.

Let $c'(v)$ be the charge of v after discharging. Since (C1.2) is forbidden in G, every 2-vertex is adjacent only to 7^+-vertices in G. By Rule 1, we immediately have $c'(v) \geq 2 + 2 \times \frac{2}{3} = \frac{10}{3}$ for every 2-vertex v. Since the absence of (C1.3) in G implies that every 3-vertex is adjacent to two 4^+-vertices in G, $c'(v) \geq 3 + 2 \times \frac{1}{6} = \frac{10}{3}$ for every 3-vertex v by Rule 2. Let v be a vertex of degree between 4 and 6. By Rule 2, one can easily deduce that $c'(v) \geq d(v) - \frac{1}{6}d(v) \geq \frac{10}{3}$. Let v be a vertex of degree between 7 and 9. Since (C1.4) is absent from G, v is adjacent to at most $d(v)-2$ 2-vertices; therefore, by Rules 1 and 2, we have $c'(v) \geq d(v) - \frac{2}{3}(d(v)-2) - 2 \times \frac{1}{6} \geq \frac{10}{3}$. At last, if $d(v) \geq 10$, then by Rules 1 and 2, $c'(v) \geq d(v) - \frac{2}{3}d(v) \geq \frac{10}{3}$. Hence, we have $\mathrm{mad}(G) \geq \frac{\sum_{v\in V(G)} c(v)}{|G|} = \frac{\sum_{v\in V(G)} c'(v)}{|G|} \geq \frac{10}{3}$, a contradiction. □

By Lemma 3.2, we have the following two immediate corollaries.

Corollary 3.3. *Every planar graph with girth at least* 5 *contains at least one of four configurations mentioned in Lemma* 3.2*.*

Corollary 3.4. *Every planar graph with girth at least* 5 *contains a vertex of degree at most* 3*.*

Theorem 3.5. *If G is a planar graph with girth at least* 5*, then G has an equitable t-tree-coloring for every $t \geq 3$, that is,* $va^{\equiv}_{\infty}(G) \leq 3$*.*

Proof. By Corollary 3.3, G contains at least one of the configurations (C1.1)–(C1.4). In what follows, we prove the theorem by induction on the order of G, via assigning t distinct vertices to $S = \{v_1, \dots, v_t\}$ as described in Lemma 3.1, where $t \geq 3$.

If G contains the configuration (C1.1), then let $x := v_1$. If G contains the configuration (C1.2), then let $x := v_1$ and $y := v_t$. If G contains the configuration (C1.3), then let $x := v_1$, $y := v_2$ and $w := v_t$. If G contains the configuration (C1.4) and $i = 7$, then let $y := v_1$, $z := v_2$ and $x := v_t$, where y and z are two 2-vertices that are adjacent to x. If G contains the configuration (C4), $8 \leq i \leq 9$ and $t \geq 4$, then let $y := v_1$, $z := v_2$ and $x := v_t$, where y and z are two 2-vertices that are adjacent to x. Now in each case we fill the remaining unspecified positions in $S = \{v_1, v_2, \dots, v_t\}$ from highest to lowest indices properly. Indeed, one can easily complete it by choosing at each step a vertex of degree at most 3 in the graph obtained from G by deleting the vertices chosen for S with higher indices. Corollary 3.4 guarantees that such vertices always exist. Meanwhile, by doing so, we would have $|N_G(v_i) \setminus \{v_{i+1}, \dots, v_t\}| \leq 2i-1$ for every $1 \leq i \leq t$. Since $G - S$ is a planar graph with girth at least 5 and with order less than G, by induction hypothesis, $G - S$ has an equitable t-tree-coloring. Hence by Lemma 3.1, G also admits an equitable t-tree-coloring.

Now, we have ignored two cases in the above discussions. There are the cases that G contains configuration (C1.4), $8 \leq i \leq 9$ and $t = 3$. Let $x_1, \dots, x_5$ be five 2-neighbors of x in G. Consider the graph $G' = G - \{x, x_1, \dots, x_5\}$. By induction, G' has an equitable 3-tree-coloring c'. If there is one color, say 1, which has not appeared on the vertex set $N_G(x) \setminus \{x_1, \dots, x_5\}$ under the coloring c', then we color x, x_1 by 1, x_2, x_3 by 2 and x_4, x_5 by 3. One can check that the extended coloring of G is an equitable 3-tree-coloring. Otherwise, since $|N_G(x) \setminus \{x_1, \dots, x_5\}| = i - 5 \leq 4$, there are two colors, say 1 and 2, which have been used only once on the vertex set $N_G(x) \setminus \{x_1, \dots, x_5\}$ under the coloring c'. Without loss of generality, denote the other neighbor of x_1 beside x was colored by 1. We now color x, x_1 by 2, x_2, x_3 by 1 and x_4, x_5 by 3. One can also check that the resulting coloring of G is an equitable 3-tree-coloring. □

Lemma 3.6. *Every graph with maximum average degree less than* 3 *contains at least one of the following configurations:*

(C2.1) *a vertex x of degree* 1*;*
(C2.2) *a* 2*-vertex x adjacent to a* 4^-*-vertex y;*
(C2.3) *a* 5*-vertex x adjacent to five* 2*-vertices.*

Proof. Suppose, to the contrary, that G contains none of the four configurations. It follows that $\delta(G) \geq 2$. Assign initial charge $c(v) = d(v)$ to every vertex $v \in V(G)$. We now redistribute the charges of vertices in G according to the following rule.

Rule. A 5^+-vertex gives $\frac{1}{2}$ to each of its 2-neighbors.

Let $c'(v)$ be the charge of v after discharging. Since G does not contain (C2.2), every 2-vertex is adjacent to two 5^+-vertices in G. Therefore, $c'(v) \geq 2 + 2 \times \frac{1}{2} = 3$ for every 2-vertex v by the discharging rule. Since 3-vertices and 4-vertices are not involved in the rule, $c'(v) = d(v) \geq 3$ for $3 \leq d(v) \leq 4$. If $d(v) = 5$, then v is adjacent to at most four 2-vertices because of the absence of (C2.3) from G, so $c'(v) \geq d(v) - 4\frac{1}{2} = 3$. If $d(v) \geq 6$, then by the discharging rule, we still have $c'(v) \geq d(v) - \frac{1}{2}d(v) \geq 3$. Hence, we have $\mathrm{mad}(G) \geq \frac{\sum_{v\in V(G)} c(v)}{|G|} = \frac{\sum_{v\in V(G)} c'(v)}{|G|} \geq 3$, a contradiction. □

By Lemma 3.6, we have the following immediate corollary.

Corollary 3.7. *Every planar graph with girth at least* 6 *contains at least one of three configurations mentioned in Lemma* 3.6*.*

Theorem 3.8. *If G is a planar graph with girth at least* 6*, then G has an equitable t-tree-coloring for every $t \geq 2$, that is,* $va^{\equiv}_{\infty}(G) = 2$ *if G is not a forest and* $va^{\equiv}_{\infty}(G) = 1$ *otherwise.*

Proof. By Theorem 3.5, we only need to show that G has an equitable 2-tree-coloring. We now apply induction on the order of G.

By Corollary 3.7, G contains one of the configurations among (C2.1), (C2.2) and (C2.3). If G contains (C2.1), then by Corollary 3.4, there exists a 3^--vertex y in $G - x$. Now let $x := v_1$ and $y := v_2$. If G contains (C2.2), then again let $x := v_1$ and $y := v_2$. In each case let $S = \{v_1, v_2\}$. We then have $|N_G(v_1) \setminus S| \leq 1$ and $|N_G(v_2) \setminus S| \leq 3$. Since $G - S$ has an equitable 2-tree-coloring by induction, G admits an equitable 2-tree-coloring by Lemma 3.1.

If G contains (C2.3), then let $x_1, \ldots, x_5$ be the five 2-neighbors of x and let $G' = G - \{x, x_1, x_2, x_3\}$. By induction, G' has an equitable 2-tree-coloring c'. If $c'(x_4) = c'(x_5) = 1$, then color x, x_1 by 2 and x_2, x_3 by 1. If $c'(x_4) = 1$ and $c'(x_5) = 2$, then denote the other neighbor of x_i beside x be x_i'. If $c'(x_1') = c'(x_2') = c'(x_3') = 1$, then color x, x_1 by 2 and x_2, x_3 by 1. Otherwise, if $c'(x_1') = 1$ and $c'(x_2') = c'(x_3') = 2$, then color x, x_1 by 2 and x_2, x_3 by 1. In each case one can check that the extended coloring of G is an equitable 2-tree-coloring. □

A graph is outerplanar if it can be drawn in the plane so that all vertices are lying on the outside face. The following structural lemma for outerplanar graphs has been proved by many authors.

Lemma 3.9 ([2]). *Every outerplanar graph with minimum degree at least two contains one of the following configurations:*

(C1) *two adjacent 2-vertices u and v;*
(C2) *a 3-cycle uvw with $d(u) = 2$ and $d(v) = 3$;*
(C3) *two intersecting 3-cycles uvw and xyw with $d(u) = d(x) = 2$ and $d(w) = 4$.*

From the above lemma, one can see that every outerplanar graph contains either a vertex x of degree 1 or an edge xy with $d(x) = 2$ and $d(y) \leq 4$. Thus by the same argument as in Theorem 3.10, we have the following theorem for outerplanar graphs.

Theorem 3.10. *Every outerplanar graph has an equitable t-tree-coloring for every $t \geq 2$, that is, $va_{\infty}^{\equiv}(G) = 2$ if G is not a forest and $va_{\infty}^{\equiv}(G) = 1$ otherwise.*

4. Open problems

To end this paper, we raise two conjectures for further research. The results in Section 3 support the first conjecture, and on the other hand, the upper bound in the second conjecture is sharp if it holds, since $va_{\infty}^{\equiv}(K_n) = \lceil \frac{n}{2} \rceil$ (this can easily be checked).

Conjecture 4.1. $va_{\infty}^{\equiv}(G) = O(1)$ *for every planar graph G.*

Conjecture 4.2. $va_{\infty}^{\equiv}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ *for every graph G.*

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