# Equitable Coloring and Equitable Choosability of Planar Graphs without 6- and 7-Cycles \*

Aijun Dong<sup>†</sup>, Guojun Li, Xiang Tan, Xin Zhang School of Mathematics, Shandong University, Jinan 250100, P. R. China

### Abstract

For any given k-uniform list assignment L, a graph G is equitably k-choosable, if and only if G is L-colorable and each color appears on at most  $\lceil \frac{|V(G)|}{k} \rceil$  vertices. A graph G is equitable k-colorable if G has a proper vertex coloring with k colors such that the size of the color classes differ by at most 1. In this paper, we prove that every planar graph G without 6- and 7-cycles is equitably k-colorable and equitably k-choosable where  $k \ge \max{\Delta(G), 6}$ .

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## 1 Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let G = (V, E) be a graph. We use V(G), E(G), F(G),  $\Delta(G)$ and  $\delta(G)$  to denote the vertex set, edge set, face set, maximum degree, and minimum degree of G, respectively. Let  $d_G(x)$  or simply d(x), denote the degree of a vertex (face) x in G. A vertex (face) x is called a k-vertex (kface),  $k^+$ -vertex ( $k^+$ -face),  $k^-$ -vertex or  $k^-$ -vertex, if d(x) = k,  $d(x) \ge k$ ,  $2 \le d(x) \le k$  or  $1 \le d(x) \le k$ . We use  $(d_1, d_2, \dots, d_n)$  to denote a face f if  $d_1, d_2, \dots, d_n$  are the degree of vertices incident to the face f. Let  $\delta(f)$  denote the minimal degree of vertices incident to f. A graph G is k-degenerate if every subgraph has a vertex of degree at most k.

A graph G = (V, E) is said to be equitably k-colorable if the vertex set V(G) can be partitioned into k independent subsets  $V_1, V_2, \dots, V_k$  such

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<sup>&</sup>lt;sup>†</sup>Corresponding author. E-mail: dongaijun@mail.sdu.edu.cn.

that  $||V_i| - |V_j|| \leq 1$   $(1 \leq i, j \leq k)$ . The equitable chromatic number of G, denoted by  $\chi_e(G)$ , is the smallest integer k such that G is equitably k-colorable. The equitable chromatic threshold of G, denoted by  $\chi_e^*(G)$ , is the smallest integer k such that G is equitably l-colorable  $(l \geq k)$ . It is obvious that  $\chi_e(G) \leq \chi_e^*(G)$  for any graph G. They might not be equal. For example, if  $K_{2n+1,2n+1}$  (n is a positive integer) is a complete bipartite graph, then  $\chi_e(K_{2n+1,2n+1}) = 2$ ,  $\chi_e^*(K_{2n+1,2n+1}) = 2n + 2$ .

In many application of graph coloring, it is desirable that the color classes are not too large. For example, when using a coloring model to find an optimal final exam schedule, one would like to have approximately equal number of final exams in each time slot because the whole exam period should be as short as possible and the number of classrooms available is limited. Recently, Pemmaraju [13] and Janson and Ruciński [6] used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence. In all of these applications, the fewer colors we use, the better the deviation bound is. Equitable coloring has a well-known property that restricts the size of each color class by its definition.

In 1970, Hajnál and Szemerédi proved that  $\chi_e^*(G) \leq \Delta(G) + 1$  for any graph G [5]. This bound is sharp as shows the example of  $K_{2n+1,2n+1}$ . In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture [11]:

**Conjecture 1** If G is a connected graph which is neither a complete graph nor odd cycle, then  $\chi_e(G) \leq \Delta(G)$ .

In 1994, Chen et al. put forth the following conjecture [2]:

**Conjecture 2** For any connected graph G, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then  $\chi_e^*(G) \leq \Delta(G)$ .

Chen et al. proved the conjecture for graphs with  $\Delta(G) \leq 3$  or  $\Delta(G) \geq \frac{|V(G)|}{2}$  or a tree [2, 3]. Yap and Zhang proved that the conjecture holds for outer planar graphs and planar graphs with  $\Delta(G) \geq 13$  [17, 18]. Lih and Wu verified  $\chi_e^*(G) \leq \Delta(G)$  for bipartite graphs other than a complete bipartite graph [9]. Wang et al. proved the conjecture for line graphs [16]. It follows from [8] that the conjecture hold for *d*-degenerate graphs with maximum degree  $\Delta(G) \geq 14d + 1$ .

For a graph G and a list assignment L assigned to each vertex  $v \in V(G)$ a set L(v) of acceptable colors, a L-coloring of G is a proper vertex coloring such that for every  $v \in V(G)$  the color on v belongs to L(v). A list assignment L for G is k-uniform if |L(v)| = k for all  $v \in V(G)$ . A graph G is equitably k-choosele if, for any k-uniform list assignment L, G is *L*-colorable and each color appears on at most  $\lceil \frac{|V(G)|}{k} \rceil$  vertices. In 2003, Kostochka, Pelsmajer and West investigated the equitable list

coloring of graphs. They proposed the following conjecture [7].

**Conjecture** 3 Every graph G is equitably k-choosable whenever  $k > \Delta(G)$ .

**Conjecture 4** If G is a connected graph with maximum degree at least 3, then G is equitably  $\Delta(G)$ -choosable, unless G is a complete graph or is  $K_{k,k}$  for some odd k.

It has been proved that Conjecture 3 holds for graphs with  $\Delta(G) < 3$ independently in [12, 14]. Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either  $G \neq K_{k+1}, K_{k,k}$  (with k odd in the later case) and  $k \geq \max\{\Delta, \frac{|V(G)|}{2}\}$ , or G is a connected inteval graph and  $k \geq \Delta(G)$  or G is a 2-degenerate graph and  $k \geq \max\{\Delta(G), 5\}$ [7]. Pelsmajer proved that every graph is equitably k-choosable for any  $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2} + 2$  [12]. There are several results for planar graphs without short cycles [10, 19].

In this paper, we show that every planar graph G without 6- and 7-cycles is equitably k-colorable and equitably k-choosable where  $k \geq$  $\max\{\Delta(G), 6\}.$ 

#### $\mathbf{2}$ Planar graphs without 6- and 7-cycles

First let us introduce some important lemma.

**Lemma 2.1** ([4]) Every planar graph without 6-cycles is 3-degenerate.

**Corollary 2.2** If G is a planar graph without 6- and 7-cycles, then  $\delta(G) \leq$ 3.

**Lemma 2.3** ([5]) Every graph has an equitable k-coloring whenever  $k \geq$  $\Delta(G) + 1.$ 

**Lemma 2.4** ([19]) Let  $S = \{v_1, v_2, \dots, v_k\}$  be a set of k different vertices in G such that G - S has an equitable k-coloring, if  $|N_G(v_i) - S| \leq k - i$ for  $1 \leq i \leq k$ , then G has an equitable k-coloring.

**Lemma 2.5** ([12, 14]) Every graph G with maximum degree  $\Delta(G) \leq 3$  is equitably k-choosable whenever  $k \ge \Delta(G) + 1$ .

**Lemma 2.6** ([7]) Let G be a graph with a k-uniform list assignment L. Let  $S = \{v_1, v_2, \dots, v_k\}$ , where  $\{v_1, v_2, \dots, v_k\}$  are distinct vertices in G. If G-S has an equitable L-coloring and  $|N_G(v_i) - S| \le k - i$  for  $1 \le i \le k$ , then G has an equitable L-coloring.

**Lemma 2.7** Let G be a connected planar graph with order at least 5 and without 6- and 7-cycles, then G has at least one of the following structures in Figure 1.



Figure 1



Figure 1



Figure 2



Each configuration in Figure 1 and Figure 2 represents subgraphs for which: (1)hollow vertices may be not distinct while solid vertices are distinct. (2)the degree of the solid vertex is fixed, and (3)except for special pointed, the degree of a hollow vertex may be any integer from  $[d, \Delta(G)]$ , where d is the number of edges incident to the hollow vertex in the configuration. (4) the order of the vertices on the boundary of a 4-face can be exchanged if the vertices are not common vertices of the 4-face and other face.

**Proof.** Let G be a minimal counterexample on the number of vertices. Then G without 6- and 7-cycles does not contain  $H_1 \sim H_{67}$  in Figure 1.

For G contains no structures  $H_1 \sim H_3$ , we can obtain the following property.

**Claim 1** Any two of  $(3, 3, 5^+)$ -faces can not simultaneously appear in G except that the structure  $G_1$  (in Figure 2).

We call a face a special face if it belongs to  $(3,3,5^+)$ -faces. In the following, we call a 3-vertex a special 3-vertex if it is incident to a special face, otherwise, it is called a simple 3-vertex. For convenience, let  $n_3(v)$ ,  $m_3(v)$  and  $m_4(v)$  denote the number of simple 3-vertices adjacent to v, the number of 3-faces incident to v and the number of 4-faces incident to v for each  $v \in V(G)$  respectively. Let  $n_i(f)$  denote the number of *i*-vertices incident to f.

Since G contains no structures  $H_4$  and  $H_5$ , we can conclude the following properties.

**Claim 2** For each  $v \in V(G)$  with  $d(v) \ge 4$ , if v is adjacent to a 3-vertex which is adjacent to two  $3^{--}$ -vertices, then it is not adjacent to other 3-vertex.

**Claim 3** For any  $v \in V(G)$ , v is adjacent to at most one simple 3-vertex which is adjacent to other  $3^{--}$ -vertex.

By Euler's formula |V|-|E|+|F|=2 and  $\sum_{v\in V(G)}d(v)=\sum_{f\in F(G)}d(f)=2|E|,$  we have

$$\sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) = -10(|V| - |E| + |F|) = -20.$$

Define an initial charge function w on  $V(G) \cup F(G)$  by setting w(v) = 3d(v) - 10 if  $v \in V(G)$  and w(f) = 2d(f) - 10 if  $f \in F(G)$ , so that  $\sum_{x \in V(G) \cup F(G)} W(x) = -20.$ 

We divide the proof into the following four cases by Corollary 2.2. Case 1  $\delta(G) = 3$ .

For G contains no structure  $H_6$ . G has the following properties.

Claim 1.1 All 3-faces in G are  $(3, 3, 5^+)$ -,  $(3, 4^+, 4^+)$ - or  $(4^+, 4^+, 4^+)$ -faces.

Now redistribute the charge according to the following discharging rules.

R1 Transfer charge 1 from each 5<sup>+</sup>-vertex to every adjacent simple 3-vertex v which is adjacent to exactly two 3<sup>--</sup>-vertices.

R2 Transfer charge  $\frac{1}{2}$  from each 4<sup>+</sup>-vertex to every adjacent simple 3-vertex v which is adjacent to exactly one 3<sup>--</sup>-vertex.

R3 Transfer charge  $\frac{1}{3}$  from each 4<sup>+</sup>-vertex to every adjacent simple 3-vertex v which is not adjacent to any 3<sup>--</sup>-vertex.

R4 Transfer charge  $\frac{3}{4}$  from each 8<sup>+</sup>-face f to every adjacent 3-face and 4-face via each common edge.

R5 If f is a 4-face incident to v, then v gives f charge  $\frac{2}{3}$  if  $d(v) \ge 6$ .

R6 If f is a 3-face incident to a vertex v, then v gives f charge  $\frac{3}{2}$  if d(v) = 4 and f is a  $(3^-, 4, 4)$ -face,  $\frac{5}{6}$  if d(v) = 4 and f is a  $(3^-, 4, 5^+)$ -face,  $\frac{2}{3}$  if d(v) = 4 and f is a  $(4, 4, 4^+)$ - or  $(4, 5, 5^+)$ -face,  $\frac{7}{4}$  if d(v) = 5 and f is a  $(3^-, 3^+, 5)$ -face,  $\frac{7}{6}$  if d(v) = 5 and f is an other face,  $\frac{7}{4}$  if d(v) = 6, 2 if  $d(v) \ge 7$ .

Let the new charge of each element x be w'(x) for each  $x \in V(G) \cup F(G)$ . Particularly, we use  $w'_s$  denotes the total new charge of all the special 3-vertices and all the special faces in G.

**Case 1.1**  $\delta(G) = 3$  and there are at least two 3-vertices in G.

Now let us check the charge of each element  $x \in V(G) \cup F(G)$ .

Suppose d(v) = 3. Then w(v) = -1. Since G contains no structure  $H_7$ , v is adjacent to at least one 5<sup>+</sup>-vertex or is adjacent to at least two 4<sup>+</sup>-vertex. If v is a simple 3-vertex, we have w'(v) = -1 + 1 = 0 by R1,  $w'(v) = -1 + \frac{1}{2} \times 2 = 0$  by R2 or  $w'(v) = -1 + \frac{1}{3} \times 3 = 0$  by R3. Otherwise, we have w'(v) = w(v) = -1.

Suppose d(v) = 4. Then w(v) = 2 and  $m_3(v) \le 3$ .

**Case 1.1.1**  $m_3(v) = 3$ , then  $n_3(v) \le 1$  for G contains no structure  $H_8$ . If  $m_3(v) = 3$ ,  $n_3(v) = 1$  and v is incident to two 3-faces each of which is incident to the 3-vertex, then we have  $w'(v) \ge 2 - \frac{5}{6} \times 2 - \frac{1}{3} = 0$  for G contains no structures  $H_9$ ,  $H_{10}$  and by R6, R3. If  $m_3(v) = 3$ ,  $n_3(v) = 1$  and v is incident to only one 3-face which is incident to a 3-vertex, then we have  $w'(v) \ge 2 - \frac{5}{6} - \frac{1}{2} = \frac{2}{3} > 0$  for G contains no structures  $H_{11}$ ,  $H_{12}$  and by R6, R2. If  $m_3(v) = 3$ ,  $n_3(v) = 0$ , then we have  $w'(v) \ge 2 - \frac{2}{3} \times 3 = 0$  by R6.

**Case 1.1.2**  $m_3(v) = 2$ , then  $n_3(v) \le 1$  for *G* contains no structure  $H_8$ . If  $m_3(v) = 2$ ,  $n_3(v) = 1$  and *v* is incident to two 3-faces each of which is incident to the 3-vertex, then we have  $w'(v) \ge 2 - \frac{5}{6} \times 2 - \frac{1}{3} = 0$  for *G* contains no structure  $H_{13}$  and by R6, R3. If  $m_3(v) = 2$ ,  $n_3(v) = 1$  and *v* is incident to only one 3-face which is incident to the 3-vertex, then we have  $w'(v) \ge 2 - \frac{5}{6} - \frac{2}{3} - \frac{1}{2} = 0$  for G contains no structures  $H_{14}$ ,  $H_{15}$  and by R6, R2. If  $m_3(v) = 2$ ,  $n_3(v) = 1$  and v is not incident to any 3-face which is incident to a 3-vertex, then we have  $w'(v) \ge 2 - \frac{2}{3} \times 2 - \frac{1}{2} = \frac{1}{6} > 0$  for G contains no structure  $H_7$  and by R6, R2. If  $m_3(v) = 2$ ,  $n_3(v) = 0$ , then we have  $w'(v) \ge 2 - \frac{2}{3} \times 2 = \frac{2}{3} > 0$  by R6.

**Case 1.1.3**  $m_3(v) = 1$ , then  $n_3(v) \le 2$  for G contains no structure  $H_8$ . If  $m_3(v) = 1$ ,  $n_3(v) = 2$ , then we have  $w'(v) \ge 2 - \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = \frac{1}{2} > 0$  for G contains no structure  $H_8$  and by R6, R2, R3 and Claim 3. If  $m_3(v) = 1$ ,  $n_3(v) \le 1$ , then  $w'(v) \ge 2 - \frac{3}{2} - \frac{1}{2} = 0$  by R6 and R2.

**Case 1.1.4**  $m_3(v) = 0$ , then  $n_3(v) \le 4$ . So  $w'(v) \ge 2 - \frac{1}{2} - \frac{1}{3} \times 3 = \frac{1}{2} > 0$ by R2, R3 and Claim 3.

Suppose d(v) = 5. Then w'(v) = 5,  $m_3(v) \le 3$ .

**Case 1.2.1**  $m_3(v) = 3$ , then  $n_3(v) \leq 1$  for G contains no structures  $H_{16} \sim H_{22}$ . If  $m_3(v) = 3$ ,  $n_3(v) = 1$  and v is incident to two 3-faces each of which is incident to the 3-vertex, then  $w'(v) \ge 5 - \frac{7}{4} \times 2 - \frac{7}{6} - \frac{1}{3} = 0$  by R6 and R3. If  $m_3(v) = 3$ ,  $n_3(v) = 1$  and v is incident to only one 3-face which is incident to the 3-vertex, then  $w'(v) \ge 5 - \frac{7}{4} - \frac{7}{6} \times 2 - \frac{1}{2} = \frac{5}{12} > 0$ by R6 and R2. If  $m_3(v) = 3$ ,  $n_3(v) = 1$  and v is not incident to any 3-face which is incident to a 3-vertex, then  $w'(v) \ge 5 - \frac{7}{6} \times 3 - 1 = \frac{1}{2} > 0$  by R6 and R1. If  $m_3(v) = 3$ ,  $n_3(v) = 0$ , then we have  $w'(v) \ge 5 - \frac{7}{6} \times 3 = \frac{3}{2} > 0$ by *R*6.

**Case 1.2.2**  $m_3(v) = 2$ , then  $n_3(v) \le 4$ . So  $w'(v) \ge 5 - \frac{7}{4} \times 2 - \frac{1}{3} \times 3 - \frac{1}{2} =$ 0 by R6, R3, R2 and Claim 4.

**Case 1.2.3**  $m_3(v) \le 1$ , then  $n_3(v) \le 5$ . So  $w'(v) > 5 - \frac{7}{4} - \frac{1}{3} \times 4 - \frac{1}{2} =$  $\frac{17}{12} > 0$  by R6, R3, R2 and Claim 4.

Suppose d(v) = 6. Then w(v) = 8,  $m_3(v) \le 4$ .

If  $m_3(v) = 4$ , then  $m_4(v) = 0$ ,  $n_3(v) \leq 3$  for G contains no structure *H*<sub>23</sub>. So  $w'(v) \ge 8 - \frac{7}{4} \times 4 - \frac{1}{3} \times 3 = 0$  for *G* contains no structure *H*<sub>24</sub> and by *R*6, *R*3. If  $m_3(v) = 3$ , then  $m_4(v) \le 1$ ,  $n_3(v) \le 4$ . So  $w'(v) \ge 8 - \frac{7}{4} \times 3 - \frac{2}{3} - \frac{1}{3} \times 4 = \frac{3}{4} > 0$  by R6, R5 and R3. If  $m_3(v) = 2$ , then  $m_4(v) \le 2, n_3(v) \le 5$ . We have  $w'(v) \ge 8 - \frac{7}{4} \times 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - \frac{1}{2} = \frac{4}{3} > 0$ by R6, R5, R3 and R2. If  $m_3(v) \leq 1$ , then  $m_4(v) \leq 6$ ,  $n_3(v) \leq 6$ . We have  $w'(v) > 8 - \frac{7}{4} - \frac{2}{3} \times 6 - \frac{1}{3} \times 5 - \frac{1}{2} = \frac{1}{12} > 0$  by R6, R5, R3 and R2. Suppose d(v) = 7. Then  $w(v) = 11, m_3(v) \le 5$ .

If  $m_3(v) = 5$ , then  $m_4(v) = 0$ ,  $n_3(v) \leq 3$  for G contains no structure  $H_{23}$ . We have  $w'(v) \ge 11 - 2 \times 5 - \frac{1}{3} \times 3 = 0$  for G contains no structure  $H_{24}$  and by R6, R3. If  $m_3(v) = 4$ , then  $m_4(v) \le 1$ ,  $n_3(v) \le 4$  for G contains no structure  $H_{23}$ . So  $w'(v) > 11 - 2 \times 4 - \frac{2}{3} - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{5}{6} > 0$ by R6, R5, R3 and R2. If  $m_3(v) = 3$ , then  $m_4(v) \le 2$ ,  $n_3(v) \le 5$ . So  $w'(v) > 11 - 2 \times 3 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - \frac{1}{2} = \frac{11}{6} > 0$  by R6, R5, R3 and R2. If  $m_3(v) = 2$ , then  $m_4(v) \le 2$ ,  $n_3(v) \le 6$ . So  $w'(v) > 11 - 2 \times 2 - \frac{2}{3} \times 2 - \frac{1}{3} \times 6 = 1$  $\frac{11}{3} > 0$  by R6, R5 and R3. If  $m_3(v) \le 1$ , then  $m_4(v) \le 7$ ,  $n_3(v) \le 7$ . So

 $w'(v) > 11 - 2 - \frac{2}{3} \times 7 - \frac{1}{3} \times 7 = 2 > 0$  by R6, R5 and R3. Suppose d(v) = 8. Then w(v) = 14,  $m_3(v) \le 6$ .

If  $m_3(v) = 6$ , then  $m_4(v) = 0$ ,  $n_3(v) \le 4$  for G contains no structure  $H_{23}$ . So  $w'(v) \ge 14 - 2 \times 6 - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{1}{2} > 0$  by R6, R3, R2 and Claim 3. If  $m_3(v) = 5$ , then  $m_4(v) = 0$ ,  $n_3(v) \le 4$  for G contains no structure  $H_{23}$ . So  $w'(v) > 14 - 2 \times 5 - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{5}{2} > 0$  by R6, R3 and R2. If  $m_3(v) = 4$ , then  $m_4(v) < 2$ ,  $n_3(v) \le 5$  for G contains no structure  $H_{23}$ . So  $w'(v) > 14 - 2 \times 4 - \frac{2}{3} \times 2 - \frac{1}{3} \times 5 = 3 > 0$  by R6, R5, R3. If  $m_3(v) = 3$ , then  $m_4(v) \le 2, n_3(v) < 8$ . We have  $w'(v) > 14 - 2 \times 3 - \frac{2}{3} \times 2 - \frac{1}{3} \times 8 = 4 > 0$ by R6, R5, R3. If  $m_3(v) \leq 2$ , then  $m_4(v) \leq 8$ ,  $n_3(v) \leq 8$ . We have  $w'(v) > 14 - 2 \times 2 - \frac{2}{3} \times 8 - \frac{1}{3} \times 8 = 2 > 0$  by R6, R5, R3. Suppose  $d(v) \ge 9$ . Then w(v) = 3d(v) - 10. Since  $n_3(v) \le m_3(v) + 1$ 

 $d(v) - \frac{4}{3}m_3(v)$ , we have

$$n_3(v) \le d(v) - \frac{1}{3}m_3(v).$$

And since  $m_4(v) \le d(v) - \frac{4}{3}m_3(v)$ , we have  $w'(v) \ge 3d(v) - 10 - 2m_3(v) - 10 - 2m_3(v)$  $\frac{2}{3} \times m_4(v) - \frac{1}{2} - \frac{1}{3}(n_3(v) - 1) \ge 3d(v) - 10 - 2m_3(v) - \frac{2}{3} \times (d(v) - \frac{4}{3}m_3(v)) - \frac{1}{2} - \frac{1}{3}(d(v) - \frac{1}{3}m_3(v) - 1) = 2d(v) - m_3(v) - \frac{61}{6} \text{ by } R6, R5, R2, R3 \text{ and } R6 + \frac{1}{3}m_3(v) - \frac{$ Claim 4. And since

$$m_3(v) \le \frac{3}{4}d(v).$$

We have  $w'(v) \ge \frac{9}{8}d(v) - \frac{61}{6} \ge \frac{13}{12} > 0.$ 

Suppose d(f) = 3. Then  $w(\tilde{f}) = -4$ ,  $n_3(f) \le 2$  by Claim 1.1.

**Case** 1.3.1  $n_3(f) = 2$ , then  $n_4(f) = 0$ , f is a special face  $(3, 3, 5^+)$ -face. **Case** 1.3.1.1 f is not adjacent to any special 3-face, we have  $w'(f) \geq c$  $-4 + \frac{7}{4} = -\frac{9}{4}$  by R6.

**Case** 1.3.1.2 f is adjacent to a special 3-faces, i.e. the cluster  $G_1$  in Figure 2 for G contains no structures  $H_2$  and  $H_3$ . If  $G_1$  is neither adjacent to a 3-face nor adjacent to a 4-face, then we have  $w'(G_1) \ge -4 \times 2 + \frac{7}{4} \times$  $2 + \frac{3}{4} \times 4 = -\frac{3}{2}$  by R6 and R4. If  $G_1$  is adjacent to a 3-face and not adjacent to a 4-face, we consider cluster  $G_2$  in Figure 2 for G contains no structure  $H_{25}$ , we have  $w'(G_2) \ge -4 \times 3 + 2 \times 4 + \frac{3}{4} \times 2 = -\frac{5}{2}$  by R6 and R4. If  $G_1$  is adjacent to a 4-face and not adjacent to a 3-face, we consider cluster  $G_3$  in Figure 2 for G contains no structure  $H_{26}$  and  $H_{27}$ , we have  $w'(G_3) \ge -4 \times 2 - 2 + 2 \times 2 + \frac{2}{3} \times 3 + \frac{3}{4} \times 4 = -1$  by R6, R5 and R4. If  $G_1$ is not only adjacent to a 3-face but also adjacent to a 4-face, we consider cluster  $G_4$  in Figure 2 for G contains no structures  $H_{25}$  and  $H_{26}$ , we have  $w'(G_4) \ge -4 \times 3 - 2 + 2 \times 4 + \frac{2}{3} \times 3 + \frac{3}{4} \times 2 = -\frac{5}{2} R_6, R_5 \text{ and } R_4.$ 

**Case** 1.3.2  $n_3(f) = 1$  and f is a (3, 4, 4)-, (3, 4, 5)-face, then f is not adjacent to any 3-face for G contains no structures  $H_{13}$ ,  $H_{14}$  and  $H_{28}$ . So  $w'(f) \ge -4 + \frac{3}{2} \times 2 + \frac{3}{4} \times 2 = \frac{1}{2} > 0$  or  $w'(f) \ge -4 + \frac{5}{6} + \frac{7}{4} + \frac{3}{4} \times 2 = \frac{1}{12} > 0$ by R5 and R4.

**Case** 1.3.3  $n_3(f) = 1$  and f is a (3, 4, 6)-face. For G contains no structure  $H_{29}$ , f is adjacent to at most two 3-faces. If f is not adjacent to any 3-face, then f is adjacent to at most one 4-face for G contains no 7-cycle. We have  $w'(f) \ge -4 + \frac{5}{6} + \frac{7}{4} + \frac{3}{4} \times 2 = \frac{1}{12} > 0$  by R6, R4. If f is adjacent to a 3-face, then f is not adjacent to any 4-face for G contains no structures  $H_{27}$  and  $H_{30}$ . We have  $w'(f) \ge -4 + \frac{5}{6} + \frac{7}{4} + \frac{3}{4} \times 2 = \frac{1}{12} > 0$ by R6, R4. If f is adjacent to two 3-faces, i.e.  $G_5$  and  $G_6$  in Figure 2 for G contains no structures  $H_{29}$ ,  $H_{31}$  and  $H_{32}$ . For G contains no 6-cycle,  $G_5$  and  $G_6$  are not adjacent to any 4-face or 5-face. We have  $w'(G_5) \ge -4 \times 3 + \frac{7}{4} + \frac{5}{6} \times 2 + \frac{7}{4} \times 3 + \frac{3}{4} \times 5 = \frac{5}{12} > 0$ ,  $w'(G_6) \ge -4 \times 3 + \frac{7}{4} + \frac{7}{4} \times 3 + \frac{3}{4} \times 5 = \frac{4}{3} > 0$  by R6 and R4. **Case** 1.3.4  $n_3(f) = 1$ , f is a  $(3, 4, 7^+)$ -face. For G contains no structure

 $H_{29}$ , f is adjacent to at most two 3-faces.

**Case** 1.3.4.1 f is not adjacent to any 3-face, we have  $w'(f) \ge -4 + \frac{5}{6} + \frac{5}{6}$  $2 + \frac{3}{4} \times 2 = \frac{1}{3} > 0$  by R6 and R4.

**Case** 1.3.4.2 f is a  $(3, 4, 7^+)$ -face and is adjacent to one 3-face. If f is not adjacent to any 4-face, then we have  $w'(f) \ge -4 + \frac{5}{6} + 2 + \frac{3}{4} \times 2 = \frac{1}{3} > 0$  by R6 and R4. Otherwise, we consider  $G_7$  and  $G_8$  (in Figure 2) for G contains no structures  $H_{26}$ ,  $H_{27}$ ,  $H_{33}$  and  $H_{34}$ . We have  $w'(G_7) \ge -4 \times 2 - 2 + \frac{5}{6} \times 2 + 2 \times 2 + \frac{2}{3} \times 3 + \frac{3}{4} \times 4 = \frac{2}{3} > 0$ ,  $w'(G_8) \ge -4 \times 2 - 2 + \frac{5}{6} + 2 \times 3 + \frac{2}{3} + \frac{3}{4} \times 4 = \frac{1}{2} > 0$ by R6 and R4.

**Case** 1.3.4.3 f is a  $(3, 4, 7^+)$ -face and adjacent to two 3-faces, then we need to consider the following three structures  $G_{20}$ ,  $G_{21}$  in Figure 2 for G contains no structures  $H_{29}$ ,  $H_{31}$  and  $H_{32}$ . We have  $w'(G_9) \ge -4 \times 3 + \frac{7}{4} \times 2 + 2 \times 2 + \frac{5}{6} \times 2 + \frac{3}{4} \times 5 = \frac{11}{12} > 0$ ,  $w'(G_{10}) \ge -4 \times 3 + \frac{7}{4} \times 2 + 2 \times 3 + \frac{5}{6} + \frac{3}{4} \times 5 = \frac{11}{25} = \frac{$  $\frac{25}{12} > 0$  by R6 and R4.

**Case** 1.3.5  $n_3(f) = 1$ , f is a  $(3, 5, 5^+)$ - or  $(3, 6, 6^+)$ -face. If f is adjacent to at most two 3-faces, we have  $w'(f) \ge -4 + \frac{7}{4} \times 2 + \frac{3}{4} = \frac{1}{4} > 0$  by R6 and R4. If f is adjacent to three 3-faces i.e.  $G_{11}$  and  $G_{12}$  in Figure 2 for G contains no structures  $H_{35}$  and  $H_{36}$ , then  $w'(G_{11}) \ge -4 \times 4 + \frac{7}{4} \times 4 + 2 \times 3 + \frac{7}{6} \times 2 + \frac{3}{4} \times 4 = \frac{7}{3} > 0$ ,  $w'(G_{12}) \ge -4 \times 4 + \frac{7}{4} \times 5 + 2 \times 3 + \frac{7}{6} + \frac{3}{4} \times 4 = \frac{35}{12} > 0$ , by R6 and R4.

**Case** 1.3.6  $n_3(f) = 1$ , f is a  $(3, 7^+, 7^+)$ -face, we have  $w'(f) \ge -4 + 2 \times$ 2 = 0 by R6.

**Case** 1.3.7  $n_3(f) = 0$ , f is a (4,4,4)-face, then f is not adjacent to any 3-face for G does not contain structure  $H_{37}$ . If f is not adjacent to any 4-face, then we have  $w'(f) = -4 + \frac{2}{3} \times 3 + \frac{3}{4} \times 3 = \frac{1}{4} > 0$  by R6 and R4. Otherwise, we consider  $G_{13}$  for  $\overset{\circ}{G}$  contains no structure  $H_{38}$ , so  $w'(G_{13}) \ge -4 - 2 + \frac{2}{3} \times 3 + \frac{2}{3} \times 2 + \frac{3}{4} \times 5 = \frac{13}{12} > 0$  by R6, R5 and R4.

**Case** 1.3.8  $n_3(f) = 0$ , f is a (4, 4, 5)- or (4, 4, 6)-face. If f is adjacent to at most one 3-face, we have  $w'(f) \ge -4 + \frac{2}{3} \times 2 + \frac{7}{6} + \frac{3}{4} \times 2 = 0$  by R6 and R4. If f is adjacent to two 3-faces i.e.  $G_{14}$ ,  $G_{15}$  and  $G_{16}$  in Figure 2 for G contains no structures  $H_{39}$ ,  $H_{40}$  and  $H_{41}$ , we have  $w'(G_{14}) \geq -4 \times 3 + \frac{7}{4} \times 2 +$   $\frac{7}{6} \times 3 + \frac{2}{3} \times 4 + \frac{3}{4} \times 5 = \frac{17}{12} > 0, \ w'(G_{15}) \ge -4 \times 3 + \frac{7}{4} \times 2 + \frac{7}{4} \times 3 + \frac{2}{3} \times 2 + \frac{3}{4} \times 5 = \frac{11}{6} > 0, \ w'(G_{16}) \ge -4 \times 3 + \frac{7}{4} \times 2 + \frac{7}{4} \times 2 + \frac{2}{3} \times 4 + \frac{3}{4} \times 5 = \frac{17}{12} > 0 \ \text{by} \ R6 = \frac{11}{6} \times 3 + \frac{7}{4} \times 2 + \frac{7}{4} \times 2 + \frac{2}{3} \times 4 + \frac{3}{4} \times 5 = \frac{17}{12} > 0 \ \text{by} \ R6 = \frac{11}{6} \times 3 + \frac{7}{4} \times 2 + \frac{7}{4} \times 2 + \frac{2}{3} \times 4 + \frac{3}{4} \times 5 = \frac{17}{12} > 0 \ \text{by} \ R6 = \frac{11}{6} \times 3 + \frac{7}{4} \times 2 + \frac{7}{4} \times 4 + \frac{3}{4} \times 5 = \frac{17}{12} \times 3 + \frac{3}{4} \times 3 + \frac{3}{4} \times 5 = \frac{17}{12} \times 3 + \frac{17}{12} \times 3 + \frac{17}{12} \times 3 + \frac{17}{12} \times 3 + \frac{17}{12}$ 

**Case** 1.3.9  $n_3(f) = 0$ , f is a  $(4, 4, 7^+)$ -face, then we have  $w'(f) \ge -4 + \frac{2}{3} \times 2 + 2 + \frac{3}{4} = \frac{1}{12} > 0$  by R6 and R4.

**Case** 1.3.10  $n_3(f) \stackrel{12}{=} 0$ , f is a (4, 5, 5)-face. If f is adjacent to at most one 3-face, we have  $w'(f) \ge -4 + \frac{7}{6} \times 2 + \frac{2}{3} + \frac{3}{4} \times 2 = \frac{1}{2} > 0$  by R6 and R4. If f is adjacent to two 3-faces i.e.  $G_{17}$  or  $G_{18}$  in Figure 2 for G contains no structures  $H_{42}$  and  $H_{43}$ . We have  $w'(G_{17}) \ge -4 \times 3 + \frac{7}{4} \times 2 + \frac{2}{3} \times 3 + \frac{7}{6} \times 4 + \frac{3}{4} \times 5 = \frac{23}{12} > 0$ , or  $w'(G_{18}) \ge -4 \times 3 + \frac{7}{4} \times 2 + \frac{2}{3} \times 2 + \frac{3}{4} \times 5 = \frac{29}{12} > 0$ by R6 and R4.

**Case** 1.3.11  $n_3(f) = 0$ , f is a  $(4, 5, 6^+)$ - or  $(4, 6^+, 6^+)$ -face, then we have  $w'(f) \ge -4 + \frac{7}{6} + \frac{7}{4} + \frac{2}{3} + \frac{3}{4} = \frac{1}{3} > 0$  by R6 and R4. **Case** 1.3.12  $n_3(f) = 0$ , f is a  $(5^+, 5^+, 5^+)$ -face, then we have  $w'(f) \ge 100 + 100$ 

 $-4 + \frac{7}{6} \times 3 + \frac{3}{4} = \frac{1}{4} > 0$  by R6 and R4.

Suppose d(f) = 4. Then w(f) = -2. If f is not adjacent to any 3-face, then all faces which are adjacent to f are  $8^+$ -faces for G contains no 6and 7-cycles. So  $w'(f) \ge -2 + \frac{3}{4} \times 4 = 1 > 0$  by R4. If f is adjacent to only one 3-face, then other faces which are adjacent to f are  $8^+$ -faces for G contains no 6- and 7-cycles. So  $w'(f) \ge -2 + \frac{3}{4} \times 3 = \frac{1}{4} > 0$  by R4. If f is adjacent to two 3-face, then f is adjacent to at least one  $6^+$ -vertex and other adjacent faces are  $8^+$ -faces for G contains no structure  $H_{44}$  and contains no 6- and 7-cycles. So  $w'(f) \ge -2 + \frac{2}{3} + \frac{3}{4} \times 2 = \frac{1}{6} > 0$  by R5 and R4..

Suppose d(f) = 5. Then w'(f) = w(f) = 0.

Suppose  $d(f) \ge 8$ . Then  $w'(f) \ge w(f) - \frac{3}{4} \times d(f) = 2d(f) - 10 - \frac{3}{4}d(f) = 2d(f) - \frac{3$  $\frac{5}{4}d(f) - 10 \ge \frac{5}{4} \times 8 - 10 = 0$  by R4.

From the above discussion, If there is at least two 3-vertices in G, we can obtain that if x is neither a special 3-vertex nor a special face, then  $w'(x) \ge 0$  for each  $x \in V(G) \cup F(G)$ . Furthermore, we have  $w'_s \ge 0$  $-1 \times 2 - \min\{-\frac{5}{2}, -\frac{9}{4}, -\frac{3}{2}, -1\} = -\frac{9}{2}$  by Claim 2. So we can obtain  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -\frac{9}{2} > -20$ , a contradiction.

**Case 1.2**  $\delta(G) = 3$  and there is only one 3-vertex in G.

There is no special 3-face and special 3-vertex in G. The discussion is similar to Case 1.1, it is clear that we only need to check the charge of 3-face in Case 1.3.3 and Case 1.3.4 of Case 1.1, i.e. (3, 4, 6)- and  $(3, 4, 7^+)$ faces such that they are adjacent to two 3-faces each of which is incident to the 3-vertex. For G contains no structure  $H_{25}$ , we only consider the structure  $G_{19}, w'(G_{19}) \ge -4 \times 2 + 2 \times 2 = -4$  by R6. So we can obtain  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -4 > -20$ , a contradiction.

**Case 2**  $\delta(G) = 2$  and there are at most two 2-vertices in G.

The discharging rules are the same as Case 1 except for the two cases. (1). A 4-face have a common 2-vertex with a 3-face, (2). Two 4-faces have a common 2-vertex. Under these situation, transfer charge from its common incident vertices to only the 3-face in the first situation and transfer charge from its common incident vertices to only one of them in the second situation. Clearly, we can guarantee the new charge of each vertex of Gis larger than or equal to zero. For convenience, let  $w'_{t1}$   $(w'_{t2})$  denote the total new charge of one 2-vertex (two 2-vertices) and the faces which are incident to the 2-vertex (the two 2-vertices).

**Case** 2.1 There exists one 2-vertex in G.

Case 2.1.1 The 2-vertex is incident to at most one 3-face and one 4-face simultaneously. Furthermore, for G contains no structure  $H_{45}$  and  $H_{46}$ , we can obtain that the 3-face and 4-face is  $(2, 3, 7^+)$ - and  $(2, 3, 6^+, 7^+)$  respectively; the 3-face and 4-face is  $(2, 4^+, 7^+)$ - and  $(2, 4^+, 6^+, 7^+)$  respectively; the 3-face and 4-face is  $(2, 5^+, 5^+)$ - and  $(2, 3^+, 5^+, 5^+)$ -face respectively. So  $w_{t1}' \ge -4 - 4 - 2 + 2 + \frac{2}{3} = -\frac{22}{3}, w_{t1}' \ge -3 - 2 - 2 + 2 + \frac{5}{6} + \frac{2}{3} = -\frac{13}{2}, w_{t1}' \ge -3 - 2 - 2 + \frac{7}{4} \times 2 = -\frac{13}{2}$  by *R*6 and *R*4.

Case 2.1.2 The 2-vertex is incident a 3-face and not incident to a 4-face, then the 3-face is  $(2, 3^+, 5^+)$ - or  $(2, 4^+, 4^+)$ -face for G contains no structure  $H_6$ . We have  $w'(t1) \ge -4 - 4 + \frac{7}{4} = -\frac{25}{4}$  or  $w'(t1) \ge -4 - 4 + \frac{3}{2} \times 2 = -5$ by *R*6.

**Case** 2.1.3 The 2-vertex is incident a 4-face and not incident to a 3face, we consider the situation such that the 2-vertex is a common vertex of two 4-faces, then the two 4-faces are adjacent to at least two  $8^+$ -face for G contains no 6- and 7-cycles. We have  $w'_{t}(t1) \geq -2 - 2 - 4 + \frac{3}{4} \times 2 = -\frac{13}{2}$  by R4. We can obtain that  $w'_{t1} \ge \min\{-\frac{22}{3}, -\frac{13}{2}, -\frac{25}{4}, -5, \}$  i.e.  $w'_{t1} \ge -\frac{22}{3}$ , so  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -\frac{9}{2} + w'_{t1} \ge -\frac{22}{3} - \frac{9}{2} = -\frac{81}{6} > -20$ , a contradiction. **Case** 2.2 There exist two 2-vertices in G.

**Case** 2.2.1 The two 2-vertices are incident to a same 3-face, then f is a  $(2,2,5^+)$ -face for G contains no structure  $H_7$ . So  $w'_{t2} \ge -2 \times 4 - 4 + \frac{7}{6} = -\frac{65}{6}$ by *R*6.

Case 2.2.2 Two 2-vertices are incident to a same 4-face. If each 2vertex is incident to another 4-face, we have  $w'_{t2} \ge -4 \times 2 - 2 \times 3 = -14$ . If one of the two 2-vertices is incident to another 3-face, the other 2-vertex is incident to another 4-face, then the 3-face which is incident to the 2vertex is a  $(2, 5^+, 5^+)$ -face for G contains no structure  $H_{46}$ . We have  $w'_{t2} \geq$  $-4 \times 3 - 2 \times 2 + \frac{7}{4} \times 2 = -\frac{25}{2}$  by R6.

Case 2.2.3 Two 2-vertices are not incident to a same face, from the discussion in **Case** 2.1, we have  $w'_{t2} \ge -\frac{22}{3} \times 2 = -\frac{44}{3}$ . Clearly, we have  $w'_{t2} \ge -\frac{44}{3}$  in the rest cases. From the above discussion, we have  $w'_{t2} \ge -\frac{44}{3}$ . So  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -\frac{9}{2} - \frac{44}{3} = -\frac{115}{6} > -20$ , a contradiction. **Case 3** There are at least three 2-vertices in G.

For G contains no structures  $H_{47} \sim H_{54}$ . G has the following properties. Claim 3.1 Any vertex v is adjacent to at most one 2-vertex.

Claim 3.2 Two 2-vertices are not adjacent to each other.

**Claim 3.3** For each  $v \in V(G)$  with  $d(v) \ge 4$ , if v is adjacent to a 2-vertex, then it is not incident to a 3-face which is incident to a 3-vertex.

**Claim 3.4** If v is adjacent to a 3-vertex, then it is not incident to a 3-face which is incident to a 2-vertex.

**Claim 3.5** If a vertex v is adjacent to a 2-vertex, then it is not adjacent to a 3-vertex which is adjacent to other  $3^{--}$ -vertex.

**Claim 3.6** 3-faces in G which is incident to a 2-vertex are  $(2, 6^+, 6^+)$ -faces.

Claim 3.7 4-faces which is incident to 2-vertices in G are  $(2, 3, 6^+, 6^+)$ -,  $(2, 4, 6^+, 6^+)$ - or  $(2, 5^+, 5^+, 5^+)$ -faces.

Claim 3.8 There is at most one 2-vertex which is adjacent to a k-vertex  $(3 \le k \le 4)$  in G.

We call a 2-vertex a special 2-vertex if it is adjacent to a k-vertex  $(3 \le k \le 4)$ , otherwise a simple 2-vertex. Let  $n_2(v)$  denote the number of simple 2-vertices adjacent to v. We call a 4-face a simple 4-face if it is incident to a 2-vertex. We use  $m'_4(v)$  denote the number of simple 4-faces which is incident to v for each  $v \in V(G)$ .

Now redistribute the charge according to the following discharging rules.

R1', R2', R3' and R4' are the same as R1, R2, R3 and R4 in Case 1.

R5' is the same as R5 except that d(v) = 5 and f is a simple 4-face. Transfer charge  $\frac{2}{3}$  from each 5-vertex to every incident simple 4-face.

R6' If f is a 3-face incident to a vertex v, then v gives f charge  $\frac{3}{2}$  if d(v) = 4 and f is a (3, 4, 4)-face,  $\frac{5}{6}$  if d(v) = 4 and f is a  $(3, 4, 5^+)$ -face,  $\frac{2}{3}$  if d(v) = 4 and f is an other face,  $\frac{7}{4}$  if d(v) = 5 and f is a  $(3, 4, 5^+)$ -face,  $\frac{2}{6}$  if d(v) = 5 and f is a (4, 4, 5)- or  $(4, 5, 5^+)$ - or  $(5, 5, 5^+)$ -face,  $\frac{2}{3}$  if d(v) = 5 and f is a  $(5, 6^+, 6^+)$ -face, 2 if d(v) = 6 and f is a  $(2, 6, 6^+)$ -face,  $\frac{7}{4}$  if d(v) = 6 and f is a  $(3, 3^+, 6)$ -face,  $\frac{3}{2}$  if d(v) = 6 and f is a (4, 4, 6)- or (4, 5, 6)-face,  $\frac{4}{3}$  if d(v) = 6 and f is an other face, 2 if d(v) = 7 and f is a  $(2, 7, 6^+)$ - or  $(3, 3^+, 7)$ -, or (4, 4, 7)-face,  $\frac{7}{4}$  if d(v) = 7 and f is an other face, 2 if  $d(v) \ge 8$ .

R7' Transfer charge 2 from each 5<sup>+</sup>-vertex to every adjacent 2-vertex. For any face  $f \in F(G)$ , if d(f) = 5,  $d(f) \ge 8$ , the discussion is similar to the corresponding situation in Case 1. For any vertex  $v \in V(G)$ , if d(v) = 3, the discussion is also similar to the corresponding situation in Case 1. In the following, we discuss the rest cases.

Suppose d(v) = 2, then w'(v) = -4. Except the special 2-vertex, we have  $w'(v) = -4 + 2 \times 2 = 0$  by R7'.

Suppose d(v) = 4. Then w(v) = 2 and  $m_3(v) \le 3$ .

First, we focus on the situation  $n_2(v) = 0$ . If  $m_3(v) = 3$ , then  $n_3(v) = 0$  for G contains no structures  $H_{55}$  and  $H_{56}$ . We have  $w'(v) \ge 2 - \frac{2}{3} \times 3 =$ 

0 by R6'. If  $m_3(v) \leq 2$ , then the following discussion is similar to the corresponding situation in Case 1.

In the following, we focus on the situation  $n_2(v) = 1$ . By Claim 3.6, we have  $m_3(v) \leq 2$ . If  $m_3(v) = 2$ , then  $n_3(v) = 0$  by Claim 3.3. We have  $w'(v) \geq 2 - \frac{2}{3} \times 2 = \frac{2}{3} > 0$  by R6'. If  $m_3(v) = 1$ , then  $n_3(v) \leq 1$  by Claim 3.3. We have  $w'(v) \geq 2 - \frac{2}{3} - \frac{1}{3} = \frac{1}{2} > 0$  by R6', R3', Claim 3.3 and Claim 3.5. If  $m_3(v) = 0$ , then  $n_3(v) \leq 3$ . So  $w'(v) \geq 2 - \frac{1}{3} \times 3 = 1 > 0$  by R3' and Claim 3.5.

Suppose d(v) = 5. Then w(v) = 5,  $m_3(v) \leq 3$ . If  $n_2(v) = 0$ , then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation  $n_2(v) = 1$ .

**Case** 3.1.1  $m_3(v) = 3$ , then  $m'_4(v) = 0$ ,  $n_3(v) = 0$  by Claim 3.3. We have  $w'(v) \ge 5 - \frac{7}{6} \times 2 - \frac{2}{3} - 2 = 0$  for G contains no structures  $H_{57}$ ,  $H_{58}$  and by R6', R7'.

**Case** 3.1.2  $m_3(v) = 2$ , then  $m'_4(v) \le 1$ . If  $m'_4(v) = 1$ , then  $n_3(v) = 0$ by Claim 3.3, Claim 3.6 and Claim 3.7. So  $w'(v) \ge 5 - \frac{7}{6} \times 2 - \frac{2}{3} - 2 = 0$ by R6', R5' and R7'. If  $m'_4(v) = 0$ , then  $n_3(v) \le 1$  by Claim 3.3. So  $w'(v) \ge 5 - \frac{7}{6} \times 2 - \frac{1}{3} - 2 = \frac{1}{3} > 0$  by Claim 3.5 and by R6', R3', R7'.

**Case** 3.1.3  $m_3(v) = 1$ , then  $m'_4(v) \le 2$ . If  $m'_4(v) = 2$ , then  $n_3(v) = 0$  by Claim 3.3 and Claim 3.7. So  $w'(v) \ge 5 - \frac{7}{6} - \frac{2}{3} \times 2 - 2 = \frac{1}{2} > 0$  by R6', R5' and R7'. If  $m'_4(v) \le 1$ , then  $n_3(v) \le 2$  by Claim 3.3. So  $w'(v) > 5 - \frac{7}{6} - \frac{2}{3} - \frac{1}{3} \times 2 - 2 = \frac{1}{2} > 0$  by R6', R5' and R7'. If  $m'_4(v) \le 1$ , then  $n_3(v) \le 2$  by Claim 3.3. So  $w'(v) > 5 - \frac{7}{6} - \frac{2}{3} - \frac{1}{3} \times 2 - 2 = \frac{1}{2} > 0$  by Claim 3.5 and by R6', R5', R3' and R7'.

**Case** 3.1.4  $m_3(v) = 0$ , then  $m'_4(v) \le 2$ ,  $n_3(v) \le 4$ . We have  $w'(v) > 5 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - 2 = \frac{1}{3} > 0$  by Claim 3.5 and by R5', R3', R7'.

Suppose d(v) = 6. Then w(v) = 8,  $m_3(v) \le 4$ . If  $n_2(v) = 0$ , then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation  $n_2(v) = 1$ .

If  $m_3(v) = 4$ , then  $m_4(v) = 0$ ,  $n_3(v) = 0$  by Claim 3.4. So  $w'(v) \ge 8 - 2 - \frac{4}{3} \times 3 - 2 = 0$  for G contains no structures  $H_{59}$ ,  $H_{60}$  and by R6' and R7'. If  $m_3(v) = 3$ , then  $m_4(v) \le 1$ ,  $n_3(v) \le 1$ . We have  $w'(v) > 8 - 2 - \frac{3}{2} \times 2 - \frac{2}{3} - 2 - \frac{1}{3} = 0$  by R6', R5', R7' and R3'. If  $m_3(v) = 2$ , then  $m_4(v) \le 2$ ,  $n_3(v) \le 2$ . We have  $w'(v) > 8 - 2 - \frac{3}{2} - \frac{2}{3} \times 2 - \frac{2}{3} - 2 - \frac{1}{3} = 0$  by R6', R5', R7' and R3'. If  $m_3(v) = 2$ , then  $m_4(v) \le 2$ ,  $n_3(v) \le 2$ . We have  $w'(v) > 8 - 2 - \frac{3}{2} - \frac{2}{3} \times 2 - 2 - \frac{1}{3} \times 2 = \frac{1}{2} > 0$  by R6', R5', R7' and R3'. If  $m_3(v) = 1$ , then  $m_4(v) \le 3$ ,  $n_3(v) \le 3$  by Claim 3.3. So  $w'(v) > 8 - 2 - \frac{2}{3} \times 3 - 2 - \frac{1}{3} \times 3 = 1 > 0$  by R6', R5', R7' and R3'. If  $m_3(v) = 0$ , then  $m_4(v) < 6$ ,  $n_3(v) \le 5$ . So  $w'(v) > 8 - \frac{2}{3} \times 6 - \frac{1}{3} \times 6 - 2 = 0$  by R5', R3' and R7'.

Suppose d(v) = 7. Then w(v) = 11,  $m_3(v) \le 5$ . If  $n_2(v) = 0$ , then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation  $n_2(v) = 1$ .

If  $m_3(v) = 5$ , then  $m_4(v) = 0$ ,  $n_3(v) = 0$ . So  $w'(v) = 11 - 2 - \frac{7}{4} \times 4 - 2 = 0$ for *G* contains no structure  $H_{59}$ ,  $H_{61}$  and by R6', R7'. If  $m_3(v) = 4$ , then  $m_4(v) \le 1$ ,  $n_3(v) = 0$ . So  $w'(v) > 11 - 2 \times 4 - \frac{2}{3} - 2 = \frac{1}{3} > 0$  by R6', R5' and R7'. If  $m_3(v) = 3$ , then  $m_4(v) \le 2$ ,  $n_3(v) \le 2$ . So w'(v) > 3 $11-2\times 3-\frac{2}{3}\times 2-2-\frac{1}{3}\times 2=1>0$  by R6', R5', R7', R3'. If  $m_3(v)=2$ , then  $m_4(v) \le 3, n_3(v) \le 6$ , we have  $w'(v) > 11 - 2 \times 2 - \frac{2}{3} \times 3 - 2 - \frac{1}{3} \times 6 = 1 > 0$ by R6', R5', R7' and R3'. If  $m_3(v) \le 1$ , then  $m_4(v) < 7$ ,  $n_3(v) \le 6$ , we have  $w'(v) > 11 - 2 - \frac{2}{3} \times 7 - 2 - \frac{1}{3} \times 6 = \frac{1}{3} > 0$  by R6', R5', R7' and R3'. Suppose  $d(v) \ge 8$ . Then w(v) = 3d(v) - 10. If  $n_2(v) = 0$ , then the

discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation  $n_2(v) = 1$ . By Claim 3.3 and Claim 3.4, we have  $n_3(v) \le d(v) - \frac{4}{3}m_3(v) - 1$ ,  $m_4(v) \le d(v) - \frac{4}{3}m_3(v)$ . So  $w'(v) \ge 3d(v) - 10 - 2m_3(v) - \frac{2}{3}m_4(v) - 2 - \frac{1}{3}n_3(v) \ge 3d(v) - 10 - 2m_3(v) - \frac{2}{3} \times (d(v) - \frac{4}{3}m_3(v)) - 2 - \frac{1}{3}(d(v) - \frac{4}{3}m_3(v) - 1) = 2d(v) - \frac{2}{3}m_3(v) - \frac{35}{3}$  by R6', R5', R7' and R3'. Since

$$m_3(v) \le \frac{3}{4}d(v).$$

We obtain  $w'(v) \ge \frac{3}{2}d(v) - \frac{35}{3} \ge \frac{1}{3} > 0$ . Suppose d(f) = 3. Then w(f) = -4 and  $n_2(f) \le 1$ .

**Case** 3.2.1  $n_2(f) = 1$ , then f is a  $(2, 6^+, 6^+)$ -face by Claim 3.6. So  $w'(f) \ge -4 + 2 \times 2 = 0$  by R6'.

**Case** 3.2.2  $n_2(f) = 0$ ,  $n_3(f) = 2$ , then  $n_4(f) = 0$ , f is a special face  $(3,3,5^+)$ -face. The discussion is similar to the corresponding situation in Case 1.

**Case** 3.2.3  $n_3(f) = 1$  and f is a (3, 4, 4)-, (3, 4, 5)-face. The discussion is similar to the corresponding situation in Case 1.

**Case** 3.2.4  $n_3(f) = 1$  and f is a (3, 4, 6)-face. For G contains no structures  $H_{62} \sim H_{64}$ , f is not adjacent to any 3-face, then  $w'(f) \geq -4 +$  $\frac{5}{6} + \frac{5}{3} + \frac{3}{4} \times 2 = 0$  by R6' and R4'.

**Case** 3.2.5  $n_3(f) = 1$ , f is a  $(3, 4, 7^+)$ -face. For G contains no structure  $H_{42}$ , f is adjacent to at most two 3-faces. If f is adjacent to at most one 3-faces, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to two 3-faces, then we need to consider the following two structures  $G_9$ ,  $G_{10}$  in Figure 2 for G contains no structures  $H_{29}, H_{31} \text{ and } H_{32}. \text{ We have } w'(G_9) \ge -4 \times 3 + \frac{7}{4} + \frac{4}{3} + \frac{7}{4} + 2 + \frac{5}{6} \times 2 + \frac{3}{4} \times 5 = \frac{1}{4} > 0, w'(G_{10}) \ge -4 \times 3 + \frac{7}{4} + \frac{4}{3} + 2 \times 2 + \frac{7}{4} + \frac{5}{6} + \frac{3}{4} \times 5 = \frac{17}{12} > 0 \text{ by } R6'$ and R4'.

**Case** 3.2.6  $n_3(f) = 1$ , f is a  $(3, 5, 5^+)$ - or  $(3, 6, 6^+)$ -face. If f is adjacent to at most two 3-faces, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to three 3-faces i.e.  $G_{11}$  and  $G_{12}$  in Figure 2 for G contains no structures  $H_{35}$  and  $H_{36}$ , then 
$$\begin{split} w'(G_{11}) &\geq -4 \times 4 + \frac{7}{4} \times 4 + 2 \times 2 + \frac{7}{4} + \frac{2}{3} \times 2 + \frac{3}{4} \times 4 = \frac{13}{12} > 0, \\ w'(G_{12}) &\geq -4 \times 4 + \frac{7}{4} \times 4 + 2 \times 2 + \frac{7}{4} + \frac{4}{3} + \frac{2}{3} + \frac{3}{4} \times 4 = \frac{7}{4} > 0, \end{split}$$
 by R6 and R4.

**Case** 3.2.7  $n_3(f) = 1$ , f is a  $(3, 7^+, 7^+)$ -face, the following discussion is similar to the corresponding situation in Case 1.

**Case** 3.2.8  $n_3(f) = 0$ , f is a (4, 4, 4)-face, the following discussion is similar to the corresponding situation in Case 1.

**Case** 3.2.9  $n_3(f) = 0$ , f is a (4, 4, 5)- or (4, 4, 6)-face. If f is adjacent to at most one 3-face, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to two 3-faces i.e.  $G_{14}$ ,  $G_{15}$  and  $\begin{array}{l} G_{16} \text{ in Figure 2 for } G \text{ contains no structures } H_{39} \sim \text{ and } H_{41}, \text{ we have} \\ w'(G_{14}) \geq -4 \times 3 + \frac{3}{2} \times 2 + \frac{7}{6} \times 3 + \frac{2}{3} \times 4 + \frac{3}{4} \times 5 = \frac{11}{12} > 0, w'(G_{15}) \geq \\ -4 \times 3 + \frac{4}{3} \times 2 + \frac{4}{3} \times 2 + \frac{3}{2} + \frac{2}{3} \times 4 + \frac{3}{4} \times 5 = \frac{5}{4} > 0, w'(G_{16}) \geq -4 \times 3 + \\ \frac{3}{2} + \frac{4}{3} \times 2 + \frac{3}{2} + \frac{2}{3} \times 5 + \frac{3}{4} \times 5 = \frac{3}{4} > 0 \text{ by } R6' \text{ and } R4'. \\ \mathbf{Case } 3.2.10 \ n_3(f) = 0, \ f \text{ is a } (4, 4, 7^+) \text{-face, the following discussion is} \end{array}$ 

similar to the corresponding situation in Case 1.

**Case** 3.2.11  $n_3(f) = 0$ , f is a (4,5,5)-face. If f is adjacent to at most one 3-face, the following discussion is similar to the corresponding situation in Case 1. If f is adjacent to two 3-faces i.e.  $G_{17}$  or  $G_{18}$  in Figure 2 for Gcontains no structures  $H_{42}$  and  $H_{43}$ . We have  $w'(G_{17}) \ge -4 \times 3 + \frac{3}{2} \times 2 + \frac{2}{3} \times 3 + \frac{7}{6} \times 4 + \frac{3}{4} \times 5 = \frac{17}{12} > 0$ , or  $w'(G_{18}) \ge -4 \times 3 + \frac{3}{2} + \frac{7}{6} \times 5 + \frac{4}{3} + \frac{2}{3} \times 2 + \frac{3}{4} \times 5 = \frac{7}{4} > 0$  by R6' and R4'.

**Case** 3.2.12  $n_3(f) = 0$ , f is a  $(4, 5, 6^+)$ - or  $(4, 6^+, 6^+)$ -face, then we have  $w'(f) \ge -4 + \frac{7}{6} + \frac{3}{2} + \frac{2}{3} + \frac{3}{4} = \frac{1}{12} > 0$  or  $w'(f) \ge -4 + \frac{2}{3} + \frac{4}{3} \times 2 + \frac{3}{4} = \frac{1}{12} > 0$ by R6' and R4'.

**Case** 3.2.13  $n_3(f) = 0$ , f is a  $(5, 5, 5^+)$ -face, then we have  $w'(f) \ge 0$ 

 $-4 + \frac{7}{6} \times 3 + \frac{3}{4} = \frac{1}{4} > 0 \text{ by } R6' \text{ and } R4'.$   $Case 3.2.14 \ n_3(f) = 0, \ f \text{ is a } (5^+, 6^+, 6^+) \text{-face, then we have } w'(v) \ge -4 + \frac{2}{3} + \frac{4}{3} \times 2 + \frac{3}{4} = \frac{1}{12} > 0 \text{ by } R6' \text{ and } R4'.$ Suppose d(f) = 4. Then  $w(f) = -2, \ n_2(f) \ge 1$  by Claim 3.7. If  $n_2(f) = 1$ 

1, then f is a  $(2,3,6^+,6^+)$ -,  $(2,4,6^+,6^+)$ - or  $(2,5^+,5^+,5^+)$ -face by Claim 3.7. Any 4-face which is incident to a 2-vertex is adjacent to at least one  $8^+$ face for G contains no 6- and 7-cycles. So  $w'(f) \ge -2 + \frac{3}{4} + \frac{2}{3} \times 2 = \frac{1}{12} > 0$  by R4' and R5'. If  $n_2(f) = 0$ , then the discussion is similar to the situation when d(f) = 4 in Case 1.

From the above discussion, we can obtain that  $w'(x) \geq 0$  for each  $x \in V(G) \cup F(G)$  and x is none of a special 3-vertex, a special 2-vertex and a special face. Furthermore, we have  $w'_s \ge -\frac{9}{2} - 4 = -\frac{17}{2}$  by Claim 1 and Claim 3.8. From the above discussion, we have  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -\frac{17}{2} e^{-12}$  $-\frac{17}{2} > -20$ , a contradiction.

Case 4  $\delta(G) = 1$ .

Case 4.1 There is one 1-vertex and at most two 2-vertices in G.

When d(v) = 1, then w(v) = -7. If there is one 1-vertex in G, then 3-faces in G are  $(3^-, 5^+, 5^+)$ -faces or  $(4^+, 4^+, 4^+)$ -faces for G contains no structure  $H_{65}$  and the 4-faces which are incident to a 2-vertex in G are  $(2, 5^+, 5^+, 5^+)$ -faces for G contains no structure  $H_{66}$ . Now there are not special 3-vertices and special faces in G. The discharging rules are the same as Case 1 except for the two cases. (1). A 4-face have a common 2-vertex with a 3-face. (2). Two 4-faces have a common 2-vertex. Under these situation, transfer charge from its common incident vertices to only the 3-face in the first situation and transfer charge from its common incident vertices to only one of them in the second situation. Clearly, we can guarantee the new charge of each vertex of G is larger than or equal to zero. For convenience, let  $w'_{t1}$  ( $w'_{t2}$ ) denote the total new charge of one 2-vertex (two 2-vertices) and the faces which are incident to the 2-vertex (the two 2-vertices).

**Case** 4.1.1 There is one 2-vertex in *G*. If the 2-vertex is incident to at most one 3-face and one 4-face simultaneously. Furthermore, the 3-face and 4-face is  $(2, 5^+, 5^+)$ - and  $(2, 5^+, 5^+, 5^+)$ -face respectively. So  $w'_{t1} \ge -4-2-4+\frac{7}{4}\times 2+\frac{3}{4}=-\frac{23}{4}>-6$  by *R*6 and *R*4. If the 2-vertex is incident a 3-face and not incident to a 4-face, then the 3-face is  $(2, 5^+, 5^+)$ -face. We have  $w'_{t1} \ge -4-4+\frac{7}{4}\times 2=-\frac{9}{2}>-6$  by *R*6. If the 2-vertex is incident a 4-face and not incident to a 3-face, we consider the situation such that the 2-vertex is a common vertex of two 4-faces, then the 4-face is a  $(2, 5^+, 5^+, 5^+)$ -face. We have  $w'_{t1} \ge -4-2-2+\frac{3}{4}\times 3=-\frac{23}{4}>-6$  by *R*4. We can obtain that  $w'_{t1} \ge -6$ , so  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -7+w'_{t1} \ge -7-6 = -13 > -20$ , a contradiction.

**Case** 4.1.2 There are two 2-vertices in G.

For two 2-vertices are not incident to a same 3- or 4-face, from the discussion in Case 4.1.1, we have  $w'_{t2} \ge -6 \times 2 = -12$ . So  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -7 - 12 = -19 > -20$ , a contradiction.

Case 4.2 There is one 1-vertex and at least three 2-vertices in G.

For G contains no structure  $H_{65}$ , the 3-faces in G are  $(3^-, 5^+, 5^+)$ -faces or  $(4^+, 4^+, 4^+)$ -faces. Now there are not special 3-vertices and special faces in G. The discussion is same as the situation in Case 3, we have  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -7 - 4 = -11 > -20$ , a contradiction.

 $\mathbf{Case}$  **4.3** There are at least two 1-vertices in G.

If there are two 1-vertices in G, then there is neither 2-vertex nor other 1-vertex in G for G contains no structure  $H_{67}$ . Furthermore, any 3-face in G is  $(3, 5^+, 5^+)$ - or  $(4^+, 4^+, 4^+)$ -face for G contains no structure  $H_{65}$ . Note that there are neither special 3-vertices nor special faces in G now. And the following discussion is the same as the situation in Case 1. These imply that  $\sum_{x \in V(G) \cup F(G)} w'(x) \ge -7 \times 2 = -14 > -20$ , a contradiction.

In the following, let us give the proof of the main theorems.

**Theorem 2.8** If G is a planar graph without 6- and 7-cycles and  $k \ge \max\{6, \Delta(G)\}$ , then G is equitably k-colorable.

Proof. Let G be a counterexample with fewest vertices. If each component of G has at most 4 vertices, then  $\Delta(G) \leq 3$ . So G is equitably k-colorable by Lemma 2.3. Otherwise, there is at least one component with at least five vertices. By Lemma 2.7, G has one of the structures  $H_1 \sim H_{67}$ , taking one and the vertices are labeled as they are in Figure 1. If there are vertices labeled repeatedly, then we take the larger  $(x_i \text{ is larger than } x_{i-1})$ . In the following, we show how to find S in Lemma 2.4. If G has one of  $H_2$ ,  $H_6$ ,  $H_{48}$  and  $H_{67}$ , then let  $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$ . If G has one of  $H_3, H_8, H_{10}, H_{12} \sim H_{15}, H_{17}, H_{1$  $H_{18}, H_{21}, H_{25}, H_{28}, H_{37}, H_{39}, H_{45} \sim H_{47}, H_{51}, H_{65}$  and  $H_{66}$ , then let  $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$ . If G has one of  $H_9$ ,  $H_{11}$ ,  $H_{16}$ ,  $H_{19}$ ,  $H_{20}$ ,  $H_{22} \sim H_{24}, H_{26}, H_{27}, H_{29} \sim H_{36}, H_{44}, H_{49}, H_{50}, H_{55} \sim H_{59}, H_{61},$ then let  $S' = \{x_k, x_{k-1}, x_{k-2}, x_3, x_2, x_1\}$ . If G has  $H_7$ , then let  $S' = \{x_k, x_{k-1}, x_{k-2}, x_3, x_2, x_1\}$ .  $\{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$ . If G has one of  $H_1, H_4, H_5, H_{38}, H_{40} \sim H_{43},$  $H_{52}, H_{53}, H_{60}, H_{62} \sim H_{64}$ , then let  $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_2, x_1\}$ . If G has  $H_{54}$ , then let  $S' = \{x_k, x_{k-1}, \dots, x_{k-4}, x_1\}$ . By Lemma 2.1, G is 3-degenerate, then we can find the remaining unspecified positions in Sfrom highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of |V(G)| and  $k \ge \Delta(G) \ge \Delta(G-S)$ . G-S is equitably k-colorable. So G is also equitably k-colorable by Lemma 2.4.

**Corollary 2.9** Let G be a planar graph without 6- and 7-cycles. If  $\Delta(G) \geq 6$ , then  $\chi_e(G) \leq \Delta(G)$ .

**Corollary 2.10** Let G be a planar graph without 6- and 7-cycles. If  $\Delta(G) \geq 6$ , then  $\chi_e^*(G) \leq \Delta(G)$ .

**Theorem 2.11** If G is a planar graph without 6- and 7-cycles and  $k \ge \max\{6, \Delta(G)\}$ , then G is equitably k-choosable.

**Proof.** Let G be a counterexample with fewest vertices. If each component of G has at most 4 vertices, then  $\Delta(G) \leq 3$ . So G is equitably k-choosable by Lemma 2.5. Otherwise, the proof is similar to the proof of Theorem 2.8 by Lemma 2.6 and Lemma 2.7.

**Corollary 2.12** Let G be a planar graph without 6- and 7-cycles. If  $\Delta(G) \geq 6$ , then G is equitable  $\Delta(G)$ -choosable.

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