BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **35**(4) (2012), 897–910

Equitable Coloring and Equitable Choosability of Planar Graphs Without 5- and 7-Cycles

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Abstract. A graph G is equitably k-choosable if for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil |V(G)|/k \rceil$ vertices. A graph G is equitable k-colorable if G has a proper vertex coloring with k colors such that the size of the color classes differ by at most 1. In this paper, we prove that if G is a planar graph without 5- and 7-cycles, then G is equitably k-choosable and equitably k-colorable where $k \geq \max\{\Delta(G), 7\}$.

2010 Mathematics Subject Classification: 05C15

Keywords and phrases: Equitable list coloring, planar graph, degenerate.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V,E)$ be a graph. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of G, respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (face) x in G. A vertex (face) x is called a k-*vertex* (k-*face*), k^+-*vertex* (k^+-*face*), k^--vertex or k^{--}-vertex, if $d(x) = k$, $d(x) \geq k$, $2 \leq d(x) \leq k$, or $1 \leq d(x) \leq k$. We use $(d_1, d_2, \cdots, d_n)$ to denote a face f if $(d_1, d_2, \cdots, d_n)$ are the degree of vertices incident to the face f. Let $\delta(f)$ denote the minimal degree of vertices incident to f. A graph G is 3-degenerate if every subgraph has a vertex of degree at most 3.

A proper k-coloring of a graph G is a mapping π from the vertex set $V(G)$ to the set of colors $\{1, 2, \cdots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. A graph G is equitable k-colorable if G has a proper k-coloring such that the size of the color classes differ by at most 1. The equitable chromatic number of G, denoted by $\chi_e(G)$, is the smallest integer k such that G is equitably k-colorable. The equitable chromatic threshold of G, denoted by $\chi_e^*(G)$, is the smallest integer k such that G is equitably l-colorable for all integers $l \geq k$. It is obvious that $\chi_e(G) \leq \chi_e^*(G)$ for any graph G. They might not be equal. For example, if $K_{2n+1,2n+1}$ (n is a positive integer) is a complete bipartite graph, then $\chi_e(K_{2n+1,2n+1}) = 2$, $\chi_e^*(K_{2n+1,2n+1}) = 2n+2$.

Communicated by Xueliang Li.
Received: October 14, 2010; *Revised:* January 11, 2011.

In 1970, Hajnál and Szemerédi proved that every graph has an equitable k-coloring whenever $k \geq \Delta(G)+1$ [4]. This bound is sharp as shows the example of $K_{2n+1,2n+1}$. In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture [8].

Conjecture 1.1. *If G is a connected graph which is neither a complete graph nor odd cycle, then $\chi_e(G) \leq \Delta(G)$.*

In 1994, Chen *et al.* put forth the following conjecture [2].

Conjecture 1.2. *For any connected graph G, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then $\chi_e^*(G) \leq \Delta(G)$.*

Chen *et al.* proved the conjecture for graphs with $\Delta(G) \leq 3$ or $\Delta(G) \geq (|V|/2)$ [2]. Yap and Zhang proved that the conjecture holds for outer planar graphs and planar graphs with $\Delta(G) \geq 13$ [14, 15]. Lih and Wu verified the conjecture for bipartite graphs [6], and Chen *et al.* verified it for tree [3]. Wang *et al.* proved the conjecture for line graphs [13], and Kostochka *et al.* proved it for d-degenerate graphs with $\Delta(G) \geq 14d+1$ [7].

For a graph G and a list assignment L assigned to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors, a L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. A list assignment L for G is k-$uniform$ if $|L(v)| = k$ for all $v \in V(G)$. A graph G is equitably k-choosable if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil |V(G)|/k \rceil$ vertices.

In 2003, Kostochka, Pelsmajer and West investigated the equitable list coloring of graphs. They proposed the following conjecture [5].

Conjecture 1.3. *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 1.4. *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{k,k}$ for some odd k.*

It has been proved that Conjecture 1.3 holds for graphs with $\Delta(G) \leq 3$ independently in [10, 11]. Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in the later case) and $k \geq \max\{\Delta, |V(G)|/2\}$, or G is a connected inteval graph and $k \geq \Delta(G)$ or G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$ [5]. Pelsmajer proved that every graph is equitably k-choosable for any $k \geq (\Delta(G)(\Delta(G)-1)/2)+2$ [10]. There are several results for planar graphs without short cycles [9, 16].

In this paper, we prove that if G is a planar graph without 5- and 7-cycles, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 7\}$.

2. Planar graphs without 5- and 7-cycles

First let us introduce some important lemma.

Lemma 2.1. [12] *Every planar graph without 5-cycles is 3-degenerate.*

Corollary 2.1. *If G is a planar graph without 5- and 7-cycles, then $\delta(G) \leq 3$.*

Lemma 2.2. [4] *Every graph has an equitable k-coloring whenever $k \geq \Delta(G)+1$.*

Lemma 2.3. [16] *Let $S = \{v_1, v_2, \cdots, v_k\}$ be a set of k different vertices in G such that $G - S$ has an equitable k-coloring. If $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable k-coloring.*

Lemma 2.4. [10,11] *Every graph G with maximum degree $\Delta(G) \leq 3$ is equitably k-choosable whenever $k \geq \Delta(G)+1$.*

Lemma 2.5. [5] *Let G be a graph with a k-uniform list assignment L. Let $S = \{v_1, v_2, \cdots, v_k\}$, where $\{v_1, v_2, \cdots, v_k\}$ are distinct vertices in G. If $G-S$ has an equitable L-coloring and $|N_G(v_i) - S| \leq k-i$ for $1 \leq i \leq k$, then G has an equitable L-coloring.*

Lemma 2.6. *Let G be a connected planar graph with order at least 5 and without 5- and 7-cycles. Then G has at least one of the structures in Figure 1.*

Proof. Let G be a minimal counterexample on the number of vertices. Then G without 5- and 7-cycles does not contain $H_1 \sim H_{43}$ in Figure 1.

For G contains no structures $H_1 \sim H_7$, then G has the following properties.

Case 1. $(3,3,5^+)$-face, $(3,4,4)$-face, $(3,4,5)$-face and $(3,4,6)$-face any two of which can not simultaneously appear in G. Furthermore, if one of them appears, then it must occur just once except that the structures G_1 (in Figure 2) or G_2 (in Figure 2).

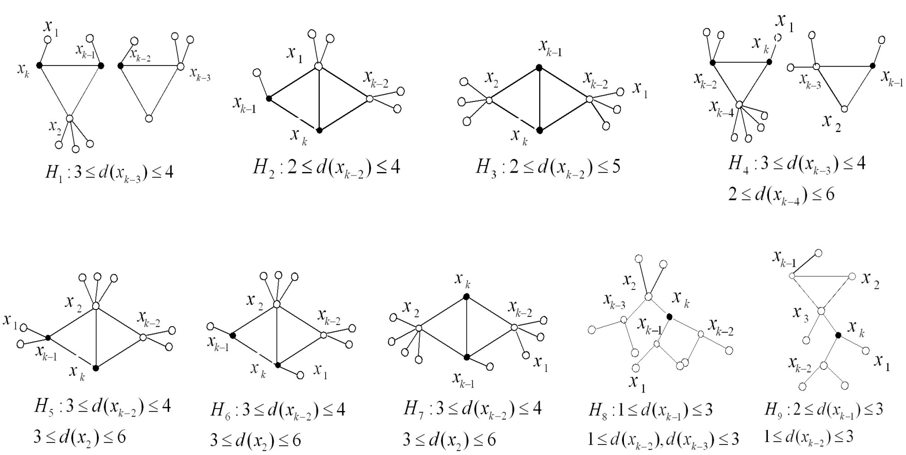
We call a face a *special face* if it belongs to one of the following structures: $(3,3,5^+)$-faces, $(3,4,4)$-faces, $(3,4,5)$-faces, $(3,4,6)$-faces, G_1 or G_2. In the following, we call a 3-vertex a *special 3-vertex* if it is incident to a special face, otherwise, it is called a *simple 3-vertex*. For convenience, let $n_3(v)$ denote the number of simple 3-vertices adjacent to v and $f_i(v)$ denote the number of i-faces incident to v for each $v \in V(G)$. For $f \in F(G)$, let $n_i(f)$ denote the number of i-vertices incident to f.

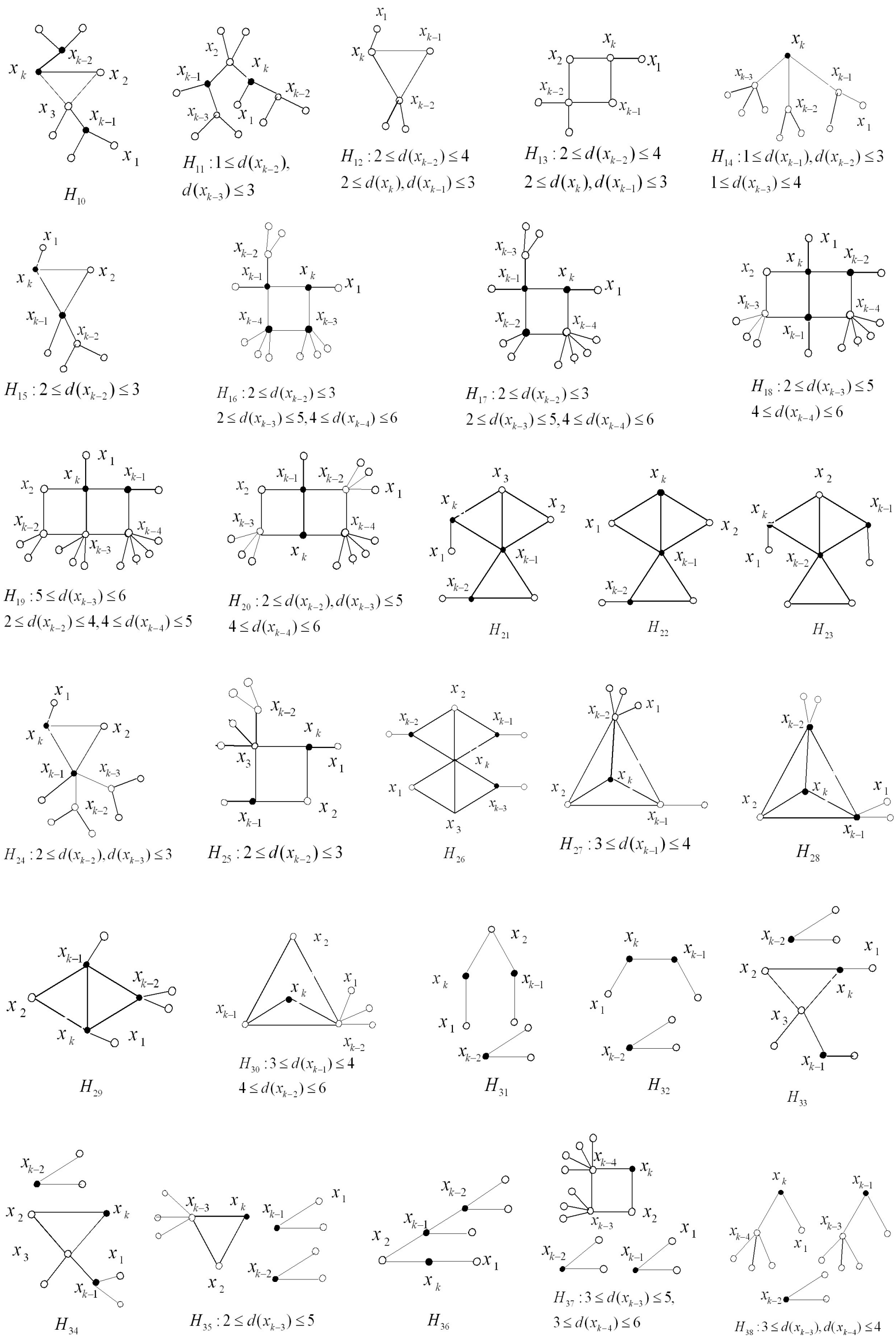
Since G contains no structure H_8 and H_{11}, we can conclude the following properties.

Case 2. For each $v \in V(G)$ with $d(v) \geq 4$, if v is adjacent to a 3-vertex which is adjacent to two 3^{--}-vertices, then it is not adjacent to other 3-vertex.

Case 3. For each $v \in V(G)$ with $d(v) \geq 4$, if v is incident to a 3-face which is incident to a 3-vertex, then it is not adjacent to other simple 3-vertex which is adjacent to a 3^{--}-vertex.

Case 4. For each $v \in V(G)$ with $d(v) \geq 4$, if v is adjacent to a 3-vertex, then it is not incident to a 3-face which is incident to other simple 3-vertex with a 3^{--}-vertex as its adjacent vertex.




H_{10}
$H_{11}: 1 \le d(x_{k-2}),\ d(x_{k-3}) \le 3$
$H_{12}: 2 \le d(x_{k-2}) \le 4$, $2 \le d(x_k), d(x_{k-1}) \le 3$
$H_{13}: 2 \le d(x_{k-2}) \le 4$, $2 \le d(x_k), d(x_{k-1}) \le 3$
$H_{14}: 1 \le d(x_{k-1}), d(x_{k-2}) \le 3$, $1 \le d(x_{k-3}) \le 4$
$H_{15}: 2 \le d(x_{k-2}) \le 3$
$H_{16}: 2 \le d(x_{k-2}) \le 3$, $2 \le d(x_{k-3}) \le 5, 4 \le d(x_{k-4}) \le 6$
$H_{17}: 2 \le d(x_{k-2}) \le 3$, $2 \le d(x_{k-3}) \le 5, 4 \le d(x_{k-4}) \le 6$
$H_{18}: 2 \le d(x_{k-3}) \le 5$, $4 \le d(x_{k-4}) \le 6$
$H_{19}: 5 \le d(x_{k-3}) \le 6$, $2 \le d(x_{k-2}) \le 4, 4 \le d(x_{k-4}) \le 5$
$H_{20}: 2 \le d(x_{k-2}), d(x_{k-3}) \le 5$, $4 \le d(x_{k-4}) \le 6$
H_{21}
H_{22}
H_{23}
$H_{24}: 2 \le d(x_{k-2}), d(x_{k-3}) \le 3$
$H_{25}: 2 \le d(x_{k-2}) \le 3$
H_{26}
$H_{27}: 3 \le d(x_{k-1}) \le 4$
H_{28}
H_{29}
$H_{30}: 3 \le d(x_{k-1}) \le 4$, $4 \le d(x_{k-2}) \le 6$
H_{31}
H_{32}
H_{33}
H_{34}
$H_{35}: 2 \le d(x_{k-3}) \le 5$
H_{36}
$H_{37}: 3 \le d(x_{k-3}) \le 5$, $3 \le d(x_{k-4}) \le 6$
$H_{38}: 3 \le d(x_{k-3}), d(x_{k-4}) \le 4$

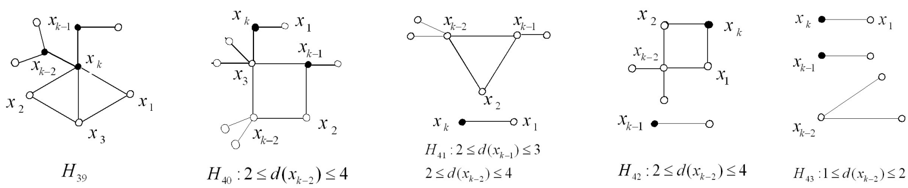


Figure 1

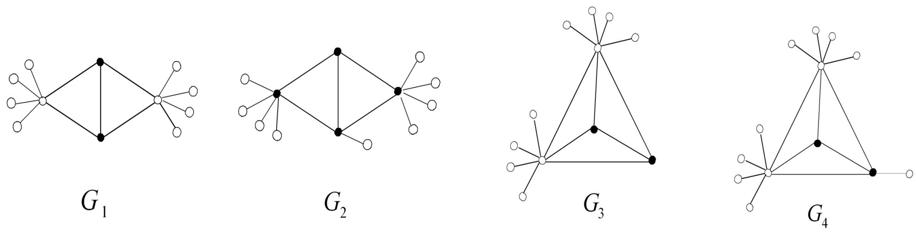


Figure 2

Remark: Each configuration in Figure 1 and Figure 2 represents subgraphs for which (1) the vertices labeled $x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}$ are distinct, (2) solid vertices have no incident edges other than the ones shown, and (3) except for being specially pointed, the degree of a hollow vertex might be an integer and in $[d, \Delta(G)]$, where d is the number of edges incident to the hollow vertex in the configuration, (4) the order of the vertices on the boundary of each 4-face can be exchanged.

Case 5. For any $v \in V(G)$, v is adjacent to at most one simple 3-vertex which is adjacent to other 3^{--}-vertex.

By Euler's formula $|V| - |E| + |F| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E|$, we have

$$\sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) = -10(|V| - |E| + |F|) = -20.$$

Define an initial charge function w on $V(G) \cup F(G)$ by setting $w(v) = 3d(v) - 10$ if $v \in V(G)$ and $w(f) = 2d(f) - 10$ if $f \in F(G)$, so that $\sum_{x \in V(G) \cup F(G)} W(x) = -20$.

Let the new charge of each element x be $w'(x)$ for each $x \in V(G) \cup F(G)$. Particularly, we use w'_s denote the total new charge of all the special vertices and all the special faces in G.

We divide the proof into the following four cases by Corollary 2.2.

Case 1. $\delta(G) = 3$. For G contains no structures H_{12}, H_{13}, G has the following properties.

Case 1.1. All 3-faces in G are $(3,3,5^+)$-, $(3,4^+,4^+)$- or $(4^+,4^+,4^+)$-faces. 4-faces are $(3,3,5^+,5^+)$-, $(3,4^+,4^+,4^+)$- or $(4^+,4^+,4^+,4^+)$-faces.

Now redistribute the charge according to the following discharging rules.

R1 Transfer charge 1 from each 5^+-vertex to every adjacent simple 3-vertex v which is adjacent to exactly two 3-vertices.

R2 Transfer charge $(1/2)$ from each 4^+-vertex to every adjacent simple 3-vertex v which is adjacent to exactly one 3-vertex.
R3 Transfer charge $(1/3)$ from each 4^+-vertex to every adjacent simple 3-vertex v which is not adjacent to any 3-vertex.
R4 Transfer charge $(3/4)$ from each 8^+-face f to every adjacent 3-face and 4-face via each common edge.
R5 If f is a 4-face incident with a vertex v, then v gives f charge $(2/3)$ if $d(v)=4$ and f is a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, $(1/2)$ if $d(v)=4$ and f is a $(4,4,4,4^+)$-face, $(1/3)$ if $d(v)=4$ and f is an other 4-face, 1 if $d(v)=5$ and f is a $(3,3,5,5)$- or $(3,3,5,6)$-face, $(2/3)$ if $d(v)=5$ and f is an other face, 1 if $d(v)=6$, $(4/3)$ if $d(v)=7^+$ and f is a $(3,3,7^+,5^+)$- or $(3,4,7^+,4^+)$-face, 1 if $d(v)=7^+$ and f is an other 4-face.
R6 If f is a 3-face incident with a vertex v, then v gives f charge $(3/4)$ if $d(v)=4$, $(3/2)$ if $d(v)=5$, $(11/6)$ if $d(v)=6$, $(9/4)$ if $d(v)=7^+$ and f is a $(3,7^+,3^+)$-face, 2 if $d(v)=7^+$ and f is an other 3-face.

In the following, let's check the charge of each element x for $x\in V(G)\cup F(G)$. Suppose $d(v)=3$. Then $w(v)=-1$. Since G contains no structure H_{14}, v is adjacent to at least one 5^+-vertex or is adjacent to at least two 4^+-vertex. If v is a simple 3-vertex, we have $w'(v)=-1+1=0$ by $R1$, $w'(v)=-1+(1/2)\times 2=0$ by $R2$ or $w'(v)=-1+(1/3)\times 3=0$ by $R3$. Otherwise, we have $w'(v)=w(v)=-1$.

Suppose $d(v)=4$. Then $w(v)=2$ and $f_3(v)\leq 2$.

Case 1.1.1 If $f_3(v)=2$, then $f_4(v)=0$, $n_3(v)\leq 1$ for G contains no 5-cycle and contains no structure H_{15}. We have $w'(v)\geq 2-(3/4)\times 2-(1/2)=0$ for G contains no configuration H_{14} and by $R6$ and $R2$.

Case 1.1.2 $f_3(v)=1$, then $f_4(v)\leq 1$. If $f_4(v)=1$, then $n_3(v)\leq 1$ for G contains no structure H_{15} and by Claim 1.1. We have $w'(v)\geq 2-(3/4)-(2/3)-(1/2)=(1/12)>0$ by $R6$, $R5$ and $R2$. Otherwise, $n_3(v)\leq 2$. So $w'(v)\geq 2-(3/4)-(1/3)-(1/2)=(5/12)>0$ by $R6$, $R3$ and $R2$.

Case 1.1.3 $f_3(v)=0$, then $f_4(v)\leq 4$.

Case 1.1.3.1 $f_4(v)=4$.

Case 1.1.3.1.1 One of the 4-faces is a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v)\leq 1$ for G contains no structures H_{16} and H_{17}. If $n_3(v)=1$, then the other 4-faces are not $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$-, $(3,4,5,5)$- or $(4,4,4,4^+)$-faces for G contains no structures $H_{18}\sim H_{20}$. We have $w'(v)\geq 2-(2/3)-(1/3)\times 3-(1/3)=0$ by $R5$ and $R3$. Otherwise, i.e. $n_3(v)=0$, then there is at most two $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-faces, and there are at least two 4-faces which are not $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$-, $(3,4,5,5)$- or $(4,4,4,4^+)$-faces for G contains no structures $H_{18}\sim H_{20}$. We have $w'(v)\geq 2-(2/3)\times 2-(1/3)\times 2=0$ by $R5$.

Case 1.1.3.1.2 Each of the 4-faces is not a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v)\leq 2$ by Claim 1.1. If $n_3(v)=2$, then we have $w'(v)\geq 2-(1/3)\times 4-(1/3)\times 2=0$ by $R5$ and $R3$. If $n_3(v)=1$, we have $w'(v)\geq 2-(1/2)\times 2-(1/3)\times 2-(1/3)=0$ by $R5$ and $R3$. If $n_3(v)=0$, we have $w'(v)=2-(1/2)\times 4=0$ by $R5$.

Case 1.1.3.2 $f_4(v)=3$.

Case 1.1.3.2.1 One of the 4-faces is a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v) \le 1$ for G contains no structures H_{16} and H_{17}. If $n_3(v) = 1$, then the other 4-faces are not $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-faces for G contains no structures H_{16} and H_{17}. There is at most one $(4,4,4,4^+)$-face in the other two 4-faces for G contains no structures $H_{18} \sim H_{20}$ and there is at least one 4-face which is not a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$-, $(3,4,5,5)$- or $(4,4,4,4^+)$-faces for G contains no structures $H_{18} \sim H_{20}$. We have $w'(v) \ge 2-(2/3)-(1/2)-(1/3)-(1/2) = 0$ by $R5$ and $R2$. Otherwise, i.e. $n_3(v) = 0$, then there is at most two $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-faces and there is at least one 4-face which is not $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$-, $(3,4,5,5)$- or $(4,4,4,4^+)$-face for G contains no structures $H_{18} \sim H_{20}$. We have $w'(v) \ge 2-(2/3)\times 2-(1/3) = (1/3) > 0$ by $R5$.

Case 1.1.3.2.2 Each of the 4-faces is not a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v) \le 2$ by Claim 1.1. If $n_3(v) = 2$, then we have $w'(v) \ge 2-(1/3)\times 2-(1/2)-(1/3)-(1/2) = 0$ by $R5$, $R3$ and $R2$. If $n_3(v) = 1$, we have $w'(v) \ge 2-(1/2)\times 2-(1/3)-(1/2) = (1/6) > 0$ by $R5$ and $R2$. If $n_3(v) = 0$, we have $w'(v) = 2-(1/2)\times 3 = (1/2) > 0$ by $R5$.

Case 1.1.3.3 $f_4(v) = 2$.

Case 1.1.3.3.1 If one of the 4-faces is a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v) \le 1$ for G contains no structures H_{16} and H_{17}. We have $w'(v) \ge 2-(2/3)-(2/3)-(1/2) = (1/6) > 0$ by $R5$ and $R2$.

Case 1.1.3.3.2 Each of the 4-faces is not a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v) \le 3$. If $n_3(v) = 3$, then we have $w'(v) \ge 2-(1/3)\times 2-(1/3)\times 2-(1/2) = (1/6) > 0$ by $R5$, $R3$ and $R2$. If $n_3(v) = 2$, then we have $w'(v) \ge 2-(1/3)-(1/2)-(1/3)-(1/2) = (1/3) > 0$ by $R5$, $R3$ and $R2$. If $n_3(v) \le 1$, then we have $w'(v) \ge 2-(1/2)\times 2-(1/2) = (1/2) > 0$ by $R5$ and $R2$.

Case 1.1.3.4 $f_4(v) = 1$. If one of the 4-faces is a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face, then $n_3(v) \le 1$ for G contains no structures H_{16} and H_{17}. We have $w'(v) \ge 2-(2/3)-(1/2) = (5/6) > 0$ by $R5$ and $R2$. Otherwise, $n_3(v) \le 3$. We have $w'(v) > 2-(1/2)-(1/2)-(1/3)\times 2 = (1/3) > 0$ by $R5$, $R3$ and $R2$.

Case 1.1.3.5 If $f_4(v) = 0$, then $n_3(v) \le 4$, we have $w'(v) \ge 2-(1/2)-(1/3)\times 3 = (1/2) > 0$ by $R2$ and $R3$.

Suppose $d(v) = 5$. Then $w'(v) = 5$, $m_3(v) \le 3$.

Case 1.2.1 If $f_3(v) = 3$, then $f_4(v) = 0$, $n_3(v) \le 1$ for G contains no structures $H_{21} \sim H_{23}$. So $w'(v) \ge 5-(3/2)\times 3-(1/2) = 0$ by $R6$ and $R2$.

Case 1.2.2 If $f_3(v) = 2$, then $f_4(v) \le 1$, $n_3(v) \le 2$ for G contains no structure H_{24}. So $w'(v) \ge 5-(3/2)\times 2-1-(1/3)-(1/2) = (1/6) > 0$ by $R6$, $R5$, $R3$ and $R2$.

Case 1.2.3 If $f_3(v) = 1$, then $f_4(v) \le 2$, $n_3(v) \le 3$ for G contains no structure H_{24}. So $w'(v) \ge 5-(3/2)-1\times 2-(1/2)-(1/3)\times 2 = (1/3) > 0$ by $R6$, $R5$, $R2$ and $R3$.

Case 1.2.4 $f_3(v) = 0$, then $f_4(v) \le 5$. If one of the 4-faces is a $(3,3,5,5)$- or $(3,3,5,6)$-face, then $n_3(v) \le 2$ for G contains no structure H_{25}. So $w'(v) \ge 5-1\times 3-(2/3)\times 2-(1/3)-(1/2) = (1/6) > 0$ by $R5$, $R3$ and $R2$. Otherwise, $n_3(v) \le 5$, we have $w'(v) \ge 5-(2/3)\times 5-(1/3)\times 5 = 0$ by $R5$ and $R3$.

Suppose $d(v) = 6$. Then $w(v) = 8$, $f_3(v) \le 4$.

Case 1.3.1 If $f_3(v) = 4$, then $f_4(v) = 0$, $n_3(v) \le 2$ for G contains no structure H_{21}. So $w'(v) \ge 8-(11/6)\times 4-(1/3)\times 2 = 0$ by $R6$, $R3$ and Claim 4.

Case 1.3.2 If $f_3(v) = 3$, then $f_4(v) = 0$, $n_3(v) \le 4$. So $w'(v) \ge 8-(11/6)\times 3-(1/3)\times 3-(1/2) = 1 > 0$ by $R6$, $R3$ and $R2$.

Case 1.3.3 If $f_3(v) = 2$, then $f_4(v) \le 2$, $n_3(v) \le 5$, So $w'(v) \ge 8-(11/6)\times 2-1\times 2-(1/3)\times 4-(1/2) = (1/2) > 0$ by $R6$, $R5$, $R3$ and $R2$.

Case 1.3.4 If $f_3(v) = 1$, then $f_4(v) \le 3$, $n_3(v) \le 5$. So $w'(v) \ge 8-(11/6)-1\times 3-(1/3)\times 4-(1/2) = (4/3) > 0$ by $R6$, $R5$, $R3$ and $R2$.

Case 1.3.5 $f_3(v) = 0$, then $f_4(v) \le 6$. If $f_4(v) = 6$, then $n_3(v) \le 3$ for G contains no structure H_{25}. So $w'(v) \ge 8-1\times 6-(1/3)\times 2-(1/2) = (5/6) > 0$ by $R5$, $R3$ and $R2$. Otherwise, $n_3(v) \le 6$, we have $w'(v) > 8-1\times 5-(1/3)\times 5-(1/2) = (5/6) > 0$ by $R5$, $R3$ and $R2$.

Suppose $d(v) = 7$. Then $w(v) = 11$, $f_3(v) \le 4$.

Case 1.4.1 If $f_3(v) = 4$, then $f_4(v) = 0$, $n_3(v) \le 5$. So $w'(v) \ge 11-(9/4)\times 4-(1/3)\times 5 = (1/3) > 0$ by $R6$, $R3$ and Claim 3.

Case 1.4.2 If $f_3(v) = 3$, then $f_4(v) \le 1$, $n_3(v) \le 5$. So $w'(v) \ge 11-(9/4)\times 3-(4/3)-(1/3)\times 5 = (5/4) > 0$ by $R6$, $R4$, $R3$ and Claim 3.

Case 1.4.3 If $f_3(v) = 2$, then $f_4(v) \le 3$, $n_3(v) \le 6$. So $w'(v) \ge 11-(9/4)\times 2-(4/3)\times 3-(1/3)\times 6 = (1/2) > 0$ by $R6$, $R4$, $R3$, Claim 3 and Claim 4.

Case 1.4.4 If $f_3(v) = 1$, then $f_4(v) \le 4$, $n_3(v) \le 6$. So $w'(v) \ge 11-(9/4)-(4/3)\times 4-(1/3)\times 6 = (17/12) > 0$ by $R6$, $R4$, $R3$ and Claim 4.

Case 1.4.5 $f_3(v) = 0$, then $f_4(v) \le 7$. If $f_4(v) = 7$, then $n_3(v) \le 3$ for G contains no structure H_{25}. So $w'(v) \ge 11-(4/3)\times 7-(1/3)\times 2-(1/2) = (1/2) > 0$ by $R5$, $R3$ and $R2$. Otherwise, $n_3(v) \le 7$. We have $w'(v) > 11-(4/3)\times 6-(1/3)\times 6-(1/2) = (1/2) > 0$ by $R5$, $R3$ and $R2$.

Suppose $d(v) \ge 8$. Then $w(v) = 3d(v)-10$. Since $n_3(v) \le f_3(v)+d(v)-(3/2)f_3(v)$, we have $n_3(v) \le d(v)-(1/2)f_3(v)$. From $f_3(v) \le (2/3)\times(d(v)-f_4(v))$, we have $(4/3)f_4(v) \le (4/3)d(v)-2f_3(v)$. So $w'(v) \ge 3d(v)-10-(9/4)f_3(v)-(4/3)f_4(v)-(1/2)-(1/3)(n_3(v)-1) \ge 3d(v)-10-(9/4)f_3(v)-((4/3)d(v)-2f_3(v))-(1/2)-(1/3)(d(v)-(1/2)f_3(v)-1) = (4/3)d(v)-(1/12)f_3(v)-(61/6)$ by $R6$, $R5$, $R2$ and $R3$. Since $f_3(v) \le (2/3)d(v)$, we obtain $w'(v) \ge (23/18)d(v)-(61/6) \ge (1/18) > 0$.

Suppose $d(f) = 3$. Then $w(f) = -4$, $n_3(f) \le 2$ by Claim 1.1.

Case 1.5.1 $n_3(f) = 2$, then $n_4(f) = 0$, f is a special face $(3,3,5^+)$-face. If f is not adjacent to any special 3-face, we have $w'(f) \ge -4+(3/2)+(3/4)\times 2 = -1$ by $R6$ and $R4$. If f is adjacent to a special 3-faces, we consider the cluster G_1 and G_3 (Figure 2) for G contains no structure H_{27} and by Claim 1, then $w'(G_1) \ge -4\times 2+(11/6)\times 2+(3/4)\times 4 = -(4/3)$ by $R6$ and $R4$, $w'(G_3) \ge -4\times 3+(9/4)\times 4+(3/4)\times 3 = -(3/4)$ by $R6$ and $R4$.

Case 1.5.2 $n_3(f) = 1$ and $n_4(f) = 1$, then f is a special face $(3,4,k)$-face $(4 \le k \le 6)$ or a $(3,4,7^+)$-face. If f is a special face $(3,4,k)$-face $(4 \le k \le 6)$ and f is not adjacent to any special 3-face, we have $w'(f) \ge -4+(3/4)\times 2+(3/4)\times 2 = -1$ by $R4$ and $R6$. If the special face f is adjacent to a special 3-face, we consider the cluster G_2 (Figure 2), then $w'(G_2) \ge -4\times 2+(11/6)\times 2+(3/4)\times 4+(3/4)\times 2 = (1/6) > 0$ by $R6$ and $R4$. If f is a $(3,4,7^+)$-face and adjacent to at most one 3-face, we have $w'(f) \ge -4+(3/4)+(9/4)+(3/4)\times 2 = (1/2) > 0$ by $R6$ and $R4$. If f is a $(3,4,7^+)$-face and adjacent to two 3-faces i.e.

G_4 (Figure 2), we have $w'(G_4) = -4 \times 3 + (3/4) \times 2 + (9/4) \times 4 + (3/4) \times 3 = (3/4) > 0$ by $R6$ and $R4$.

Case 1.5.3 $n_3(f) = 1$ and $n_4(f) = 0$. If f is a $(3,5,5)$-face, then f is adjacent to at most one 3-face for G contains no structure H_{28}. So $w'(f) \geq -4 + (3/2) \times 2 + (3/4) \times 2 = (1/2) > 0$ by $R6$ and $R4$. If f is a $(3,5^+,6^+)$-face, then f is adjacent to at most two 3-faces, we have $w'(f) \geq -4 + (11/6) + (3/2) + (3/4) = (1/12) > 0$ by $R6$ and $R4$.

Case 1.5.4 $n_3(f) = 0$. If f is a $(4,4,4)$-face, then f is not adjacent to any 3-face for G does not contain structure H_{29}. So $w'(f) = -4 + (3/4) \times 3 + (3/4) \times 3 = (1/2) > 0$ by $R6$ and $R4$. If f is a $(4^+,4^+,5^+)$-face, then f is adjacent to at most one 3-face, we have $w'(f) \geq -4 + (3/4) \times 2 + 1 + (3/4) \times 2 = 0$ by $R6$ and $R4$.

Suppose $d(f) = 4$. Then $w(f) = -2$ and $n_3(f) \leq 2$ by Claim 1.1.

Case 1.6.1 $n_3(f) = 2$, then f is a $(3,3,5^+,5^+)$-face. If f is a $(3,3,5,5)$- or $(3,3,5,6)$-face, then we have $w'(f) \geq -2 + 1 \times 2 = 0$ by $R5$. Otherwise, we have $w'(f) \geq -2 + (2/3) + (4/3) = 0$ by $R5$.

Case 1.6.2 $n_3(f) = 1$ and $n_4(f) = 2$, then f is a $(3,4,4,4^+)$-face. If f is a $(3,4,4,4)$-, $(3,4,4,5)$- or $(3,4,4,6)$-face, then we have $w'(f) \geq -2 + (2/3) \times 3 = 0$ by $R5$. Otherwise, we have $w'(f) \geq -2 + (1/3) \times 2 + (4/3) = 0$ by $R5$.

Case 1.6.3 $n_3(f) = 1$ and $n_4(f) = 1$, then f is a $(3,4,5^+,5^+)$-face. If f is a $(3,4,5,5)$-face, then we have $w'(f) \geq -2 + (2/3) \times 3 = 0$ by $R5$. Otherwise, we have $w'(f) \geq -2 + (1/3) + (2/3) + 1 = 0$ by $R5$.

Case 1.6.4: If $n_3(f) = 1$ and $n_4(f) = 0$, then f is a $(3,5^+,5^+,5^+)$-face, we have $w'(f) \geq -2 + (2/3) \times 3 = 0$ by $R5$.

Case 1.6.5 If $n_3(f) = 0$, then f is a $(4^+,4^+,4^+,4^+)$-face, we have $w'(f) \geq -2 + (1/2) \times 4 = 0$ by $R5$.

Suppose $d(f) = 6$. Then $w'(f) = w(f) = 2 > 0$.

Suppose $d(f) \geq 8$. Then $w'(f) \geq w(f) - (3/4)d(f) = 2d(f) - 10 - (3/4)d(f) = (5/4)d(f) - 10 \geq 0$ by $R4$.

From the above discussion, we can obtain that $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$ and x is neither a special 3-vertex nor a special face. Furthermore, we have $w'_s \geq \min\{-1 \times 2 - (4/3), -1 \times 2 - (3/4), -1 \times 2 - 1\} = -1 \times 2 - (4/3) = -(10/3) > -4$ by Claim 1. So we can obtain $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -4 > -20$, a contradiction.

Case 2. $\delta(G) = 2$ and there are at most two 2-vertices in G.

The discharging rules and the discussion are the same as Case 1 except for the two cases. (1) A 4-face have a common 2-vertex with a 3-face. (2) Two 4-faces have a common 2-vertex. Under these situation, transfer charge from its common incident vertices to only the 3-face in the first situation and transfer charge from its common incident vertices to only one of them in the second situation. Clearly, we can guarantee the new charge of each vertex of G is larger than or equal to zero. For convenience, let w'_{t1} (w'_{t2}) denote the total new charge of one 2-vertex (two 2-vertices) and the faces which are incident to the 2-vertex (the two 2-vertices).

Case 2.1. There exists one 2-vertex in G. For the 2-vertex is incident to at most one 3-face and one 4-face simultaneously. Furthermore, for G contains no structures H_{13} and H_{30}, we can obtain that the 3-face and 4-face is $(2,3,7^+)$- and $(2,3,5^+,7^+)$ respectively; the 3-face and 4-face is $(2,4,7^+)$- and $(2,4,4^+,7^+)$ respectively; the 3-face and 4-face is $(2,5^+,5^+)$-

and $(2,3^+,5^+,5^+)$-face respectively. So $w'_{t1} \geq -4-2-4+(9/4)+(2/3) = -(85/12)$, $w'_{t1} \geq -4-2-4+(9/4)+(3/4)+(1/3) = -(20/3)$, $w'_{t1} \geq -4-2-4+(3/2)\times 2 = -7$ by $R6$ and $R5$. If the 2-vertex is incident a 3-face and not incident to a 4-face, then the 3-face is $(2,3,5^+)$- or $(2,4^+,4^+)$-face for G contains no structure H_{12}. We have $w'_{t1} \geq -4-4+(3/2) = -(13/2)$ or $w'_{t1} \geq -4-4+(3/4)\times 2 = -(13/2)$ by $R6$. If the 2-vertex is incident a 4-face and not incident to a 3-face, we consider the situation such that the 2-vertex is a common vertex of two 4-faces, then the 4-face is a $(2,3,5^+,5^+)$-or $(2,4^+,4^+,4^+)$-face for G contains no structure H_{13}. We have $w'_{t1} \geq -4-2-2+(2/3)\times 2 = -(20/3)$ or $w'_{t1} \geq -4-2-2+(1/3)\times 4 = -(20/3)$. We can obtain that $w'_{t1} \geq \min\{-(85/12), -(20/3), -7\} = -(85/12)$, so $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -4+w'_{t1} \geq -4-(85/12) = -(133/12) > -20$, a contradiction.

Case 2.2. There exist two 2-vertices in G.

Case 2.2.1 The two 2-vertices are incident to a same 3-face, then f is a $(2,2,5^+)$-face for G contains no structure H_{12}. So $w'_{t2} \geq -4\times 3+1 = -11$ by $R6$.

Case 2.2.2 Two 2-vertices are incident to a same 4-face, then the 4-face is a $(2,2,5^+,5^+)$-face for G contains no structure H_{13}. If each 2-vertex is incident to another 4-face, we have $w'_{t2} \geq -2-2-2-4-4+(2/3)\times 2 = -(38/3)$ by $R5$. If one of the two 2-vertices is incident to another 3-face, the other 2-vertex is incident to another 4-face, we have $w'_{t2} \geq -2\times 2-4\times 3+(3/2)\times 2 = -13$ by $R6$.

Case 2.2.3 If the two 2-vertices are not incident to a same face, from the discussion in *Case 2.1*, we have $w'_{t2} \geq -(85/12)\times 2 = -(85/6) > -15$. Clearly, we have $w'_{t2} \geq -15$ in the rest cases. From the above discussion, we have $w'_{t2} \geq \min\{-11, -(38/3), -13, -15\} = -15$. So $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -4-15 = -19 > -20$, a contradiction.

Case 3. $\delta(G) = 2$ and there are at least three 2-vertices in G.

For G contains no structures $H_{31} \sim H_{38}$. G has the following properties:

Case 3.1. Any vertex v is adjacent to at most one 2-vertex.

Case 3.2. Two 2-vertices are not adjacent to each other.

Case 3.3. For each $v \in V(G)$ with $d(v) \geq 4$, if v is adjacent to a 2-vertex, then it is not incident to a 3-face which is incident to a 3-vertex.

Case 3.4. If v is adjacent to a 3-vertex, then it is not incident to a 3-face which is incident to a 2-vertex.

Case 3.5. 3-faces in G which is incident to a 2-vertex are $(2,6^+,6^+)$-faces.

Case 3.6. If a vertex v is adjacent to a 2-vertex, then it is not adjacent to a 3-vertex which is adjacent to other 3-vertex.

Case 3.7. 4-faces which is incident to a 2-vertex in G are $(2,3,7^+,7^+)$-, $(2,4,7^+,7^+)$-, $(2,5,7^+,7^+)$- or $(2,6^+,6^+,6^+)$-faces.

Case 3.8. There is at most one 2-vertex which is adjacent to a k-vertex $(3 \leq k \leq 4)$ in G.

We call a 2-vertex a *special 2-vertex* if it is adjacent to a k-vertex $(3 \leq k \leq 4)$, otherwise a simple 2-vertex. Let $n_2(v)$ denote the number of simple 2-vertices adjacent to v.

Now redistribute the charge according to the following discharging rules:

$R1'$, $R2'$, $R3'$ and $R4'$ are the same as $R1$, $R2$, $R3$, $R4$ in Case 1.

$R5'$ is the same as $R5$ except for the situation when $d(v)=4$ and the 4-face is a $(2,4,7^+,7^+)$-face; $d(v)=5$ and the 4-face is a $(2,5,7^+,7^+)$-face; $d(v)=6$ and the 4-face is a $(2,6,6^+,6^+)$-face. Transfer charge 0 from v to the $(2,4,7^+,7^+)$-face if $d(v)=4$ in $R5'$. Transfer charge 0 from v to the $(2,5,7^+,7^+)$-faces if $d(v)=5$ in $R5'$. Transfer charge $(2/3)$ from v to the $(2,6,6^+,6^+)$-faces if $d(v)=6$ in $R5'$.

$R6'$ is the same as $R6$ except for the situation when $d(v)=6$ and the 3-face is a $(2,6,6^+)$- or $(4^+,4^+,6)$-face. Transfer charge 2 from v to the $(2,6,6^+)$-faces if $d(v)=6$ in $R6'$. Transfer charge 1 from v to the $(4^+,4^+,6)$-faces if $d(v)=6$ in $R6'$.

$R7'$. Transfer 2 from each 5^+-vertex to every adjacent 2-vertex.

For any face $f\in F(G)$, if $d(f)=6$, $d(f)\geq 8$, the discussion is same as the corresponding situation in Case 1. For any vertex $v\in V(G)$, if $d(v)=3$, then the discussion is also same as the corresponding situation in Case 1. In the following, we discuss the rest cases.

Suppose $d(v)=2$. Then $w'(v)=-4$. Except the special 2-vertex, we have $w'(v)=-4+2\times 2=0$ by $R7'$.

Suppose $d(v)=4$. Then $w(v)=2$, $f_3(v)\leq 2$. If $n_2(v)=0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v)=1$.

Case 3.1.1 If $f_3(v)=2$, then $f_4(v)=0$, the discussion is similar to the corresponding situation in Case 1.

Case 3.1.2 If $f_3(v)=1$, then $f_4(v)\leq 1$, $n_3(v)\leq 1$ by Claim 3.3, we have $w'(v)>2-(3/4)-(1/2)=(3/4)>0$ by $R6'$, $R5'$, $R2'$.

Case 3.1.3 $f_3(v)=0$, then $f_4(v)\leq 4$, there are at most two 4-faces which are not incident to the 2-vertex and each of the 4-face is not a $(3,4,4,4)$-, $(3,4,4,5)$-, $(3,4,4,6)$- or $(3,4,5,5)$-face for G contains no structures H_{18} and H_{19}. If one of the 4-face is a $(4,4,4,4^+)$-face, then $n_3(v)\leq 1$, we have $w'(v)>2-(1/2)\times 2-(1/2)=(1/2)>0$ by $R5$ and $R2$. Otherwise, $n_3(v)\leq 3$, we have $w'(v)\geq 2-(1/3)\times 2-(1/3)\times 2-(1/2)=(1/6)>0$ by $R5$, $R3$ and $R2$.

Suppose $d(v)=5$. Then $w(v)=5$, $f_3(v)\leq 3$. If $n_2(v)=0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v)=1$.

Case 3.2.1 If $f_3(v)=3$, then $f_4(v)=0$, $n_2(v)=0$ by Claim 3.5. The discussion is similar to the corresponding situation in Case 1.

Case 3.2.2 If $f_3(v)=2$, then $f_4(v)\leq 1$, $n_3(v)=0$ for G contains no structure H_{39}. So $w'(v)\geq 5-(3/2)\times 2-2=0$ by $R6'$, $R5'$ and $R7'$.

Case 3.2.3 If $f_3(v)=1$, then $f_4(v)\leq 2$, $n_3(v)\leq 2$ by Claim 3.3. For G contains no structure H_{25}, v is not incident to a $(3,3,5,5)$- or $(3,3,5,6)$-face. So $w'(v)>5-(3/2)-(2/3)-(1/3)\times 2-2=(1/6)>0$ by $R6'$, $R5'$, $R3'$ and $R7'$.

Case 3.2.4 $f_3(v)=0$, then $f_4(v)\leq 5$. For G contains no structure H_{25}, v is not incident to a $(3,3,5,5)$- or $(3,3,5,6)$-face. There are at most three 4-faces which are not incident to the 2-vertex. If there are three 4-faces which are not incident to the 2-vertex, then $n_3(v)\leq 2$ for G contains no structure H_{67}. We have $w'(v)\geq 5-(2/3)\times 3-(1/3)\times 2-2=(1/3)>0$ by $R5'$, $R3'$ and $R2'$. Otherwise, $n_3(v)\leq 4$, we have $w'(v)>5-(2/3)\times 2-(1/3)\times 4-2=(1/3)>0$ by $R5'$, $R3'$ and $R2'$.

Suppose $d(v)=6$. Then $w(v)=8$, $f_3(v)\leq 4$. If $n_2(v)=0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v)=1$.

Case 3.3.1 If $f_3(v)=4$, then $f_4(v)=0$, $n_3(v)=0$ by Claim 3.4, we have $w'(v)\geq 8-1\times 3-2-2=1>0$ by $R6'$ and $R7'$.

Case 3.3.2 If $f_3(v)=3$, then $f_4(v)\leq 1$, $n_3(v)=0$ by Claim 3.3 and Claim 3.4. So $w'(v)>8-1\times 2-2-1-2=1>0$ by $R6'$, $R5'$ and $R7'$.

Case 3.3.3 $f_3(v)=2$, then $f_4(v)\leq 3$. If one of the 3-faces is incident to the 2-vertex, then $n_3(v)=0$ by Claim 3.4, we have $w'(v)\geq 8-2-1-1\times 3-2=0$ by $R6'$, $R5'$ and $R7'$. Otherwise, $n_3(v)\leq 2$, $f_4(v)\leq 2$, we have $w'(v)>8-1-1-1\times 2-(1/3)\times 2-2=(4/3)>0$ by $R6'$, $R5'$, $R3'$ and $R7'$.

Case 3.3.4 $f_3(v)=1$, then $f_4(v)\leq 4$. If the 3-face is incident to the 2-vertex, then $n_3(v)=0$. So $w'(v)\geq 8-2-1\times 4-2=0$ by $R6'$, $R5'$ and $R7'$. Otherwise, $n_3(v)\leq 3$, $f_4(v)\leq 3$, we have $w'(v)>8-1-1\times 3-(1/3)\times 3-2=1>0$ by $R6'$, $R5'$, $R3'$ and $R7'$.

Case 3.3.5 $f_3(v)=0$, then $f_4(v)\leq 6$. If $f_4(v)=6$, then $n_3(v)\leq 2$ by Claim 3.7 and for G contains no structure H_{25}. So $w'(v)\geq 8-(2/3)\times 2-1\times 4-(1/3)\times 2-2=0$ by $R5'$, $R3'$ and $R7'$. If $f_4(v)=5$, then $n_3(v)\leq 2$ by Claim 3.7 and for G contains no structure H_{25}. So $w'(v)\geq 8-(2/3)-1\times 4-(1/3)\times 2-2=(2/3)>0$ by $R5'$, $R3'$ and $R7'$. Otherwise, $f_4(v)\leq 4$, $n_3(v)\leq 6$, we have $w'(v)>8-1\times 4-(1/3)\times 6-2=0$ by $R5'$, $R3'$ and $R7'$.

Suppose $d(v)\geq 7$. Then $w(v)=3d(v)-10$. If $n_2(v)=0$, then the discussion is similar to the corresponding situation in Case 1. In the following, we focus on the situation $n_2(v)=1$. For G contains no structure H_{40}, each 4-face is incident to v is not a $(3,3,7^+,5^+)$- or $(3,4,7^+,4^+)$-face. Since $n_3(v)+(3/2)f_3(v)+1\leq d(v)$, we have

$$n_3(v)\leq d(v)-(3/2)f_3(v)-1.$$

For G contains no 5-cycle, then $f_3(v)\leq (2/3)(d(v)-f_4(v))$, we have

$$f_4(v)\leq d(v)-(3/2)f_3(v).$$

So $w'(v)\geq 3d(v)-10-2f_3(v)-f_4(v)-(1/3)n_3(v)-2\geq 3d(v)-10-2f_3(v)-d(v)+(3/2)f_3(v)-(1/3)d(v)+(1/2)f_3(v)+(1/3)-2=(5/3)d(v)-(35/3)\geq 0$ by $R6'$, $R5'$, $R3'$ and $R7'$.

Suppose $d(f)=3$. Then $w(f)=-4$ and $n_2(f)\leq 1$. If $n_2(f)=1$, then f is a $(2,6^+,6^+)$-face by Claim 3.5. So $w'(f)\geq 2\times 2=0$ by $R6'$. Otherwise, the discussion on other cases is similar to the corresponding situation when $d(f)=3$ in Case 1.

Suppose $d(f)=4$. Then $w(f)=-2$. If $n_2(f)=1$, then f is a $(2,3,7^+,7^+)$- $(2,4,7^+,7^+)$-, $(2,5,7^+,7^+)$- or $(2,6^+,6^+,6^+)$-face by Claim 3.7. So $w'(f)\geq -2+1\times 2=0$ or $w'(f)\geq -2+(2/3)\times 3=0$ by $R5'$. If $n_2(f)=0$, then the discussion is similar to the situation when $d(f)=4$ in Case 1.

From the above discussion, we can obtain that $w'(x)\geq 0$ for each $x\in V(G)\cup F(G)$ and x is none of a special 3-vertex, a special 2-vertex and a special face. From the above discussion, we have $\sum_{x\in V(G)\cup F(G)} w'(x)\geq -4-4=-8>-20$, a contradiction.

Case 4. $\delta(G)=1$.

Case 4.1. There is one 1-vertex and at most two 2-vertices in G.

When $d(v)=1$, then $w(v)=-7$. If there is one 1-vertex in G, then 3-faces in G are $(3^-,5^+,5^+)$-faces or $(4^+,4^+,4^+)$-faces for G contains no structure H_{41} and the 4-face which is incident to a 2-vertex is $(2,5^+,5^+,5^+)$-face for G contains no structure H_{42}. Now there are not special 3-vertices and special faces in G. The discharging rules are the same as Case 1 except for the two cases. (1) A 4-face have a common 2-vertex with a 3-face. (2) Two 4-faces have a common 2-vertex. Under these situation, transfer charge from its common incident vertices to only the 3-face in the first situation and transfer charge from its common incident vertices to only one of them in the second situation. Clearly, we can guarantee the new charge of each vertex of G is larger than or equal to zero. For convenience, let w'_{t1} (w'_{t2}) denote the total new charge of one 2-vertex (two 2-vertices) and the faces which are incident to the 2-vertex (the two 2-vertices).

Case 4.1.1 There is one 2-vertex in G. If the 2-vertex is incident to at most one 3-face and one 4-face simultaneously. Furthermore, the 3-face and 4-face is $(2,5^+,5^+)$- and $(2,5^+,5^+,5^+)$-face respectively. So $w'_{t1} \geq -4-2-4+(3/2)\times 3 = -(11/2) > -6$ by $R6$ and $R5$. If the 2-vertex is incident a 3-face and not incident to a 4-face, then the 3-face is $(2,5^+,5^+)$-face. We have $w'_{t1} \geq -4-4+(3/2)\times 2 = -5 > -6$ by $R6$. If the 2-vertex is incident a 4-face and not incident to a 3-face, we consider the situation such that the 2-vertex is a common vertex of two 4-faces, then the 4-face is a $(2,5^+,5^+,5^+)$-face. We have $w'_{t1} \geq -4-2-2+(2/3)\times 4 = -(16/3) > -6$ by $R5$. We can obtain that $w'_{t1} \geq -6$, so $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7+w'_{t1} \geq -7-6 = -13 > -20$, a contradiction.

Case 4.1.2 There are two 2-vertices in G.

For two 2-vertices are not incident to a same 3- or 4-face, from the discussion in *Case 4.1.1*, we have $w'_{t2} \geq -6\times 2 = -12$. So $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7-12 = -19 > -20$, a contradiction.

Case 4.2 There is one 1-vertex and at least three 2-vertices in G.

For G contains no structure H_{41}, the 3-faces in G are $(3^-,5^+,5^+)$-faces or $(4^+,4^+,4^+)$-faces. Now there are not special 3-vertices and special faces in G. The discussion is same as the situation in Case 3, we have $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7-4 = -11 > -20$, a contradiction.

Case 4.3 There are at least two 1-vertices in G.

If there are two 1-vertices in G, then there is neither 2-vertex nor other 1-vertex in G for G contains no structure H_{43}. Furthermore, any 3-face in G is $(3,5^+,5^+)$- or $(4^+,4^+,4^+)$-face for G contains no structure H_{41}. Note that there are neither special 3-vertices nor special faces in G now. And the following discussion is the same as the situation in Case 1. These imply that $\sum_{x\in V(G)\cup F(G)} w'(x) \geq -7\times 2 = -14 > -20$, a contradiction. ∎

In the following, let us give the proof of the main theorems.

Theorem 2.1. *If G is a planar graph without 5- and 7-cycles, then G is equitably k-colorable where $k \geq \max\{7,\Delta(G)\}$.*

Proof. Let G be a counterexample with fewest vertices. If each component of G has at most four vertices, then $\Delta(G) \leq 3$. So G is equitably k-colorable by Lemma 2.3. Otherwise, there is at least one component with at least five vertices. By Lemma 2.7, G has one of the structures $H_1 \sim H_{43}$, taking one and the vertices are labeled as they are in Figure 1. If there are vertices labeled repeatedly, then we take the larger (x_i is larger than x_{i-1}). In the following, we show how to find S in Lemma 2.4. If G has one of H_2, H_{12}, H_{32} and H_{43}, then let $S' = \{x_k,x_{k-1},x_{k-2},x_1\}$. If G has one of H_3, $H_5 \sim H_{13}$, H_{15}, H_{22}, H_{23}, $H_{28} \sim H_{31}$, H_{36}, H_{41} and H_{42}, then let $S' = \{x_k,x_{k-1},x_{k-2},x_2,x_1\}$. If G has one of H_9, H_{10}, H_{21}, H_{25}, H_{33}, H_{34}, H_{39} and H_{40}, then let $S' = \{x_k,x_{k-1},x_{k-2},x_3,x_2,x_1\}$.

If G has H_{14}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$. If G has one of H_1, H_8, H_{11}, H_{24} and H_{35}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_2, x_1\}$. If G has structure H_{26}, then let $S' = \{x_k, x_{k-1}, \cdots, x_{k-3}, x_3, x_2, x_1\}$. If G has H_{16}, H_{17} and H_{38}, then let $S' = \{x_k, x_{k-1}, \cdots, x_{k-4}, x_1\}$. If G has one of H_4, $H_{18} \sim H_{20}$, H_{37}, then let $S' = \{x_k, x_{k-1}, \cdots, x_{k-4}, x_2, x_1\}$. By Lemma 2.1, G is 3-generate, then we can find the remaining unspecified positions in S from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of $|V(G)|$ and $k \geq \Delta(G) \geq \Delta(G-S)$, $G-S$ is equitably k-colorable. So G is equitably k-colorable too by Lemma 2.4. ∎

Corollary 2.2. *Let G be a planar graph without 5- and 7-cycles. If $\Delta(G) \geq 7$, then $\chi_e(G) \leq \Delta(G)$.*

Theorem 2.2. *If G is a planar graph without 5- and 7-cycles and $k \geq \max\{7, \Delta(G)\}$, then G is equitably k-choosable.*

Proof. Let G be a counterexample with fewest vertices. If each component of G has at most four vertices, then $\Delta(G) \leq 3$. So G is equitably k-choosable by Lemma 2.5. Otherwise, the proof is similar to the proof of Theorem 2.8 by Lemma 2.6 and Lemma 2.7. ∎

Corollary 2.3. *Let G be a planar graph without 5- and 7-cycles. If $\Delta(G) \geq 7$, then G is equitable $\Delta(G)$-choosable.*

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