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Edge covering pseudo-outerplanar graphs with forests

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\textbf{A B S T R A C T}

A graph is pseudo-outerplanar if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this paper, we prove that each pseudo-outerplanar graph admits edge decompositions into a linear forest and an outerplanar graph, or a star forest and an outerplanar graph, or two forests and a matching, or max\{Δ(G), 4\} matchings, or max\{\lceil Δ(G)/2 \rceil, 3\} linear forests. These results generalize known results on outerplanar graphs and \(K_{2,3}\)-minor-free graphs, since the class of pseudo-outerplanar graphs is larger than the class of \(K_{2,3}\)-minor-free graphs.

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1. Introduction

In this paper, all graphs considered are finite, simple and undirected. We use \(V(G)\), \(E(G)\), \(\delta(G)\) and \(\Delta(G)\) to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph \(G\), respectively. Let \(d_G(v)\) (or \(d(v)\) for simplicity) denote the degree of a vertex \(v \in V(G)\). A block of a graph \(G\) is a maximal 2-connected subgraph of \(G\). A graph \(H\) is a minor of a graph \(G\) if a copy of \(H\) can be obtained from \(G\) via repeated edge deletion and/or edge contraction. For a subset \(S \subseteq V(G) \cup E(G)\), \(G[S]\) denotes the subgraph of \(G\) induced by \(S\). The connectivity of a graph \(G\), denoted by \(\kappa(G)\), is the minimum number of vertices whose deletion from \(G\) disconnects it. For other undefined concepts we refer the readers to [3].

An outerplanar graph is a graph that can be embedded on the plane in such a way that it has no crossings and that all its vertices lie on the outer face. In this paper, we introduce an extension of this concept. A graph is pseudo-outerplanar if each block has an embedding on the plane in such a way that the vertices lie on a fixed circle and the edges lie inside the disk of this circle with each of them crossing at most one another. In this embedding, the edges bounding the disk(s) are boundary edges and a disk is closed or open according to whether or not it contains the circle that constitutes its boundary. For example, Fig. 1 exhibits a pseudo-outerplanar embedding of a graph with two blocks: one is \(K_4\) and the other is \(K_{2,3}\). The drawing of \(K_4\) in this embedding lies inside a closed disk but the one of \(K_{2,3}\) in this embedding lies inside an open disk. In Fig. 1, the edges in bold are the boundary edges. A pseudo-outerplanar graph is maximal if it is not possible to add an edge such that the resulting graph is still pseudo-outerplanar. Thus, \(K_{2,3}\) is not a maximal pseudo-outerplanar graph, since we can add two edges to \(K_{2,3}\) and remain its pseudo-outerplanarity. One can easily check that each pseudo-outerplanar graph has a planar embedding, so the class of pseudo-outerplanar graphs forms a subclass of planar graphs. Actually, the definition of pseudo-outerplanar graphs is similar to that of 1-planar graphs (i.e. graphs that can be drawn on the plane so that each edge is crossed by at most one other edge), which was introduced by Ringel [10].

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Many classic problems in graph theory are considered for the class of planar graphs and its subclasses, such as the class of series-parallel graphs and the class of outerplanar graphs. Consider the problem of covering graphs with forests and a graph of bounded maximum degree, for example. We say that a graph is \((t, d)\)-coverable if its edges can be covered by at most \(t\) forests and a graph of maximum degree \(d\). In [2], et al. conjectured that every simple planar graph is \((2, 4)\)-coverable and gave an example to show that there are infinitely many planar graphs that are not \((2, 3)\)-coverable. This conjecture was recently confirmed by Gonçalves in [5]. In [2], it is also proved that every series-parallel graph is \((2, 0)\)-coverable and that every \(K_{2,3}\)-minor-free graph is both \((1, 3)\)-coverable and \((2, 0)\)-coverable. Since a graph is outerplanar if and only if it is \(\{K_4, K_{2,3}\}\)-minor-free [6], every outerplanar graph is both \((1, 3)\)-coverable and \((2, 0)\)-coverable. It is interesting to know what can be said about pseudo-outerplanar graphs, a larger class than outerplanar graphs.

Edge-coloring is another classic problem in graph theory. In fact, we can regard edge-coloring problems as an edge decomposition problem. When we color the edges of a graph \(G\), our actual task is to decompose the edge set \(E(G)\) into many parts such that the graph induced by each part satisfies a property \(\mathcal{P}\). Different properties \(\mathcal{P}\) correspond to different types of edge-coloring. For example, a proper edge \(k\)-coloring of \(G\) is a decomposition of \(E(G)\) into \(k\) subsets such that the graph induced by each subset is a matching in \(G\). The minimum integer \(k\) such that \(G\) has a proper edge \(k\)-coloring, denoted by \(\chi'(G)\), is the edge chromatic number of \(G\). Vizing’s Theorem states that for any graph \(G\), \(\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1\). A graph \(G\) is of class 1 if \(\chi'(G) = \Delta(G)\) and of class 2 if \(\chi'(G) = \Delta(G) + 1\). Sanders and Zhao [11] showed that each planar graph with maximum degree at least 7 is of class 1. Juvan et al. [9] proved that each series-parallel graph (and each outerplanar graph) with maximum degree at least 3 is of class 1. We ask whether each pseudo-outerplanar graph with large maximum degree is of class 1.

On the other hand, one can consider improper edge-colorings. Concerning this topic, Harary [7] introduced the concept of linear arboricity. A linear forest is a forest in which every connected component is a path. A tree \(k\)-coloring of \(G\) is a decomposition of \(E(G)\) into \(k\) subsets such that the graph induced by each subset is a matching in \(G\). The linear arboricity \(\text{la}(G)\) of a graph \(G\) is the minimum integer \(k\) such that \(G\) has a tree \(k\)-coloring. Akiyama et al. [1] conjectured that \(\text{la}(G) = \lceil \Delta(G) + 1 \rceil / 2\) for any regular graph \(G\). It is obvious that \(\text{la}(G) \geq \lceil \Delta(G) / 2 \rceil\) for any graph \(G\) and that \(\text{la}(G) \geq \lceil (\Delta(G) + 1) / 2 \rceil\) for any regular graph \(G\). Hence the conjecture is equivalent to the following one.

**Conjecture 1.1 (Linear Arboricity Conjecture).** For any graph \(G\), \(\lceil \Delta(G) / 2 \rceil \leq \text{la}(G) \leq \lceil \Delta(G) + 1 \rceil / 2\).

**Conjecture 1.1** has been proved true for all planar graphs [13,15]. However, it is still interesting to determine which planar graphs satisfy \(\text{la}(G) = \lceil \Delta(G) / 2 \rceil\). Wu [13] proved that it holds for planar graphs with maximum degree at least 13, and this bound 13 was later improved to 9 by Cygan et al. [4]. For subclasses of planar graphs, Wu [14] proved that \(\text{la}(G) = \lceil \Delta(G) / 2 \rceil\) for all series-parallel graphs (hence also for all outerplanar graphs) with maximum degree at least 3. Can the same conclusion extend to the class of pseudo-outerplanar graphs?

In Section 2, we give some relationships among the three classes containing the outerplanar graphs; they are the \(K_{2,3}\)-minor-free graphs, the series-parallel graphs, and the pseudo-outerplanar graphs. In Section 3, we investigate the problem of covering pseudo-outerplanar graphs with forests and a graph of bounded maximum degree. In Section 4, some unavoidable structures of pseudo-outerplanar graphs are obtained. These structures will be applied to determine the edge chromatic number and the linear arboricity of pseudo-outerplanar graphs in Section 5.

### 2. Basic properties

Let \(G\) be a pseudo-outerplanar graph. In the remainder of this paper, we always assume that \(G\) has been drawn on the plane such that (1) for each block \(B\) of \(G\), the vertices of \(B\) lie on a fixed circle and the edges of \(B\) lie inside the disk of this circle with each of them crossing at most one another; (2) the number of crossings in \(G\) is as small as possible. We call such a drawing a pseudo-outerplanar diagram of \(G\). Let \(G\) be a pseudo-outerplanar diagram and let \(B\) be a block of \(G\). Denote by \(v_1, v_2, \ldots, v_{|B|}\) the vertices of \(B\), which are lying in a clockwise sequence. Let \(V(v_i, v_j) = \{v_i, v_{i+1}, \ldots, v_j\}\) and \(V(v_i, v_j) = V[v_i, v_j] \setminus \{v_i, v_j\}\), where the subscripts and the additions are taken modular \(|B|\).
Lemma 2.1 ([6]). If $G$ is an outerplanar graph, then

(a) $\delta(G) \leq 2$.
(b) $\kappa(G) \leq 2$.

Theorem 2.2. If $G$ is a pseudo-outerplanar graph, then

(a) $\delta(G) \leq 3$.
(b) $\kappa(G) \leq 2$ unless $G \cong K_4$.

Proof. The proof of (a) is left to Corollary 4.3; we only prove (b) here. If $|G| \leq 4$, then this claim is trivial. Hence we may assume that $G$ is a pseudo-outerplanar diagram with $|G| \geq 5$ and $\kappa(G) \geq 3$. If $G$ has no crossings, then $G$ is an outerplanar graph; thus Lemma 2.1 yields $\kappa(G) \leq 2$, a contradiction. Therefore, we assume that there are two chords $v_i v_j$ and $v_k v_l$ in $G$ that cross each other, and that $v_i, v_k, v_j, v_l$ are lying in a clockwise sequence. Since $|G| \geq 5$, at least one of $V(v_i, v_k), V(v_k, v_j), V(v_j, v_i)$ and $V(v_i, v_j)$ is nonempty. Without loss of generality, assume that $V(v_i, v_k) \neq \emptyset$. Since $v_i v_j$ crosses $v_k v_l$, there is no edge between the two vertex sets $V(v_i, v_k)$ and $V(v_k, v_j)$. Thus, $\{v_i, v_k\}$ separates $V(v_i, v_k)$ and $V(v_k, v_j)$, contradicting $\kappa(G) \geq 3$. □

It is well-known that every connected outerplanar graph is Hamiltonian, but this does not hold for 2-connected pseudo-outerplanar graphs. The complete bipartite graph $K_{2, 3}$ is just a counterexample. If the disk of a circle $C$ is closed, then we call $C$ a closed circuit. A 2-connected pseudo-outerplanar diagram is a Hamiltonian diagram if it is drawn so that all its vertices lie on a closed circuit $C$, and this closed circuit $C$ is the Hamiltonian boundary of the diagram. By this definition, one can easily see that a non-Hamiltonian 2-connected pseudo-outerplanar graph cannot have a Hamiltonian diagram. We ask whether each Hamiltonian pseudo-outerplanar graph has a Hamiltonian diagram.

Theorem 2.3. Let $G$ be a pseudo-outerplanar diagram and let $C$ be a Hamiltonian cycle of $G$. If $C$ is not the boundary of $G$, then $G$ has a Hamiltonian diagram such that $C$ is the Hamiltonian boundary of this diagram.

Proof. We proceed by induction on the order of $G$. Since $G$ has a Hamiltonian cycle $C$ with vertices $v_1, \ldots, v_n$ that is not the boundary of the pseudo-outerplanar diagram of $G$, there exists at least one crossing in the drawing of $C$, which is a sub-diagram of $G$. Suppose that $v_j v_{j+1}$ and $v_k v_{k+1}$ for $j < k$ cross each other and that $v_j, v_l, v_{k+1}, v_{j+1}$ lie in a clockwise order. Denote respectively by $U$ and $W$ the set of vertices from $v_j$ to $v_{k+1}$ and from $v_{j+1}$ to $v_l$ in the cyclic clockwise sequence of vertices on the outer boundary of $G$. Take the first graph in Fig. 2 for example, we have $C = v_1 v_2, \ldots, v_n v_1, U = \{v_1, v_{j-1}, \ldots, v_{j+1}, v_l, v_{l+1}, v_{l+2}, \ldots, v_m, v_{m-1}, \ldots, v_{k+1}\}$ and $W = \{v_{j+1}, v_{j+2}, \ldots, v_k, v_{k-1}, v_k\}$. Note that besides $v_j v_{j+1}$ and $v_k v_{k+1}$, there is no other edge $uv$ such that $u \in U$ and $w \in W$, by the definition of $G$. One can see that $G_1$ is a pseudo-outerplanar diagram with a Hamiltonian cycle $C_1$ having vertices $v_{k+1}, \ldots, v_n, v_1, \ldots, v_j$, while $G_2$ is a pseudo-outerplanar diagram with a Hamiltonian cycle $C_2$ having vertices $v_{j+1}, \ldots, v_k$. By the induction hypothesis, $G_1$ and $G_2$ have Hamiltonian diagrams with Hamiltonian boundaries $C_1$ and $C_2$, respectively. We now combine these two Hamiltonian diagrams and add two edges $v_j v_{j+1}$ and $v_k v_{k+1}$ (see the second graph in Fig. 2) to obtain a Hamiltonian diagram of $G$ with Hamiltonian boundary $v_{k+1} v_k v_{k+2}, \ldots, v_n v_1, \ldots, v_j v_{j+1} v_{j+2}, \ldots, v_{k-1} v_k v_{k+1}$, which is the cycle $C$. □

Corollary 2.4. Each Hamiltonian pseudo-outerplanar graph has a Hamiltonian diagram.

A graph $G$ is quasi-Hamiltonian if each block of $G$ is Hamiltonian. Denote the class of pseudo-outerplanar graphs, quasi-Hamiltonian pseudo-outerplanar graphs, series-parallel graphs, $K_{2, 3}$-minor-free graphs, and outerplanar graphs by $\mathcal{P}, \mathcal{P}_H, \mathcal{D}, \mathcal{M}_{2, 3},$ and $\Theta$, respectively. The following basic relationship is obvious.

Remark 2.5. $\mathcal{P} \supset \mathcal{P}_H \supset \Theta, \mathcal{M}_{2, 3} \cap \delta = \Theta$.

In the following, we prove other relationships among these five classes of graphs.

Theorem 2.6. $\mathcal{P}_H \cap \delta = \Theta$. 
is Hamiltonian, there are four pairwise disjoint paths $P_{ux}, P_{xw}, P_{wy}$ and $P_{yu}$ that connect $u$ to $x$, $x$ to $y$, $v$ to $y$ and $y$ to $u$. Thus, the two edges $uw$ and $vy$ and the four paths $P_{ux}, P_{xw}, P_{wy}, P_{yu}$ form a $K_4$-minor, which is impossible in a series–parallel graph. Hence $B$ is an outerplanar graph.

**Lemma 2.7 ([8])** Let $H$ be a graph obtained from $K_{2,3}$ by adding an edge joining two vertices of degree $2$. If $G$ is a $H$-minor-free graph, then each block of $G$ is either $K_4$-minor-free or isomorphic to $K_4$.

**Corollary 2.8.** For any $2$-connected graph $G \in \mathcal{M}_{2,3}$, either $G \in \Theta$ or $G \simeq K_4$.

**Proof.** Since $G \in \mathcal{M}_{2,3}$, $G$ is $H$-minor-free, where $H$ is the graph in Lemma 2.7. Thus, by Remark 2.5 and Lemma 2.7, either $G \in \Theta$ or $G \simeq K_4$.

**Theorem 2.9.** $\mathcal{M}_{2,3} \subset \mathcal{P}_H$.

**Proof.** The inclusion of $\mathcal{M}_{2,3}$ in $\mathcal{P}_H$ directly follows from Corollary 2.8. The inequality comes from the graph $(K_1 \cup K_2) \lor K_2$ that belongs to $\mathcal{P}_H$ but not to $\mathcal{M}_{2,3}$.

### 3. Decomposability

Let $G$ be a pseudo-outerplanar diagram and let $B$ be a block of $G$. Denote by $v_1, v_2, \ldots, v_{|B|}$ the vertices of $B$, which are lying in a clockwise sequence. The edges joining consecutive vertices in this list are boundary edges and other edges are chords of $G$. Since $G$ is a pseudo-outerplanar diagram, all of the crossings are generated by one chord crossing another chord. Let $C[v_1, v_j]$ be the set of chords $xy$ with $x, y \in C[v_1, v_j]$ and let $C(G)$ be the set of crossed chords in $G$.

**Theorem 3.1.** Let $G$ be a Hamiltonian pseudo-outerplanar diagram and let $C$ be the Hamiltonian boundary of this diagram. If $y \in C(C)$ and $yx, yz \in E(C)$, then there exists a linear forest $T$ in $G$ such that $E(T) \subseteq C(G), d_T(y) = 0$, $\max\{d_T(x), d_T(z)\} \leq 1$, and $G = E(T)$ is an outerplanar diagram.

**Proof.** We proceed by induction on the order of $G$. One can see that this claim holds for $|G| \leq 4$, since the case $G = K_4$ is trivial. Hence, we may assume that $|G| > 4$ and $x, y, z$ occur on C in a clockwise sequence.

First, we consider the case when $d_T(y) = 2$. If the edge $xz$ already exists in $G$, then let $G' = G - y$ and $C' = C - y$; otherwise, let $C' = (G - y) + xz$ and $C' = (C - y) + xz$. It is easy to see that $G'$ is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary $C'$. Let $x' \neq z$ be a vertex such that $xx' \in E(C')$ ($x'$ exists because $|V(G)| \geq 5$). By induction on $(G', C', x', x, z)$ (as $(G, C, x, y, z)$, respectively), there exists a linear forest $T'$ in $G'$ such that $E(T') \subseteq C(G'), d_{T'}(x) = 0$, $\max\{d_{T'}(x), d_{T'}(z)\} \leq 1$, and $G' - E(T')$ is an outerplanar diagram. Let $T = T'. Then $C(C) - E(T') \subseteq C(G)$, we have $E(T) \subseteq C(G)$, $d_{T'}(x) = d_T(y) = 0$, and $d_{T'}(z) \leq 1$. Furthermore, one can easily see that $G = E(T)$ is an outerplanar diagram.

If $d_T(y) = 3$ and $xz \in E(G)$, then the edge $xz$ is crossed by another edge $yw$. Assume first that $V(z, w) = \emptyset$. We then immediately have $zw \in E(C)$. Let $G' = G[V[w, x]] + uw$ and let $C'$ be the cycle consisting of the edge $wv$ and the clockwise subpath around $C$ from $w$ to $x$. We assume that $V(x) \setminus \{w\} \neq \emptyset$, because otherwise $G$ would have less than five vertices, a contradiction. Let $x' \neq w$ be a vertex such that $xx' \in E(C)$ (see 1st graph of Fig. 3). Note that $G'$ is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary $C'$. By induction on $(G', C', x', x, w)$, there exists a linear forest $T'$ in $G'$ such that $E(T') \subseteq C(G'), d_{T'}(x) = 0$, $\max\{d_{T'}(x'), d_{T'}(w)\} \leq 1$, and $G' - E(T')$ is an outerplanar diagram. Let $T = T' + xz$. One can easily check that $E(T) \subseteq C(G), d_T(y) = 0$, $d_T(x) = d_T(z) = 1$, and $G = E(T)$ is an outerplanar diagram. Thus, a linear forest $T$ as required can be constructed. In the following, we assume that $V(z, w) \neq \emptyset$ and $V(w, x) \neq \emptyset$. Let $z' \neq y, w$ be a vertex such that $zz' \in E(C_1)$ and let $x' \neq y, w$ be a vertex such that $xx' \in E(C)$ (see 2nd graph of Fig. 3). Let $G_1 = G[V[z, w]] + zw$ and $G_2 = G[V[w, x]] + wu$. By $C_1$ and $C_2$, we respectively denote the cycle that consists of $u$ and the clockwise subpath around $C$ from $z$ to $w$, and that consists of the edge $xw$ and the clockwise subpath around $C$ from $w$ to $x$. For $i = 1, 2$, $G_i$ is a Hamiltonian pseudo-outerplanar diagram with Hamiltonian boundary $C_i$. By inductions on $(G_1, C_1, w, z, z')$ and $(G_2, C_2, w, x, x')$, there exist a linear forest $T_1$ in $G_1$ with...
$E(T_1) \in C(G_1), d_{T_1}(z) = 0, \max\{d_{T_1}(w), d_{T_1}(z')\} \leq 1,$ and $G_1 - E(T_1)$ is an outerplanar diagram, and a linear forest $T_2$ in $G_2$ with $E(T_2) \in C(G_2), d_{T_2}(x) = 0, \max\{d_{T_2}(w), d_{T_2}(x')\} \leq 1,$ and $G_2 - E(T_2)$ being an outerplanar diagram. Let $T = T_1 \cup T_2 \cup \{x\}$. One can easily see that $E(T) \subseteq C(G), d_{T}(y) = 0, d_{T}(x) = d_{T}(z) = 1, d_{T}(w) \leq 2,$ and $G - E(T)$ is an outerplanar diagram. Since $T_1$ and $T_2$ intersect on at most one vertex, $w,$ of degree at most one in each forest and there is no edge between $V(T_1) \setminus \{w\}$ and $V(T_2) \setminus \{w\},$ $T_1 \cup T_2$ is a linear forest. Furthermore, since $x, y$ and $z$ have degree 0 in $T_1 \cup T_2,$ $T_1 \cup T_2 \cup \{x\}$ is the required linear forest.

The last case is when $d_{T}(y) \geq 3$ and $x \not\in E(G).$ We label the neighbors of $y$ by $y_1, y_2, \ldots, y_k$ in a clockwise sequence on $C,$ where $y_1 = z, y_k = x$ and $k \geq 3.$ If $y_2$ is not a crossed chord in $G,$ then let $G_1 = G[\{y, y_2\}]$ and $G_2 = G[\{y, y_1\}]$. Denote by $C_1$ (resp. $C_2$) the cycle consisting of the edge $y_2y$ and the clockwise subpath around $C$ from $y$ to $y_2$ (resp. from $y_2$ to $y$). For $i = 1, 2, G_i$ is a Hamiltonian pseudo-outplanar diagram. By inductions on $(G_1, C_1, y_2, y, z)$ and $(G_2, C_2, y_2, y, x),$ it is easy to construct a linear forest as required. Hence, we may assume that $y_{y_2}$ is crossed by another edge $y_{y_2}^2$ in $G,$ where $y_{y_2}, y_{y_2}^2$ are labeled clockwise. Since there is no edge between $V(y_{y_2}^2)$ and $V(y_{y_2}^2), y_2$ can add two edges $y_{y_2}^2$ and $y_{y_2}^2$ to $G$ if they do not really exist so that they do not generate new crossings in $G,$ and thus the resulting graph is still pseudo-outplanar (see the 3rd graph of Fig. 3). By $C_1, C_2$ and $C_3,$ we respectively denote the cycle that consists of the edge $y_{y_2}^2y$ and the clockwise subpath around $C$ from $y$ to $y_{y_2}^2,$ and that consists of the edge $y_{y_2}^2$ and the clockwise subpath around $C$ from $y_{y_2}^2$ to $y.$ Let $G_i$ be the subgraph of $G$ contained in the closed disk of $C_1$ for $i = 1, 2, 3.$ Here one should be careful that if $y_{y_2} = y_1$ (resp. $y_{y_2} = y_k$), then $C_1$ (resp. $C_3$) is not a cycle and $C_1$ (resp. $C_3$) is defined to be a null graph. However, $G_1$ and $G_3$ cannot simultaneously be null graphs, since $y_1y_k \not\in E(G).$ Hence any of $G_1$ for $i = 1, 2, 3$ is a subgraph of $G$ with smaller order. Moreover, every non-null graph $G_i$ is a Hamiltonian pseudo-outplanar diagram with Hamiltonian boundary $C_i.$ Without loss of generality, we assume that none of $G_i$ for $i = 1, 2, 3$ is a null graph. By inductions on $(G_1, C_1, y_1, y, y_{y_2})$, $(G_2, C_2, y_{y_2}^2, y, y_{y_2}^2)$ and $(G_3, C_3, y_1, y, y_{y_2}^2)$, there exist a linear forest $T_i$ in $G_i$ for $i = 1, 2, 3$ such that $E(T_i) \subseteq C(G_i), d_{T_i}(y) = 0,$ and $G - E(T_i)$ is an outerplanar diagram. Meanwhile, we have $\max\{d_{T_i}(y), d_{T_i}(y_2^2), d_{T_i}(y_2^2), d_{T_i}(y_2^2), d_{T_i}(y_2^2), d_{T_i}(y_2^2)\} \leq 1.$ Let $T = T_1 \cup T_2 \cup T_3.$ Since there is no edge whose end points are belong to different parts of the vertex partition $\{V(y, y_{y_2}), V(y_{y_2}^2, y_{y_2}^2), V(y_{y_2}^2, y_{y_2}^2)\}$ (because otherwise either $y_{y_2}^2$ or $y_{y_2}^2$ may be crossed twice), $T$ is a forest. Since $d_{T_i}(y_{y_2}^2) \leq d_{T_i}(y_{y_2}) + d_{T_i}(y_{y_2}) \leq 2$ and $d_{T_i}(y_{y_2}^2) \leq d_{T_i}(y_{y_2}^2) + d_{T_i}(y_{y_2}) \leq 2,$ $\Delta(T) \leq 2$ and thus $T$ is a linear forest. Since $C(G) \subseteq C(G_i)$ for $i = 1, 2, 3,$ $E(T) = E(T_1) \cup E(T_2) \cup E(T_3) \subseteq C(G(1)) \cup C(G(2)) \cup C(G(3)) \subseteq C(G).$ Meanwhile, $E(T) = E(T_1) \cup E(T_2) \cup E(T_3) \subseteq C(G(1)) \cup C(G(2)) \cup C(G(3)) \subseteq C(G).$ Since $G - E(T) \subseteq \bigcup_{i=1}^{3}(G_i - E(T_i)),$ $G - E(T)$ is an outerplanar diagram. Hence we construct a linear forest $T$ as required in $G$ and completes the proof of the theorem. □

A star forest is a graph in which every component is a star. The root of a star is the vertex of maximum degree. Note that $K_2$ has two roots. The roots of a star forest is the union of the root of each star component. The following Theorem 3.2 is an analog of Theorem 3.1 (note that the condition $\max\{d_{T}(x), d_{T}(z)\} \leq 1$ in Theorem 3.1 is equivalent to that $x$ or $z$ is a vertex of $T$ if and only if $x$ or $z$ is a leaf of $T$), whose proof is almost the same with that of Theorem 3.1. Actually, we can still proceed by induction on the order of $G$ and split the proofs into three cases: the first is $d_{T}(y) = 2$, the second is $d_{T}(y) = 3$ and $x \in E(G),$ and the last is $d_{T}(y) \geq 3$ and $x \not\in E(G).$ In each case we can construct a star forest $T$ as required by the same way as in the proof of the proof of Theorem 3.2 is left to the readers.

Theorem 3.2. Let $G$ be a Hamiltonian pseudo-outplanar diagram and let $C$ be the Hamiltonian boundary of this diagram. If $y \in V(C)$ and $xy, yz \in E(C),$ then there exists a star forest $T$ in $G$ such that $E(T) \subseteq C(G), d_{T}(y) = 0,$ $x$ or $z$ is a vertex of $T$ if and only if $x$ or $z$ is a root of $T,$ and $G - E(T)$ is an outerplanar diagram.

Corollary 3.3. Each pseudo-outplanar graph can be decomposed into an outerplanar graph and a linear forest, or an outerplanar graph and a star forest.

Proof. Without loss of generality, let $G$ be a quasi-Hamiltonian pseudo-outplanar diagram (otherwise we can add some edges to close the circumferential boundary of each block). In what follows, we proceed by induction on the number of blocks, $\omega(G),$ in $G.$ In the base case when $\omega(G) = 1$ follows directly from Theorems 3.1 and 3.2, so we assume that $\omega(G) \geq 2.$ Choose a block $B$ of $G$ that contains only one cut vertex $y$ (i.e. $B$ is an end-block). By Theorems 3.1 and 3.2, $B$ can be decomposed into an outerplanar graph $H_1$ and a linear forest $T_1$ with $d_{T_1}(y) = 0,$ or an outerplanar graph $H_2$ and a star forest $T_2$ with $d_{T_2}(y) = 0.$ Meanwhile, by the induction hypothesis, $G - B$ can also be decomposed into an outerplanar graph $H_3$ and a linear forest $T_3,$ or an outerplanar graph $H_4$ and a star forest $T_4.$ Therefore, $G$ can be covered by the linear forest $T = T_1 \cup T_3$ and the outerplanar graph $H = H_1 \cup H_3,$ or the star forest $T = T_2 \cup T_4$ and the outerplanar graph $H = H_2 \cup H_4.$ □

Theorem 3.4. For every integer $n \geq 12,$ there exists a 2-connected pseudo-outplanar graph with order $n$ that cannot be decomposed into an outerplanar graph and a matching.

Proof. We show the last graph $G$ in Fig. 3 is a graph that cannot be decomposed into an outerplanar graph and a matching. Otherwise, we may assume that $E(G) = E(H) \cup E(M),$ where $H$ is an outerplanar and $M$ is matching. Set $S_i = \{u_iu_{i+1}, u_iu_{i+2}, u_{i+1}u_{i+3}, u_{i+1}u_{i+3}, u_{i+2}u_{i+3}\}$ (mod 12) for $i = 1, 4, 7, 10.$ We now prove that there exists an edge set $S$ that is contained in $E(H).$
If this claim does not hold, then we may meet one of the following two cases. If \( v_1v_2 \in E(M) \), then \( v_1v_k \in E(H) \) for \( k = 3, 4, 7, 10, 11, 12 \) and exactly one of \( v_10v_11 \) and \( v_10v_12 \) should be contained in \( E(M) \), say \( v_10v_11 \). It follows that \( v_kv_{10} \in E(H) \) for \( k = 4, 7, 12 \). However, the five vertices \( v_1, v_4, v_7, v_{10}, v_{12} \) and the three disjoint paths \( v_1v_4v_{10}, v_1v_7v_{10}, v_1v_{12}v_{10} \) form a copy of \( K_{2,3} \) in \( H \); this is a contradiction. If \( v_1v_4 \in E(M) \), then \( v_1v_2, v_1v_3, v_1v_7, v_1v_4, v_1v_4, v_1v_7 \in E(H) \) and the graph induced by \( \{v_1, v_2, v_3, v_4, v_7\} \) is a \( K_{2,3} \), which is impossible in an outerplanar graph.

Hence, in the following, we may assume that \( S_1 \subseteq E(H) \). If \( \{v_1v_7, v_4v_7\} \subseteq E(H) \), then the five vertices \( v_1, v_2, v_3, v_4, v_7 \) and the three disjoint paths \( v_1v_2v_4, v_1v_3v_4, v_1v_7v_4 \) form a copy of \( K_{2,3} \) in \( H \), a contradiction. Thus, exactly one of \( v_1v_7 \) and \( v_1v_7 \) shall be contained in \( E(M) \), say \( v_1v_7 \). Similarly, we can prove that \( \{v_1v_{10}, v_4v_{10}\} \subseteq E(H) \). Thus, we have \( v_1v_10 \in E(H), v_4v_{10} \in E(M) \), and \( v_1v_{10} \in E(H) \). Now the six vertices \( v_1, v_2, v_3, v_4, v_7, v_{10} \) and the three disjoint paths \( v_1v_4v_{10}, v_1v_2v_4, v_1v_10v_{10}v_4 \) form a \( K_{2,3} \)-minor in \( H \). This contradiction completes the proof of this theorem.

\[ \square \]

**Theorem 3.5.** Every maximal pseudo-outerplanar graph \( G \) is obtained from a maximal pseudo-outerplanar diagram \( H \) by gluing a \( K_3 \) or a \( K_4 \) along a boundary edge of \( H \).

**Proof.** Without loss of generality, we assume that \( G \) is a 2-connected maximal pseudo-outerplanar diagram. Since \( G \) is maximal, \( G \) is Hamiltonian and has at least one chord. Let \( C \) be the Hamiltonian boundary of the diagram of \( G \) with vertices \( v_1, v_2, \ldots, v_G \). We split the proof into two cases.

**Case 1.** There exists a crossed chord in \( G \).

Let \( v_kv_l \) be a chord in \( G \) that crosses another chord \( v_iv_j \) for \( 1 \leq i < k < j < l \leq |G| \). Actually, we can properly choose \( i \) and \( j \) such that there is no pair of mutually crossed chords in \( E(v_1, v_j) \setminus \{v_iv_j, v_kv_l\} \), because otherwise we can change the value of \( i \) or \( j \) to meet this condition.

Assume first that there is no non-crossed chord in \( E(v_1, v_j) \setminus \{v_iv_j, v_kv_l\} \). In this case, we shall have \( k = i + 1 \). Otherwise, since \( v_1v_k \notin E(G) \), we can add \( v_kv_k \) to \( G \) so that \( G \) is still pseudo-outerplanar, contradicting the fact that \( G \) is maximal. Similarly, by the maximality of \( G \), we have \( j = k + 1 \), \( l = j + 1 \), and \( v_1v_l \in E(G) \). Furthermore, \( d(v_k) = d(v_l) = 3 \). Remove the vertices \( v_k \) and \( v_l \) from \( G \) and denote the resulting graph by \( H \). Actually, \( H \) is a maximal pseudo-outerplanar diagram, because otherwise we can add an edge \( e = v_kv_l \notin E(H) \), where \( a, b \neq k \) or \( j \), to \( H \) so that \( H + e \) is pseudo-outerplanar, and thus \( G + e \) is pseudo-outerplanar, since \( e \notin E(G) \), contradicting the fact that \( G \) is maximal. At this stage, one can easily see that \( G \) is obtained from \( H \) by gluing a \( K_4 \) along the boundary edge \( v_kv_l \).

Second, assume that there is a non-crossed chord \( v_i, v_l \) in \( E(v_1, v_j) \setminus \{v_iv_j, v_kv_l\} \). Since there is no crossed chord in \( E(v_1, v_j) \), we can properly choose \( r \) and \( s \) such that \( \{v_1, v_j\} \setminus \{v_i, v_l\} = \emptyset \). By the maximality of \( G \), we have \( s = r + 2 \), otherwise we can add an edge \( v_{r+2}v_{r+2} \) to \( G \) so that the resulting graph is still pseudo-outerplanar, a contradiction. Since \( v_1v_j \) is a non-crossed chord, \( d(v_{r+1}) = 2 \). Remove the vertex \( v_{r+1} \) from \( G \) and denote the resulting graph by \( H' \). By a similar argument as above, one can prove that \( H' \) is a maximal pseudo-outerplanar diagram. Furthermore, one can easily see that \( G \) is obtained from \( H' \) by gluing a \( K_3 \) along the boundary edge \( v_kv_l \).

**Case 2.** There exists a non-crossed chord in \( G \).

Let \( v_1v_j \) for \( 1 \leq i < j \leq |G| \) be a non-crossed chord in \( G \). In this case, we shall assume that there is no crossed chord in \( E(v_1, v_j) \setminus \{v_1v_j\} \), because otherwise we are in Case 1. We choose \( i \) and \( j \) such that \( \{v_1, v_j\} \setminus \{v_1v_j\} = \emptyset \), and then we are in the second subcase of Case 1, where we can set \( r := i \) and \( s := j \).

**Corollary 3.6.** Each pseudo-outerplanar graph can be decomposed into two forests and a matching.

**Proof.** Let \( G \) be a pseudo-outerplanar graph. In the following, we proceed by induction on the size of \( G \) and assume that \( G \) is a maximal pseudo-outerplanar diagram. By Theorem 3.5, there respectively exists a \( K_3 \) with vertices \( x, y \) and \( z \) or a \( K_4 \) with vertices \( x, y, u \) and \( v \) contained in \( G \) such that \( H = G - \{xv, yz\} \) or \( H = G - \{xv, xu, yu, uy, uv\} \) is a maximal pseudo-outerplanar graph with \( xy \) being the resulting edge. By induction on \( H \), there exists two forests \( F_1, F_2 \) and a matching \( M \) such that \( E(H) = E(F_1) \cup E(F_2) \cup E(M) \). In the former case, let \( F_1 = F_1 + xz, F_2 = F_2 + yz \) and \( M' = M \); and in the latter case, let \( F_1 = F_1 + xu, F_2 = F_2 + yu, yv, uv \) and \( M' = M + uv \). One can easily check that the two forests \( F_1, F_2 \) and the matching \( M' \) is the desired decomposition of \( G \).

**Theorem 3.7.** For every integer \( n \geq 6 \), there exists a 2-connected pseudo-outerplanar graph with order \( n \) that cannot be decomposed into two forests.

**Proof.** Let \( C \) be a cycle with \( n \) vertices \( v_1, \ldots, v_n \), where \( n \geq 6 \). Add edges \( v_1v_i \) for all \( 3 \leq i \leq n \) and edges \( v_nv_{n+i} \) for all \( 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1 \). One can easily check that the resulting graph \( G_n \) is a 2-connected pseudo-outerplanar graph with order \( n \) and size \( \lfloor \frac{3n}{2} \rfloor - 4 \). If \( G_n \) can be decomposed into two forests \( F_1 \) and \( F_2 \), then \( |E(G_n)| = |E(F_1)| + |E(F_2)| \leq |V(F_1)| + |V(F_2)| - 2 \leq 2n - 2 \). However, for \( n \geq 6 \), \( |E(G_n)| = \lfloor \frac{3n}{2} \rfloor - 4 > 2n - 2 \). Hence, the graph \( G_n \) for \( n \geq 6 \) cannot be covered by two forests.

From Corollary 3.6 and Theorem 3.7, we directly have the following two corollaries.

**Corollary 3.8.** Every pseudo-outerplanar graph is \((2, 1)\)-coverable: and the two parameters given here are best possible.

**Corollary 3.9.** The arboricity of a pseudo-outerplanar graph is at most 3; and this bound is sharp.
4. Unavoidable structures

In this section, a vertex set \( V[v_i, v_j] \) is a non-edge if \( j = i + 1 \) and \( v_i v_j \notin E(G) \), is a path if \( v_k v_{k+1} \in E(G) \) for all \( i \leq k < j \), and is a subpath if \( j > i + 1 \) and some edges in the form \( v_k v_{k+1} \) for \( i \leq k < j \) are missing. We say that a chord \( v_i v_j \) is contained in a chord \( v_l v_j \) if \( i \leq k \leq l \leq j \). In any figure of this section, the solid vertices have no edges of \( G \) incident with them other than those shown.

**Lemma 4.1 ([12]).** If \( G \) is a 2-connected outerplanar graph, then

1. \( G \) has two adjacent 2-vertices \( u \) and \( v \), or
2. \( G \) has a 3-cycle \( u v u \) such that \( d(u) = 2 \) and \( d(v) = 3 \), or
3. \( G \) has a 4-vertex \( w \), where \( N(w) = \{u, v, x, y\} \), such that \( d(u) = d(v) = 2 \), \( N(u) = \{w, x\} \) and \( N(v) = \{w, y\} \).

For the class of pseudo-outerplanar graphs, we have a similar structural theorem as Lemma 4.1.

**Theorem 4.2.** If \( G \) is a pseudo-outerplanar diagram with \( \delta(G) \geq 2 \), then \( G \) contains one of the following configurations \( G_1 - G_{17} \). Moreover,

(a) if \( G \) contains some configuration among \( G_6 - G_{17} \), then the drawing of this configuration in the figure is a part of the diagram of \( G \) with its bending edges corresponding to the chords;
(b) if \( G \) contains the configuration \( G_3 \) and \( x y \notin E(G) \), where \( x \) and \( y \) are the vertices of \( G_3 \) as described in the figure, then we can properly add an edge \( xy \) to \( G \) so that the resulting diagram is still pseudo-outerplanar.

![Diagram](image)

**Proof.** We first consider the case when \( G \) is a 2-connected pseudo-outerplanar diagram. Let \( v_1, v_2, \ldots, v_{|G|} \) be the vertices of the diagram lying in a clockwise sequence. If there is no crossing in \( G \), then \( G \) is an outerplanar graph, and thus \( G \) satisfies this claim by Lemma 4.1. Otherwise, we can properly choose one chord \( v_i v_j \) such that

1. \( v_i v_j \) crosses \( v_k v_l \) in \( G \);
2. \( v_i, v_k, v_l \) and \( v_j \) are lying in a clockwise sequence;
3. \( v_k, v_l \) and \( v_j \), there is no crossed chord in \( C[v_i, v_j] \).

The condition (3) can be easily fulfilled, because otherwise we could change the values of \( i \) and \( j \) to meet this condition (note that the values of \( k \) and \( l \) are determined by \( i \) and \( j \)). Without loss of generality, assume that \( 1 \leq i < k < j \leq |G| \), because otherwise we can adjust the labellings of the vertices in \( G \) to meet it.

**Claim 1.** \( V[v_i, v_k] \) is either non-edge or path, and so do \( V[v_k, v_j] \) and \( V[v_j, v_i] \).

We only need to prove that \( V[v_i, v_k] \) cannot be subpath. Otherwise, there exists two vertices \( v_m \) and \( v_{m+1} \), where \( i \leq m \leq k - 1 \), such that \( v_m v_{m+1} \notin E(G) \). If there are chords in the form \( v_a v_{m+1} \) such that \( i \leq a \leq m - 1 \), then we choose one among them such that \( a \) is maximum. One can see that \( v_a \) is a vertex cut of \( G \), because there is no edge between \( V[v_{a+1}, v_m] \) and \( V[v_{m+1}, v_{a-1}] \), by the choice of \( a \) and by (3). This contradicts the fact that \( G \) is 2-connected. Thus, there is no chord in the form \( v_m v_{m+1} \) such that \( i \leq a \leq m - 1 \). Similarly, there is no chord in the form \( v_m v_b \) such that \( m + 2 \leq b \leq k \). Let \( p = \max\{i|v_{m+1} v_m \in E(G), m + 1 < n \leq k\} \) and \( q = \min\{n|v_m v_n \in E(G), i \leq n < m\} \). Since \( V[v_i, v_k] \) is neither non-edge nor path, we have \( k - i \geq 2 \), and thus at least one of the integers \( p \) and \( q \) exists. Without loss of generality, suppose that \( p \) exists. In this case, \( v_p \) is a vertex cut of \( G \), because there is no edge between \( V[v_{m+1}, v_{p-1}] \) and \( V[v_{p+1}, v_m] \), by the choices of \( m \), \( p \) and by (3). This contradiction completes the proof of Claim 1.

**Claim 2.** If \( V[v_i, v_k] \) is a path and \( k - i \geq 3 \), then \( G \) has a subgraph isomorphic to one of the configurations \( G_1, G_2 \) and \( G_4 \). This result also holds for \( V[v_k, v_j] \) or \( V[v_j, v_i] \) when \( j - k \geq 3 \) or \( l - j \geq 3 \), respectively.
If there is no other chord except \( v_i v_k \) (if exists) in \( V[v_i, v_k] \), then the configuration \( G_1 \) occurs in \( G \), since \( k - i \geq 3 \). Hence we may assume that \( S := C[v_i, v_k] \setminus \{v_i v_k\} \neq \emptyset \).

We now prove that there exists at least one chord in \( S \) that contains at least one other chord. If such a chord does not exist, then we first choose a chord \( v_m v_n \in S \) for \( m < n \). Without loss of generality, assume that \( n \neq k \). If \( n - m \geq 3 \), then we can easily find a copy of \( G_1 \) in \( G \), since \( v_m v_n \) contains no other chord by our assumption. If \( n - m = 2 \), then it is trivial to see that \( d(v_{m+1}) \geq 2 \). In this case, if \( \min(d(v_m), d(v_n)) \leq 3 \), then a copy of \( G_2 \) would be found in \( G \). Hence we may assume that \( \min(d(v_m), d(v_n)) \geq 4 \). Since \( v_m v_n \) cannot be contained in a chord in the form \( v_q v_i \) for \( q < n \) by the assumption, there exists another chord \( v_p v_q \) for \( p < n \) in \( S \). If \( p = n - 2 \) or \( d(v_n+1) = 2 \), then \( d(v_q) = 4 \), because otherwise there would be chord in \( S \) that contains either \( v_m v_n \) or \( v_p v_q \), a contradiction. At this stage, one can see that the graph induced by \( V[v_m, v_p] \) contains the configuration \( G_4 \).

By the above arguments, we can choose one chord \( v_m v_n \in S \) for \( a < b \) such that \( v_m v_n \) contains at least one chord, and moreover, every chord contained in \( v_m v_n \) contains no other chords (this condition can be easily fulfilled by properly changing the values of \( a \) and \( b \) if necessary). Let \( v_m v_n \) for \( m < b \) be the chord contained in \( v_m v_n \). By a similar argument as above, we only need to consider the case when \( n - m = 2 \), \( d(v_{m+1}) = 2 \), and \( \min(d(v_m), d(v_n)) \geq 4 \). Without loss of generality, assume that \( n \neq b \). Since \( d(v_n) \geq 4 \) and \( v_m v_n \) cannot be contained in a chord in the form \( v_q v_i \) for \( q < n \) by the choices of \( a \) and \( b \), there is a chord \( v_m v_k \) for \( n < p \leq b \) in \( S \). If \( G \) contains no copies of \( G_1 \) or \( G_2 \), then \( p = n - 2 \) and \( d(v_{n+1}) = 2 \). Furthermore, by the choices of \( a \) and \( b \), one can similarly prove that \( d(v_n) = 4 \). Thus, we would find a copy of \( G_4 \) in the graph induced by \( V[v_m, v_p] \).

**Claim 3.** At most one of \( V[v_i, v_j], V[v_k, v_j] \) and \( V[v_i, v_k] \) can be non-edge.

If \( V[v_i, v_j] \) and \( V[v_k, v_j] \) are non-edge, then \( v_j \) is a vertex cut vertex of \( G \), contradicting the fact that \( G \) is 2-connected. If \( V[v_i, v_k] \) and \( V[v_i, v_j] \) are non-edge, then we can adjust the drawing of \( G \) by replacing the vertices order \( v_i, v_j, v_k, v_i \) with \( v_i, v_j, v_j, \ldots, v_{k-1}, v_i, v_j \). This operation can reduce the number of crossings in the drawing of \( G \) by one, contradicting the assumption that this diagram minimizes the number of crossings.

**Claim 4.** If one of \( V[v_i, v_j], V[v_k, v_j] \) and \( V[v_i, v_k] \) is non-edge, then \( G \) has a subgraph isomorphic to one of the configurations \( G_1, G_2 \) and \( G_3 \).

Suppose first that \( V[v_i, v_j] \) is a non-edge. By **Claims 1–3**, both \( V[v_k, v_j] \) and \( V[v_i, v_j] \) are paths with \( 1 \leq j - k \leq 2 \) and \( 1 \leq l - j \leq 2 \). If \( j - k = 2 \) and \( v_k v_j \in E(G) \), then it is clear that \( d(v_k) = 3 \) and \( d(v_{k+1}) = 2 \), implying that the configuration \( G_2 \) occurs in \( G \). If \( j - k = 2 \) and \( v_k v_j \notin E(G) \), then \( d(v_k) = d(v_{k+1}) = 2 \), implying that the configuration \( G_1 \) occurs in \( G \). Hence we may assume that \( j + 1 \). If \( j = 2 \), then \( d(v_{j+1}) = 2 \) whenever \( v_j v_l \) is a chord or not. In this case, the configuration \( G_3 \) occurs in \( G \), since \( d(v_k) = 2 \), and moreover, \( G + v_j v_l \) is still pseudo-outplanar if \( v_j v_l \notin E(G) \). Thus, we shall assume that \( l = j + 1 \). Since \( v_k, v_j, v_l \) form a triangle satisfying \( d(v_k) = 2 \) and \( d(v_j) = 3 \), the configuration \( G_2 \) occurs in \( G \). The case when \( V[v_k, v_j] \) is a non-edge can be dealt with similarly, so we omit it here.

Second, suppose that \( V[v_k, v_j] \) is a non-edge. By **Claims 1–3**, both \( V[v_i, v_k] \) and \( V[v_j, v_k] \) are paths with \( 1 \leq k - i \leq 2 \) and \( 1 \leq l - j \leq 2 \). If \( k - i = 2 \) or \( l - j = 2 \), then by a similar argument as before, we either have \( d(v_{k-1}) = d(v_k) = 2 \) or \( d(v_{j+1}) = d(v_j) = 2 \), implying that the configuration \( G_1 \) occurs in \( G \). If \( k - i = l - j = 1 \), then the four vertices \( v_i, v_j, v_k \) and \( v_l \) form a quadrilateral with \( d(v_i) = d(v_k) = 2 \), which implies that the configuration \( G_3 \) occurs in \( G \), and moreover, \( G + v_i v_j \) is still pseudo-outplanar if \( V[v_i, v_j] \notin E(G) \).

In the following, we assume that \( V[v_i, v_k], V[v_k, v_j] \) and \( V[v_i, v_j] \) are all paths, where \( \max[k - i, j - k, l - j] \leq 2 \). Set \( X = C[v_i, v_k] \setminus \{v_i v_j, v_k v_l\} \) and let \( x = |X| \). It is clear that \( x \leq 3 \).

**Claim 5.** If \( x = 0 \), then \( G \) has a subgraph isomorphic to one of the configurations \( G_6-G_{11} \), if \( x = 1 \), then \( G \) has a subgraph isomorphic one of the configurations \( G_{12}, G_{13}, G_{14} \), if \( x = 2 \), then \( G \) has a subgraph isomorphic to one of the configurations \( G_5, G_{15}, G_{16} \) and if \( x = 3 \), then \( G \) has a subgraph isomorphic to the configuration \( G_{17} \).

We prove for the case when \( x = 2 \) and \( v_k v_j, v_j v_i \in X \) for example, and leave the discussions on other cases to the readers, since they are quite similar. In fact, if \( k - i = 1 \) (resp. \( k - i = 2 \)), then the configuration \( G_{15} \) (resp. \( G_5 \)) would occur in \( G \), since \( d(v_k) = 4 \) and \( d(v_{i+1}) = d(v_{j+1}) = d(v_{j+i+1}) = 2 \), and moreover, the drawing of the configuration \( G_{15} \) (resp. \( G_5 \)) in the figure is just a part of the diagram of \( G \) with its bending edges corresponding to the chords.

**Claims 1–5** complete the proof of this theorem for the case when \( G \) is 2-connected. We now suppose that \( G \) is a counterexample that has at least two blocks. Let \( B \) be an end block and let \( v_1, v_2, \ldots, v_{|B|} \) be the vertices of \( B \) that lies in a clockwise sequence. Without loss of generality, let \( v_1 \) be the unique cut vertex of \( B \).

**Claim 6.** \( B \) is an outerplanar graph.

We prove that there is no crossing in \( B \). Suppose, to the contrary, that there is a chord \( v_i v_j \) that crosses another chord \( v_k v_l \), where \( 1 \leq i < k < j < l \). Note that the chord \( v_i v_j \) satisfies (1) and (2) now. If it does not fulfill (3) at this stage, then there must be at least one pair of mutually crossed chords contained in either \( C[v_i, v_k], C[v_k, v_l], \) or \( C[v_j, v_l] \). We choose one pair \( v_k v_l \) and \( v_i v_j \) among them, where \( a < c \leq b < d \), such that there is no other crossed chord in \( C[v_i, v_k] \) besides \( v_k v_l \) and \( v_i v_j \), and then set \( i := a, j := b, k := c \) and \( l := d \). In any case, we can find a pair of mutually crossed chords.
Claim 7. $B$ has a subgraph isomorphic to one of the configurations $G_1, G_2$ and $G_4$ in such a way that $v_1$ is not a solid vertex.

Since $B$ is a 2-connected outerplanar graph, $B$ is Hamiltonian and $\mathcal{V}[v_1, v_2]$ is a path. The proof of Claim 2 implies that if $\mathcal{V}[v_i, v_j], \text{ where } k - i \geq 3$, is a path such that there is no cross edge in $C[v_i, v_j]$ and no edge between $\mathcal{V}[v_i, v_k]$ and $\mathcal{V}[v_k, v_j]$, then $G$ contains one of $G_1, G_2$ and $G_4$ in such a way that $v_1$ and $v_k$ are not the solid vertices. In this claim, if $|B| \geq 4$, then we set $i := 1, k := |B|$ and come back to the proof of Claim 2. If $|B| \leq 3$, then it is trivial to see that $G_4$ would appear in $G$. This contradiction completes the proof of the theorem for the case when $G$ has cut vertices. $
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The following is a straightforward corollary of Theorem 4.2.

Corollary 4.3. Each pseudo-outerplanar graph contains a vertex of degree at most 3.

5. Edge Chromatic Number and Linear Arboricity

In this section, we consider the problems of covering a pseudo-outerplanar graph $G$ with matchings or linear forests. A graph $G$ is $\chi'$-critical if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) \leq \Delta(G)$ for any proper subgraph $H \subseteq G$, and is $\chi$-critical if $\chi(G) > \lceil \Delta(G)/2 \rceil$ and $\chi(H) \leq \lceil \Delta(G)/2 \rceil$ for any proper subgraph $H \subseteq G$.

Lemma 5.1. If $G$ is $\chi'$-critical and $uw \in E(G)$, then $d(u) + d(v) \geq \Delta(G) + 2$.

Lemma 5.2. If $G$ is $\chi$-critical and $uw \in E(G)$, then $d(u) + d(v) \geq 2\lceil \Delta(G)/2 \rceil + 2$.

The above two lemmas are very classic and useful; their proofs can be found in [3, 14], respectively. Given a coloring $\varphi$ of $G$, $\varphi(v)$ denotes the number of edges incident with $v$ colored by $y$. Let $C^i_y(v) := \{|v \in C^i_y(v) = i\}$ for $i = 0, 1, 2$. If $y$ is a proper edge $k$-coloring, then $C^0_y(v) \cup C^1_y(v) = \{1, 2, \ldots, k\}$; if $y$ is a tree $k$-coloring, then $C^0_y(v) \cup C^1_y(v) \cup C^2_y(v) = \{1, 2, \ldots, k\}$. For brevity, we use the notion $k$-coloring to replace the statements of proper edge $k$-coloring or tree $k$-coloring, and use the notion PO-graph to replace the statement of pseudo-outerplanar graph. For a graph $G$ and two distinct vertices $u, v \in V(G)$, denote by $G + xy$ the graph obtained from $G$ by adding an new edge $xy$ if $xy \notin E(G)$, or $G$ itself if $xy \in E(G)$.

Theorem 5.3. Let $G$ be a pseudo-outerplanar graph. If $\Delta(G) \geq 4$, then $\chi'(G) = \Delta(G)$.

Proof. Suppose, for a contradiction, that there exists a minimal (in terms of the size) pseudo-outerplanar graph $G$ with $\Delta(G) \geq 4$ that has no $\Delta(G)$-colorings. One can easily see that $G$ is 2-connected and $\chi'$-critical. By Theorem 4.2 and Lemma 5.1, $G$ contains at least one of the configurations $G_1, G_3, G_5, G_6, G_{12}, G_{13}, G_{16}$ and $G_{17}$. Set $S = \{1, 2, \ldots, \Delta(G)\}$.

If $G \supseteq G_3$, then the pseudo-outerplanar graph $G' = G \setminus \{u, v\}$ admits a $\Delta(G)$-coloring $\varphi$, by the minimality of $G$ (when $\Delta(G') = \Delta(G)$) or Vizing’s Theorem (when $\Delta(G') \leq \Delta(G) - 1$). Construct a $\Delta(G)$-coloring $\varphi$ of $G$ as follows. If $C^i_y(x) = C^i_y(y) := L$ (note that $|L| = \Delta(G) - 2$ by Lemma 5.1), let $\varphi(ux) = \varphi(uy) \in S \setminus L$ and $\varphi(uy) = \varphi(ux) \in S \setminus (L \cup \{\varphi(uw)\})$. If $C^i_y(x) \neq C^i_y(y)$, then $S \setminus C^i_y(x) \cap C^i_y(y) \neq \emptyset$, since $d(x) = d(y) = \Delta(G)$ by Lemma 5.1. Let $\varphi(ux) \in (S \setminus C^i_y(x)) \cap C^i_y(y)$, $\varphi(ux) \in S \setminus (C^i_y(x) \cup \{\varphi(uw)\})$, $\varphi(uy) \in S \setminus (C^i_y(y) \cup \{\varphi(ux)\})$, and $\varphi(uy) \in S \setminus (C^i_y(y) \cup \{\varphi(ux)\})$. In each case, we color the remaining edges of $G$ by the same colors used in $\varphi$. Thus, we have constructed a $\Delta(G)$-coloring of $G$ from the $\Delta(G)$-coloring of $G'$. In the discussions of the next seven cases, while constructing a coloring $\varphi$ of $G$, we only give the colorings for the edges in $E(G) \setminus E(G')$, since for every edge $e \in E(G) \cap E(G')$, we always let $\varphi(e) = \varphi(e)$.

If $G \supseteq G_4$, then we shall assume that $d(v) = d(u) = \Delta(G) = 4$ because of Lemma 5.1. By the minimality of $G$, the PO-graph $G' = G \setminus \{x, y, u\}$ admits a 4-coloring $\varphi$. Construct a 4-coloring $\varphi$ of $G$ as follows, where two cases are considered without loss of generality (wlog. for short). If $C^i_y(v) = C^i_y(w) = \{1, 2\}$, then let $\varphi(uy) = 1, \varphi(ux) = 2, \varphi(ux) = \varphi(uy) = 3$, and $\varphi(uy) = \varphi(uy) = 4$. If $C^i_y(v) = \{1, 2\}$, then let $\varphi(uy) = 1, \varphi(ux) = 2, \varphi(ux) = \varphi(uy) = 3$, $\varphi(uy) = 4$, and $\varphi(uy) \in \{2, 3, 4\} \setminus C^i_y(w)$. If $G \supseteq G_3$, then we shall assume that $\varphi(uy) = \Delta(G) = 4$ because of Lemma 5.1. By the minimality of $G$, the PO-graph $G' = G \setminus \{u, v\}$ admits a 4-coloring $\varphi$. One can easily see that $(C^i_y(v) \cup C^i_y(w)) \setminus \{\varphi(uw)\} \neq \emptyset$, because otherwise $uw$ would be incident with four colors under $\varphi$. Assume that $C^i_y(v) = \{1, 2, 3\}$ and $\varphi(uu) = 3$ wlog. If $C^i_y(w) \neq C^i_y(v)$, then assume that $C^i_y(v) = \{1, 3, 4\}$ wlog. Whereafter, we extend $\varphi$ to a 4-coloring of $G$ by taking $\varphi(uu) = 4$ and $\varphi(uw) = 2$. If $C^i_y(w) = C^i_y(v)$, then we consider two subcases. If $\varphi(xz) = 4$, then construct a 4-coloring of $G$ by recoloring $ux$ and $uw$ with...
3 and 4, and coloring $uv$ and $uw$ with 3 and 2, respectively. If $\phi(xz) \neq 4$, then construct a 4-coloring of $G$ by recoloring $ux$ with 4 and coloring $uv$ and $uw$ with 2 and 4, respectively.

If $G \supseteq G_0$, then we shall assume that $\min(d(x_0), d(y_0)) \geq 3$ and $\Delta(G) = 4$ by Lemma 5.1. Assume that $d(x_0) = d(y_0) = 4$. If $x_0 \not\in E(G)$, then let $N(x_0) = \{u, v, x_1, x_2\}$ and $N(y_0) = \{u, v, y_1, y_2\}$. Let $G' = G \setminus \{u, v, x_0, y_0\}$. By Theorem 4.2, the configuration $G_0$ is a part of the pseudo-outplanar diagram of $G$, so $G'$ is a PO-graph that admits a 4-coloring $\phi$ by the minimality of $G$. Set $M = \{\phi(x_0x_1), \phi(x_0x_2), \phi(y_0y_1), \phi(y_0y_2)\}$ and let $m = |M|$. Since the colors used in $\phi$ is at most four and $x_0, y_0 \in E(G)$, $m \leq 3$ (otherwise the edge $x_0y_0$ cannot be colored under $\phi$, because it is already incident with four colored edges). If $m = 3$, then assume that $\phi(x_0x_1) = \phi(y_0y_1) = 1$, $\phi(x_0x_2) = 2$ and $\phi(y_0y_2) = 3$. Now we extend $\phi$ to a 4-coloring $\psi$ of $G$ by taking $\psi(uv) = 1$, $\psi(vy) = 2$, $\psi(ux) = 3$, and $\psi(uy) = 4$. If $m \leq 2$, then assume that $\phi(x_0x_1) = \phi(y_0y_1) = 1$ and $\phi(x_0x_2) = \phi(y_0y_2) = 2$. Now we extend $\phi$ to a 4-coloring $\psi$ of $G$ by taking $\psi(uv) = 1$, $\psi(vy) = \psi(ux) = 3$, and $\psi(uy) = 4$. On the other hand, if $x_0 \not\in E(G)$, then let $N(x_0) = \{u, v, x_0, y_1\}$ and $N(y_0) = \{u, v, y_0, x_1\}$. One can see that $x_1 \neq y_1$, because otherwise we have $G \supseteq G_0/\{u, v, x_0, y_0, x_1\}$, by the 2-connectivity of $G$, and thus $G$ can be 4-colorable. Consider the graph $G' = G \setminus \{u, v\}$, which admits a 4-coloring $\phi$ by the minimality of $G$. If $\phi(x_0x_1) = \phi(y_0y_1) = 1$, then let $\psi(uv) = 1$, $\psi(x_0y_0) = 2$, $\psi(x_0u) = \psi(x_0y) = 3$, and $\psi(vy) = \psi(uy) = 4$. If $\phi(x_0y_0) = 1$ and $\phi(y_0y_1) = 2$, then let $\psi(uv) = 1$, $\psi(x_0y_0) = 2$, $\psi(uv) = \psi(x_0y) = 3$, and $\psi(vy) = \psi(uy) = 4$. Second, assume that $\phi(x_0y_0) = 3$ wlog. Let $N(x_0) = \{u, v, w\}$. Consider the PO-graph $G' = G - x_0y_0$. By the minimality of $G$, $G'$ has a 4-coloring $\phi$. If $A := S \setminus \{\phi(xu), \phi(xw), \phi(yu), \phi(yw)\} \not= \emptyset$ (recall that $S = \{1, 2, 3, 4\}$), then let $\phi(xu) \in A$. Otherwise, assume that $\phi(xu) = 1$, $\phi(xw) = 2$, $\phi(yy) = 3$, and $\phi(yw) = \psi(uy) = 4$. If $x \in E(G)$, then let $N(x) = \{u, v, x_0, y_1\}$ and $N(y) = \{u, v, x_0, y_1\}$. We shall also assume that $x_1 \neq y_1$, because otherwise $G \supseteq G_1/\{u, v, x, y, x_1, y_1\}$, by the 2-connectivity of $G$, and thus $G$ admits a 4-coloring. Now we remove $u, v$ and $w$ from the diagram of $G$. Denote by $G''$ the resulting diagram. One can see that $G''$ is a PO-graph with $x$ and $y$ being of degree 2 in $G''$. Since the diagram of $G$ minimizes the number of crossings, $x_1$ does not cross $yy_1$ in $G$ (and thus in $G''$). Denote by $G'$ the graph obtained from $G''$ by contracting the edge $xy$. From the above arguments, one can see that $G'$ is a PO-graph with $E(G') \subseteq \{uu, uv, uw, vx, vx, vy, uy, wy, xy\}$. Furthermore, by the minimality of $G'$, $G'$ admits a 4-coloring $\phi$ with $\phi(xz) \neq \phi(yu)$. Suppose that $\phi(xz) = 1$ and $\phi(yu) = 2$. We construct a 4-coloring $\psi$ of $G$ by taking $\psi(uw) = \psi(vy) = 1$, $\psi(uv) = \psi(ux) = 2$, $\psi(uy) = \psi(ux) = 3$, and $\psi(vw) = \psi(ux) = 4$. If $x \in E(G)$, then let $N(x) = \{u, v, x_0, y_1\}$. Consider the PO-graph $G' = G - uv$, which admits a 4-coloring $\phi$ by the minimality of $G$. If $A := S \setminus \{\phi(uw), \phi(vy), \phi(vy), \phi(ux)\} \not= \emptyset$ (recall that $S = \{1, 2, 3, 4\}$), then let $\phi(ux) \in A$. Otherwise, assume that $\phi(uw) = 1$, $\phi(uv) = 2$, $\phi(vy) = 3$, and $\phi(ux) = 4$. Next it follows that $\phi(uw) = 3$ and $\phi(uy) = 4$. If $\phi(xz) = 1$, then we construct a 4-coloring of $G$ by recoloring $ux$ and $uw$ with 2, recoloring $uw$ with 1 and coloring $uv$ with 4. If $\phi(xz) = 1$, then we construct a 4-coloring of $G$ by recoloring $ux$ with 1 and coloring $uw$ with 4.
and $C_1^k(y) = \{3, 4\} \text{ wlog.}$, then let $\varphi(vw) = \varphi(ay) = 1, \varphi(wz) = \varphi(vy) = 2, \varphi(vx) = \varphi(uw) = 3, \varphi(wx) = \varphi(av) = 4, \text{ and } \varphi(xz) = \varphi(uv) = \varphi(wy) = 5$. □

**Theorem 5.4.** For each integer $n \geq 1$, there exists a 2-connected pseudo-outer-planar graph $G$ with order $2n + 5$ and $\Delta(G) = 3$ so that $\chi'(G) = \Delta(G) + 1$.

**Proof.** Let $x_0, \ldots, x_n, y_0, \ldots, y_0v x_0$ be a cycle denoted by $C$. Add edges $x_i y$, for all $1 \leq i \leq n$ and add another two edges $x_0 v$ and $y_0 u$ to $C$. Denote the resulting graph by $P_n$ (see Fig. 4). One can easily check that $P_n$ is a 2-connected pseudo-outer-planar graph with $|P_n| = 2n + 5$ and $\Delta(P_n) = 3$. If $P_n$ has a 3-coloring $\phi$, then we shall have $\phi(x_0 v) = \phi(y_0 u)$ and $\phi(x_0 u) = \phi(y_0 v)$ (otherwise we cannot color $uv$ properly). Thereby we would deduce that $\phi(x_i x_{i+1}) = \phi(y_i y_{i+1})$ for all $0 \leq i \leq n - 1$ and $\phi(x_n u, y)$. This contradiction implies that $\chi'(P_n) = \Delta(P_n) + 1 = 4$. □

**Theorem 5.5.** Let $G$ be a pseudo-outer-planar graph. If $\Delta(G) = 3$ or $\Delta(G) \geq 5$, then $\lambda(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

**Proof.** Since Conjecture 1.1 has already been proved for planar graphs and every PO-graph is planar (cf. Section 1), this claim holds trivially when $\Delta(G)$ is odd. Hence in the following we may assume that $\Delta(G)$ is an even not less than 6. For brevity, we write $k = \frac{\Delta(G)}{2}$. Suppose, for a contradiction, that there exists a minimal (in terms of the size) pseudo-outer-planar graph $G$ that has no $k$-colorings. One can easily observe that $G$ is 2-connected and $k$-critical. By Theorem 4.2 and Lemma 5.2, $G$ contains the configuration $G_3$.

If $xy \not\in E(G)$, then by (b) of Theorem 4.2, $G' = G \setminus \{v\} + xy$ is still a PO-graph. By the minimality of $G$, $G'$ admits a $k$-coloring $\psi$. We now construct a $k$-coloring $\varphi$ of $G$ by taking $\varphi(vx) = \varphi(vy) = \varphi(xy)$ and $\varphi(e) = \varphi(e)$ for every $e \in E(G) \cap E(G')$.

If $xy \in E(G)$, then consider the PO-graph $G' = G \setminus \{v\}$, which has a $k$-coloring $\psi$ by the minimality of $G$. It is easy to see that $|C_1^\psi(x)| = |C_1^\psi(y)| = 1$, since $d(x) = d(y) = \Delta(G) = 2k$ by Lemma 5.2. We now construct a coloring $\varphi$ of $G$ by taking $\varphi(vx) \in C_1^\psi(x), \varphi(vy) \in C_1^\psi(y), \varphi(e) = \varphi(e)$ for every $e \in E(G) \cap E(G')$. If $C_1^\psi(x) \neq C_1^\psi(y)$, then it is easy to see that $\varphi$ is a $k$-coloring. If $C_1^\psi(x) = C_1^\psi(y)$, then $\varphi(vx) = \varphi(vy)$ and $\varphi$ is also a $k$-coloring unless $\varphi(xy) = \varphi(uv)$ or $\varphi(uv) = \varphi(ux)$ if $\psi(xy) = \varphi(ux)$, then we can exchange the colors on $ux$ and $uv$. One can easily check that the resulting coloring of $G$ is a $k$-coloring. If $\varphi(uv) = \varphi(ux)$, then we recolor $xy$ with $\varphi(ux)$ and recolor both $ux$ and $uy$ with $\varphi(xy)$. The resulting coloring of $G$ is also a $k$-coloring. □

**Theorem 5.6.** For each integer $m \geq 1$, there exists a 2-connected pseudo-outer-planar graph $G$ with order $10m + 5$ and $\Delta(G) = 4$ so that $\lambda(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$.

**Proof.** Let $z_1, \ldots, z_{2n} \subset T_0$ be a cycle denoted by $T_0$ and let $u_i v_i w_i u_i$ be a triangle denoted by $T_i$ for each $1 \leq i \leq n$. Assume that any two of $T_0, \ldots, T_n$ are vertex-disjoint. For each $1 \leq i \leq n$, add fours edges $v_i z_{2i}, v_i z_{2i-1}, w_i z_{2i-1}$ and $w_i z_{2i}$. Denote the resulting graphs by $Q_{m+1}$ (see Fig. 4). One can easily check that $Q_{m+1}$ is a 2-connected pseudo-outer-planar graph with $\Delta(Q_{m+1}) = 4$. Consider the graph $Q_{m+1}$, for $m \geq 1$. It is trivial that $|Q_{m+1}| = 10m + 5$ and $\lambda(Q_{m+1}) \leq 3$ by Lemma 5.2. If $Q_{m+1}$ has a 2-coloring $\phi$, then we shall have $\phi(z_2 z_{2i-2}) \neq 0$ for all $1 \leq i \leq 2m + 1$, where $z_0 = z_{2m+2}$ and $z_{4m+3} = z_1$ (otherwise we cannot properly color the set of edges $u_i v_i, v_i w_i, u_i u_i, v_i z_{2i} - 1, v_i z_{2i} - 1, v_i z_{2i-1}, z_{2i} - 1, w_i z_{2i}$ for some $i$). However, the size of the set $\{z_2 z_3, z_3 z_4, \ldots, z_{4m+1} z_1\}$ is $2m + 1$, which is odd, but there are only two colors that can be used in $\phi$. This contradiction implies that $\lambda(Q_{m+1}) = \lceil \frac{\Delta(Q_{m+1})}{2} \rceil + 1 = 3$. □

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