

(1, λ)-EMBEDDED GRAPHS AND THE ACYCLIC EDGE CHOOSABILITY

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ABSTRACT. A $(1, \lambda)$ -embedded graph is a graph that can be embedded on a surface with Euler characteristic λ so that each edge is crossed by at most one other edge. A graph G is called α -linear if there exists an integral constant β such that $e(G') \leq \alpha v(G') + \beta$ for each $G' \subseteq G$. In this paper, it is shown that every $(1, \lambda)$ -embedded graph G is 4-linear for all possible λ , and is acyclicly edge- $(3\Delta(G) + 70)$ -choosable for $\lambda = 1, 2$.

1. Introduction and basic definitions

In this paper, all graphs considered are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G , respectively. Let $e(G) = |E(G)|$ and $v(G) = |V(G)|$. Moreover, for an embedded graph G (i.e., a graph that can be embedded on a surface), by $F(G)$ we denote the face set of G . Let $f(G) = |F(G)|$. The girth $g(G)$ of a graph G is the length of the shortest cycle of G . A k -, k^+ - and k^- -vertex (or face) is a vertex (or face) of degree k , at least k and at most k , respectively. A graph G is called α -linear if there exists an integral constant β such that $e(G') \leq \alpha v(G') + \beta$ for each $G' \subseteq G$. Furthermore, if $\beta \geq 0$, then G is said to be α -nonnegative-linear; and if $\beta < 0$, then G is said to be α -negative-linear. For other undefined concepts we refer the reader to [3].

A mapping c from $E(G)$ to the sets of colors $\{1, \dots, k\}$ is called a proper edge- k -coloring of G provided that any two adjacent edges receive different colors. A proper edge- k -coloring c of G is called an acyclic edge- k -coloring of G if there are no bichromatic cycles in G under the coloring c . The smallest number of colors such that G has an acyclic edge coloring is called the *acyclic*

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edge chromatic number of G , denoted by $\chi'_a(G)$. A graph is said to be acyclic edge- f -choosable, whenever we give a list L_e of $f(e)$ colors to each edge $e \in E(G)$, there exists an acyclic edge- k -coloring of G , where each element is colored with a color from its own list. If $|L_e| = k$ for edge $e \in E(G)$, we say that G is acyclicly edge- k -choosable. The minimum integer k such that G is acyclicly edge- k -choosable is called the *acyclic edge choice number* of G , denoted by $\chi'_c(G)$.

Acyclic coloring problem introduced in [8] has been extensively studied in many papers. One of the famous conjectures on the acyclic edge chromatic number is due to Alon, Sudakov and Zaks [2]. They conjectured that $\chi'_a(G) \leq \Delta(G) + 2$ for any graph G . Alon et al. [1] proved that $\chi'_a(G) \leq 64\Delta(G)$ for any graph G by using probabilistic arguments. This bound for arbitrary graph was later improved to $16\Delta(G)$ by Molloy and Reed [9] and recently improved to $9.62\Delta(G)$ by Ndreca et al. [10]. In 2008, Fiedorowicz et al. [7] proved that $\chi'_a(G) \leq 2\Delta(G) + 29$ for each planar graph G by applying a combinatorial method. Nowadays, acyclic coloring problem has attracted more and more attention since Coleman et al. [4, 5] identified acyclic coloring as the model for computing a Hessian matrix via a substitution method. Thus to consider the acyclic coloring problems on some other special classes of graphs seems interesting.

A graph is called 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. The notion of 1-planar graph was introduced by Ringel [11] while trying to simultaneously color the vertices and faces of a planar graph such that any pair of adjacent/incident elements receive different colors. In fact, from a planar graph G , we can construct a 1-planar graph G' with its vertex set being $V(G) \cup F(G)$, and any two vertices of G' being adjacent if and only if their corresponding elements in G are adjacent or incident. Now we generalize this concept to $(1, \lambda)$ -embedded graph, namely, a graph that can be embedded on a surface S with Euler characteristic λ so that each edge is crossed by at most one other edge. Actually, a $(1, 2)$ -embedded graph is a 1-planar graph. It is shown in many papers such as [6] that $e(G) \leq 4v(G) - 8$ for every 1-planar graph G . Whereafter, to determine whether the number of edges in the class of $(1, \lambda)$ -embedded graphs is linear or not linear in the number of vertices for every $\lambda \leq 2$ might be interesting.

In this paper, we first investigate some structures of $(1, \lambda)$ -embedded graph G in Section 2 and then give a relationship among the three parameters $e(G)$, $v(G)$ and $g(G)$ of G , which implies that every $(1, \lambda)$ -embedded graph is 4-linear for any $\lambda \leq 2$. In Section 3, we will introduce a linear upper bound for the acyclic edge choice number of $(1, \lambda)$ -embedded graphs with special given λ .

2. The linearity of $(1, \lambda)$ -embedded graphs

Given a “good” graph G (i.e., one for which all intersecting edges intersect in a single point and arise from four distinct vertices), the crossing number,

denoted by $cr(G)$, is the minimum possible number of crossings with which the graph can be drawn.

Let G be a $(1, \lambda)$ -embedded graph. In the following we always assume that G has been embedded on a surface with Euler characteristic λ so that each edge is crossed by at most one other edge and the number of crossings of G in this embedding is minimum. Thus, G has exactly $cr(G)$ crossings. Sometimes we say such an embedding proper for convenience.

Theorem 2.1. *Let G be a $(1, \lambda)$ -embedded graph. Then $cr(G) \leq v(G) - \lambda$.*

Proof. Suppose G has been properly embedded on a surface with Euler characteristic λ . Then for each pair of edges ab, cd that cross each other at a crossing point s , their end vertices are pairwise distinct. For each such pair, we add new edges ac, cb, bd, da (if it does not exist originally) to close s , then arbitrarily delete one edge ab or cd from G . Denote the resulting graph by G^* and then we have $cr(G^*) = 0$. By Euler's formula $v(G^*) - e(G^*) + f(G^*) = \lambda$ and the well-known relation $\sum_{v \in V(G^*)} d_{V(G^*)}(v) = \sum_{f \in F(G^*)} d_{V(G^*)}(f) = 2e(G^*)$, $f(G^*) \leq 2v(G^*) - 2\lambda$. Since each crossing point s (note that s is not a real vertex in G) lies on a common boundary of two faces of G^* and each face of G^* is incident with at most one crossing point (recall the definition of G^*), we deduce that $2cr(G) \leq f(G^*)$. Since $v(G) = v(G^*)$, we have $cr(G) \leq \frac{f(G^*)}{2} \leq v(G^*) - \lambda = v(G) - \lambda$ in final. \square

Theorem 2.2. *Let G be a $(1, \lambda)$ -embedded graph with girth at least g . Then $e(G) \leq \frac{2g-2}{g-2}(v(G) - \lambda)$.*

Proof. Suppose G has been properly embedded on a surface with Euler characteristic λ . Now for each pair of edges ab, cd that cross each other, we arbitrarily delete one from G . Let G' be the resulting graph. One can easily see that $cr(G') = 0$. By Euler's formula $v(G') - e(G') + f(G') = \lambda$ and the relations $v(G') = v(G)$, $e(G') = e(G) - cr(G)$, we have

$$(2.1) \quad v(G) - e(G) + f(G') = v(G') - e(G') + f(G') - cr(G) = \lambda - cr(G)$$

and

$$(2.2) \quad \sum_{f \in F(G')} d_{G'}(f) = 2e(G') = 2(e(G) - cr(G)) \geq g(G')f(G') \geq g(G)f(G') \geq g \cdot f(G').$$

Now combine equations (2.1) and (2.2) together, we immediately have $e(G) \leq \frac{g}{g-2}(v(G) - \lambda) + cr(G) \leq \frac{2g-2}{g-2}(v(G) - \lambda)$ by Theorem 2.1. \square

By Theorem 2.2, the following two corollaries are natural.

Corollary 2.3. *Every $(1, \lambda)$ -embedded graph is 4-linear for any $\lambda \leq 2$.*

Corollary 2.4. *Every triangle-free $(1, \lambda)$ -embedded graph is 3-linear for any $\lambda \leq 2$.*

3. Acyclic edge choosability of $(1, \lambda)$ -embedded graphs

In this section we mainly investigate the acyclic edge choosability of $(1, \lambda)$ -embedded graphs with special given λ . In [7], Fiedorowicz et al. proved the following two results.

Theorem 3.1. *If G is a graph such that $e(G') \leq 2v(G') - 1$ for each $G' \subseteq G$, then $\chi'_a(G) \leq \Delta(G) + 6$.*

Theorem 3.2. *If G is a graph such that $e(G') \leq 3v(G') - 1$ for each $G' \subseteq G$, then $\chi'_a(G) \leq 2\Delta(G) + 29$.*

In fact, these two theorems respectively imply that the acyclic edge chromatic number of 2-negative-linear graph G is at most $\Delta(G) + 6$ and that the acyclic edge chromatic number of 3-negative-linear graph G is at most $2\Delta(G) + 29$.

Note that every triangle-free $(1, \lambda)$ -embedded graph is 3-negative-linear for any $1 \leq \lambda \leq 2$ by Theorem 2.2. Hence the following corollary is trivial.

Corollary 3.3. *Let G be a triangle-free $(1, \lambda)$ -embedded graph with $1 \leq \lambda \leq 2$. Then $\chi'_a(G) \leq 2\Delta(G) + 29$.*

The following main theorem in this section is dedicated to giving a linear upper bound for the acyclic edge choice number of 4-negative-linear graphs.

Theorem 3.4. *If G is a graph such that $e(G') \leq 4v(G') - 1$ for each $G' \subseteq G$, then $\chi'_c(G) \leq 3\Delta(G) + 70$.*

As an immediately corollary of Theorems 2.2 and 3.4, we have the following result.

Corollary 3.5. *Let G be a $(1, \lambda)$ -embedded graph with $1 \leq \lambda \leq 2$. Then $\chi'_c(G) \leq 3\Delta(G) + 70$.*

Before proving Theorem 3.4, we first show an useful structural lemma.

Lemma 3.6. *Let G be a graph such that $e(G) \leq 4v(G) - 1$ and $\delta(G) \geq 4$. Then at least one of the following configurations occurs in G :*

- (C1) a 4-vertex adjacent to a 19^- -vertex;
- (C2) a 5-vertex adjacent to two 19^- -vertices;
- (C3) a 6-vertex adjacent to four 19^- -vertices;
- (C4) a 7-vertex adjacent to six 19^- -vertices;
- (C5) a vertex v such that $20 \leq d(v) \leq 22$ and at least $d(v) - 3$ of its neighbors are 7^- -vertices;
- (C6) a vertex v such that $23 \leq d(v) \leq 25$ and at least $d(v) - 2$ of its neighbors are 7^- -vertices;
- (C7) a vertex v such that $26 \leq d(v) \leq 28$ and at least $d(v) - 1$ of its neighbors are 7^- -vertices;
- (C8) a vertex v such that $29 \leq d(v) \leq 31$ and all its neighbors are 7^- -vertices;
- (C9) a vertex v such that at least $d(v) - 7$ of its neighbors are 7^- -vertices and at least one of them is of degree 4.

Proof. Suppose, to the contrary, that none of the nine configurations occurs in G . We assign to each vertex v a charge $w(v) = d(v) - 8$, then $\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} (d(v) - 8) \leq -2$. In the following, we will reassign a new charge denoted by $w'(x)$ to each $x \in V(G)$ according to some discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$(3.1) \quad \sum_{v \in V(G)} w'(v) = \sum_{v \in V(G)} w(v) \leq -2.$$

We next show that $w'(v) \geq 0$ for each $v \in V(G)$, which leads to a desired contradiction. We say a vertex big (resp. small) if it is a 20^+ -vertex (resp. 7^- -vertex). The discharging rules are defined as follows.

(R1) Each big vertex gives 1 to each adjacent 4-vertex.

(R2) Each big vertex gives $\frac{3}{4}$ to each adjacent vertex of degree between 5 and 7.

Let v be a 4-vertex. Since (C1) does not occur, v is adjacent to four big vertices. So v totally receives 4 by (R1). This implies that $w'(v) = w(v) + 4 = d(v) - 4 = 0$. Similarly, we can also prove the nonnegativity of $w'(v)$ if v is a k -vertex with $5 \leq k \leq 7$. Let v be a k -vertex with $8 \leq k \leq 19$. Since v is not involved in the discharging rules, $w'(v) = w(v) = d(v) - 8 \geq 0$. Let v be a k -vertex with $20 \leq k \leq 22$. If v is adjacent to a 4-vertex, then v is adjacent to at most $d(v) - 8$ small vertices since (C9) does not occur. Since v sends each small vertex at most 1 by (R1) and (R2), $w'(v) \geq w(v) - (d(v) - 8) = 0$. If v is adjacent to no 4-vertices, then v sends each small vertex $\frac{3}{4}$ by (R2). Since (C5) does not occur either, v is adjacent at most $d(v) - 4$ small vertices. So $w'(v) \geq w(v) - \frac{3}{4}(d(v) - 4) = \frac{1}{4}(d(v) - 20) \geq 0$. By similar arguments as above, we can also respectively show the nonnegativity of $w'(v)$ if v is a k -vertex with $k \geq 23$. \square

Proof of Theorem 3.4. Let K stands for $3\Delta(G) + 70$. We prove the theorem by contradiction. Let G be a counterexample to the theorem with the number of edges as small as possible. So there exists a list assignment L of K colors such that G is not acyclicly edge- L -choosable. For each coloring c of G , we define $c(uv)$ to be the color of edge uv and set $C(u) = \{c(uv) \mid uv \in E(G)\}$ for each vertex u . For $W \subseteq V(G)$, set $C(W) = \bigcup_{w \in W} C(w)$. If $uv \in E(G)$, we let $W_G(v, u)$ stands for the set of neighbors w of v in G such that $c(vw) \in C(u)$. Now, we first prove that $\delta(G) \geq 4$.

Suppose that there is a 3-vertex $v \in V(G)$. Denote the three neighbors of v by x, y and z . Then by the minimality of G , the graph $H = G - vx$ is acyclicly edge- L -choosable. Let c be an acyclic edge coloring of H . We can extend c to uv by defining a list of available colors for uv as follows:

$$A(uv) = L(uv) \setminus (C(x) \cup C(y) \cup C(z)).$$

Since $|C(x)| \leq \Delta(G) - 1$, $|C(y)| \leq \Delta(G)$ and $|C(z)| \leq \Delta(G)$, we have $|A(uv)| \geq K - 3\Delta(G) + 1 > 0$. So we can color uv by a color in $A(uv) \subseteq L(uv)$, a

contradiction. Similarly, one can also prove the absences of 1-vertices and 2-vertices in G . Hence $\delta(G) \geq 4$. Then by Lemma 3.6, G contains at least one of the configurations (C1)-(C9). In the following, we only show that if one of the configurations (C4), (C5) and (C9) appears, then we would get a contradiction. That is because the proofs are similar and easier for another six cases.

Configuration (C4): Suppose that there is a 7-vertex v that is adjacent to six 19^- -vertices, say x_1, x_2, \dots, x_6 . Denote another one neighbor of v by x_7 . Then by the minimality of G , the graph $H = G - vx_7$ is acyclicly edge- L -choosable. Let c be an acyclic edge coloring of H . Suppose $c(vx_j) \notin C(x_7)$ for some $1 \leq j \leq 6$. Then we can extend c to vx_7 by defining a list of available colors for vx_7 as follows:

$$A(vx_7) = L(vx_7) \setminus \bigcup_{1 \leq i \neq j \leq 7} C(x_i).$$

Since $|C(x_i)| \leq \min\{19, \Delta(G)\}$ for every $1 \leq i \leq 6$ and $|C(x_7)| \leq \Delta(G) - 1$, we have $|A(vx_7)| \geq K - \min\{\Delta(G) + 94, 6\Delta(G) - 1\} = \max\{2\Delta(G) - 24, -3\Delta(G) + 69\} > 0$. So we can color vx_7 by a color in $A(vx_7) \subseteq L(vx_7)$, a contradiction. Thus we shall assume that $c(vx_j) \in C(x_7)$ for every $1 \leq j \leq 6$. This implies that $|\bigcup_{1 \leq i \leq 7} C(x_i)| \leq \Delta(G) - 1 + 6 \times \min\{19, \Delta(G)\} - 6 = \min\{\Delta(G) + 107, 7\Delta(G) - 7\}$. Now we extend c to vx_7 by defining a list of available colors for vx_7 as follows:

$$A(vx_7) = L(vx_7) \setminus \bigcup_{1 \leq i \leq 7} C(x_i).$$

Note that $|A(vx_7)| \geq K - \min\{\Delta(G) + 107, 7\Delta(G) - 7\} = \max\{2\Delta(G) - 37, -4\Delta(G) + 77\} > 0$. So we can again color vx_7 by a color in $A(vx_7) \subseteq L(vx_7)$, also a contradiction.

Configuration (C5): If there is a vertex v such that $20 \leq d(v) \leq 22$ and at least $d(v) - 3$ of its neighbors are 7^- -vertices. Without loss of generality, we assume that $d(v) = 22$ and that v have nineteen 7^- -neighbors. Denote another three neighbors of v by x, y and z . Choose one 7^- -neighbor, say u , of v . Without loss of generality, we assume that $d(u) = 7$. Then by the minimality of G , the graph $H = G - uv$ is acyclicly edge- L -choosable. Let c be an acyclic edge coloring of H . Then we can extend c to uv by defining a list of available colors for uv as follows:

$$A(uv) = L(uv) \setminus \{C(u) \cup C(v) \cup C(x) \cup C(y) \cup C(z) \cup C(W_H(v, u))\}.$$

Since c is an acyclic (and thus it is proper), $|W_H(v, u)| \leq d(u) - 1 = 6$. Since $|C(x)| \leq \Delta(G)$, $|C(y)| \leq \Delta(G)$, $|C(z)| \leq \Delta(G)$ and $|C(w)| \leq 7$ for each $w \in W_H(v, u)$, we have $|A(uv)| \leq K - (3\Delta(G) + 6 + 6 \times 6 + 21 - 9) > 0$. So we can color uv by a color in $A(uv) \subseteq L(uv)$, a contradiction.

Configuration (C9): If there is a vertex v such that at least $d(v) - 7$ of its neighbors are 7^- -vertices and at least one of them is of degree 4, say u . Denote another three neighbors of u by x, y and z . Let $C_1 = \{c(vw) \mid w \in N_H(v) \text{ and } d_H(w) > 7\}$ and $C_2 = \{c(vw) \mid w \in N_H(v) \text{ and } d_H(w) \leq 7\}$. Then

$|C_1| \leq 7$. By the minimality of G , the graph $H = G - uv$ is acyclicly edge- L -choosable. Let c be an acyclic edge coloring of H . Suppose $C(u) \cap C_1 \neq \emptyset$. Without loss of generality, we assume that $c(uv) \in C_1$. Now we erase the color of the edge uv from c and recolor it from the list defined as follows:

$$A(uv) = L(uv) \setminus \{C(x) \cup C(y) \cup C(z) \cup C_1\}.$$

Since $|C(x)| \leq \Delta(G)$, $|C(y)| \leq \Delta(G)$, $|C(z)| \leq \Delta(G)$ and $|C_1| \leq 7$, we have $|A(uv)| \geq K - (3\Delta(G) + 7) > 0$. Note that $A(uv)$ is just a sub-list of the original list given at the beginning of the proof and the new color of uv preserves the acyclicity of the coloring of H . So we can assume that $C(u) \cap C_1 = \emptyset$. In this case, we can extend c to the edge uv by defining a list of available colors for uv as follows:

$$A(uv) = L(uv) \setminus \{C(u) \cup C_1 \cup C_2 \cup C(W_H(v, u))\}.$$

Since $C(u) \cap C_1 = \emptyset$, we have $c(vw) \in C_2$ for each $w \in W_H(v, u)$ and thus $|C(W_H(v, u))| \leq 7d_H(v) = 21$. Since $|C_1 \cup C_2| = d_H(v) \leq \Delta(G) - 1$ and $|C(u)| = 3$, we have $|A(uv)| \geq K - (\Delta(G) + 23) > 0$. So we can color uv by a color in $A(uv) \subseteq L(uv)$. This contradiction completes the proof of Theorem 3.4. \square

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