On \( r\)-acyclic edge colorings of planar graphs

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**Abstract**

A proper edge coloring of \( G \) is \( r\)-acyclic if every cycle \( C \) contained in \( G \) is colored with at least \( \min(|C|, r) \) colors. The \( r\)-acyclic chromatic index of a graph, denoted by \( \chi'_r(G) \), is the minimum number of colors required to produce an \( r\)-acyclic edge coloring. In this paper, we study \( 4\)-acyclic edge colorings by proving that \( \chi'_4(G) \leq 37\Delta(G) \) for every planar graph, \( \chi'_4(G) \leq 2\Delta(G) - 4 \) for every series–parallel graph, and \( \chi'_4(G) \leq 2\Delta(G) \) for every outerplanar graph. Additionally, we prove that every planar graph with maximum degree at least 4 and girth at least 5 has \( \chi'_r(G) = \Delta(G) \) for every \( r \geq 4 \).

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. We use \( V(G) \), \( E(G) \), \( \delta(G) \) and \( \Delta(G) \) to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph \( G \). The order \( |G| \) of \( G \) is the value of \( |V(G)| \) and the size of \( G \) is the value of \( |E(G)| \). Let \( N_C(v) \) be the set of neighbors of \( v \) and let \( d_C(v) \) be the degree of \( v \) in \( G \). Sometimes we write \( N(v) \) and \( d(v) \) instead of \( N_C(v) \) and \( d_C(v) \) for short. A \( k \)-vertex is a vertex of degree \( k \). A \( k \)-cycle or a \( k^r \)-cycle is a cycle with length \( k \) or at least \( k \). For a subset \( U \subseteq V(G) \), \( G[U] \) denotes the subgraph induced by \( U \). For other undefined concepts we refer the readers to [4].

In this paper we aim to study generalized acyclic edge colorings of graphs. We begin by reviewing some separate concepts. A mapping \( c \) from \( E(G) \) to the set of colors \( \{1, \ldots , k\} \) is a proper edge \( k \)-coloring of \( G \) if any two adjacent edges receive different colors. A proper edge coloring of \( G \) is \( r\)-acyclic if every cycle \( C \) contained in \( G \) is colored with at least \( \min(|C|, r) \) colors. The \( r\)-acyclic chromatic index of a graph \( G \), denoted by \( \chi'_r(G) \), is the minimum number of colors required to produce an \( r\)-acyclic edge coloring. Clearly we have \( \chi'_r(G) \geq \Delta(G) \) for every graph \( G \).

A 2-acyclic edge coloring coincides with a proper edge coloring, thus by the well-known Vizing’s Theorem, \( \Delta(G) \leq \chi'_2(G) \leq \Delta(G) + 1 \) for every graph \( G \). For a planar graph \( G \), Sanders and Zhao [19], and Zhang [21], independently proved that \( \chi'_2(G) = \Delta(G) \) if \( \Delta(G) \geq 7 \). This bound of 7 for \( \Delta(G) \) can be improved to 3 (being sharp) if we restrict \( G \) to be a series–parallel graph [15].

A 3-acyclic edge coloring, which is also known as an acyclic edge coloring in the literature, was introduced by Alon et al. [1], who proved for every graph \( G \) that \( \chi'_3(G) \leq 64\Delta(G) \) using the Local lemma. This upper bound was later improved to \( 16\Delta(G) \) by Molloy and Reed [16] and to \( 9.62\Delta(G) \) by Ndreca et al. [18]. In 2001, Alon, Sudakov and Zaks conjectured a stronger bound of \( \Delta(G) + 2 \) for \( \chi'_3(G) \). For a planar graph \( G \), Fiedorowicz, Haluszczak and Narayanav [9] proved an upper
bound of \(2\Delta(G) + 29\) for \(\alpha'_r(G)\) which was later improved to \(\max(2\Delta(G) - 2, \Delta(G) + 22)\) by Hou et al. [13], to \(\Delta(G) + 22\) by Cohen, Havet and Müller [7], and to \(\Delta(G) + 12\) by Basavaraju et al. [3]. Hou et al. also proved that \(\alpha'_r(G) \leq \Delta(G) + 1\) for every series–parallel graph \(G\) [13] and \(\alpha'_r(G) = \Delta(G)\) for every outerplanar graph with maximum degree at least 5 [14].

In view of the results mentioned in the previous two paragraphs, we can see that \(\alpha'_r(G) = O(\Delta(G))\) for every \(r \leq 3\). However, the \(r\)-acyclic chromatic index of \(G\) may not be linear in \(\Delta(G)\) for some \(r \geq 4\). For example, the complete bipartite graph \(H := K_{m,n}\) satisfies \(\alpha'_r(H) = mn\) for every \(r \geq 4\), since every edge of \(G\) is contained in a 4-cycle and the edges in each 4-cycle should be colored all distinctly. In general, there exist \(\Delta\)-regular graphs \(G\) with \(\alpha'_r(G) \geq c_r \Delta^{(\Delta/2)}\), where \(c_r\) depends on \(r\) but is constant with respect to \(\Delta\), for \(r \geq 6\) by Greenhill and Pikhurko [12], and for \(r \geq 4\) by Aravind and Subramanian [2]. It is natural to guess that the graph with its \(r\)-acyclic chromatic index superlinear in \(\Delta(G)\) may have very special structures. The question of finding the classes of graphs with \(\alpha'_r(G) = \Delta(G)\) for some (or all) \(r \geq 4\) is also interesting. Until now, the only two known classes of graphs with linear bounds for the \(r\)-acyclic chromatic index are the random regular graphs [10] and the graphs with large enough girth [11].

In this paper, we mainly work with 4-acyclic edge colorings by proving that \(\alpha'_4(G) \leq 37\Delta(G)\) for every planar graph, \(\alpha'_4(G) \leq \max(2\Delta(G), 3\Delta(G) - 4)\) for every series–parallel graph and \(\alpha'_4(G) \leq 2\Delta(G)\) for every outerplanar graph. Moreover, we show that the \(r\)-acyclic chromatic index of every planar graph with maximum degree at least \(r\) and girth at least \(5r + 1\) is exactly \(\Delta(G)\) for every \(r \geq 4\).

2. Planar graphs

A graph is planar if it can be drawn in the plane so that there are no crossed edges. It is well-known that each planar graph contains a vertex of degree at most 5. In [5], Borodin et al. established a more general structural theorem, which is quite useful for dealing with coloring problems for planar graphs. For example, Cohen et al. [7] have used this “configuration existence” theorem in combination with minimum counterexample arguments to show that \(\alpha'_r(G) \leq \Delta(G) + 22\) for every planar graph \(G\). Before stating this theorem, we recall two important definitions in [7,5].

A vertex \(v\) in a planar graph \(G\) is good if it is of degree at most 5 and there is a vertex \(w \in N(v)\) such that \(d(u) \leq 25\) for all \(u \in N(v) \setminus \{w\}\) and \(\sum_{u \in N(v) \setminus \{w\}} d(u) \leq 38\). We say \(G\) has a bunch of length \(m \geq 3\) with poles \(p\) and \(q\) if \(G\) contains a sequence of paths \(P_1, \ldots, P_m\), joining \(p\) with \(q\), which satisfy the following properties. First, each \(P_i\) has length 1 or 2. Furthermore, for each \(1 \leq i \leq m - 1\) the cycle formed by \(P_i\) and \(P_{i+1}\) is not separating in \(G\), i.e., there is no vertex of \(G\) inside this cycle. Moreover, this sequence of paths is maximal in the sense that there is no path \(P_0\) or \(P_{m+1}\) that can be added to the bunch preserving the above properties. A path \(P_i = pq\) of length 1 in the bunch will be referred to as a parental edge. If a path \(P_i = pq\) in the bunch has length 2, then the middle vertex is called bunch vertex. If the cycle bounded by \(P_i\) and \(P_{m}\) separates \(G\), then the vertices \(v_1\) and \(v_m\) (if they exist) are called end vertices of the bunch. The vertex \(v_i\) in the bunch is interior if \(2 \leq i \leq m - 1\) and strictly interior if \(3 \leq i \leq m - 2\). Note that each strictly interior vertex has degree 2, 3 or 4 and is adjacent to the poles and possibly to one or two interior vertices, whose degrees are at most 4. For an example of a bunch, see Fig. 1.

The “configuration existence” theorem of Borodin et al. is as follows.

**Theorem 1** ([5]). Every planar graph contains either a good vertex or a vertex \(v\) of degree \(d(v) \geq 26\) that is a pole for a bunch of length at least \(d(v)/5\).

We say two bunches are internally disjoint if they do not share any bunch vertices. The following corollary of Theorem 1 is due to Cohen et al. [7].

**Theorem 2** ([7]). Every planar graph contains either a good vertex or a vertex \(v\) that is a pole for \(1 \leq k \leq 6\) internally disjoint bunches of length at least 6 and has at most \(24 - 4k\) neighbors that do not belong to these bunches.

In what follows, we use Theorem 2 to prove one of the main results in this paper.

**Theorem 3.** The 4-acyclic chromatic index of every planar graph \(G\) is at most \(37\Delta(G)\).

**Proof.** Suppose that \(G\) is a counterexample to the theorem with the smallest number of vertices. First of all, one can verify that \(\delta(G) \geq 2\), because otherwise there is a 1-vertex \(v\) in \(G\) and a 4-acyclic edge coloring of \(G - v\) can be easily extended to \(G\) by coloring the edge incident with \(v\) properly. By Theorem 2, we split our proofs into two cases.

First, we assume that \(G\) contains a good vertex \(v\) with its neighbors \(v_1, \ldots, v_d\) satisfying \(\max_{2 \leq i \leq d} d(v_i) \leq 25\) and \(\sum_{i=2}^d d(v_i) \leq 38\), where \(2 \leq d := d(v) \leq 5\). By the minimality of \(G\), \(G' = G - v\) admits a 4-acyclic edge coloring \(\phi\) using 37 \(\Delta\) colors (here and in the following, we use \(\Delta\) instead of \(\Delta(G)\) for brevity). Let \(C\) be the color set of \(\phi\). For a vertex \(u\) in \(G'\) define \(\Phi(u) = \{\phi(ww) \mid w \in N_C(u)\}\) and for a vertex set \(S \in V(G')\) define \(\Phi(S) = \bigcup_{v \in S} \Phi(v)\). We first color \(vv_1\) by a color \(\phi(vv_1) \in C \setminus F(vv_1)\), where

\[
F(vv_1) = \bigcup_{i=2}^d \Phi(N_C(v_i)) \cup \Phi(v_1).
\]
Since $\sum_{i=1}^{d} d_c(v_i) \leq 38$ and $\max_{2 \leq i \leq d} d(v_i) \leq 25$, $|F(vv_i)| \leq (38 - d + 1)\Delta + \Delta - 1 \leq 37\Delta - 1$ if $d \geq 3$ and $|F(vv_i)| \leq (25 - 1)\Delta + \Delta - 1 \leq 26\Delta - 1 \leq 37\Delta - 1$ if $d = 2$. Thus $|C \setminus F(vv_i)| \geq 1$ and hence the coloring of $vv_i$ chosen above does exist. We now color $vv_2, \ldots, vv_d$ greedily by assigning to $vv_i$ ($2 \leq i \leq d$) a color $\phi(vv_i) \in C \setminus F(vv_i)$, where

$$F(vv_i) = \bigcup_{i=2}^{d} \Phi(N_C(v_i)) \cup \bigcup_{j=1}^{i-1} \phi(vv_j)).$$

One can also verify that $|F(vv_i)| \leq (38 - d + 1)\Delta + i - 1 \leq 36\Delta - 1$ here. Thus $|C \setminus F(vv_i)| \geq \Delta - (i - 1) \geq 1$ since $\Delta \geq d_c(v) \geq i$, and the edges $vv_2, \ldots, vv_d$ can be colored properly. Since $\bigcup_{i=1}^{d} \phi(vv_i) \cap \bigcup_{i=2}^{d} \Phi(N_C(v_i)) = \emptyset$, every cycle $C$ through $v$ in $G$ must be colored with at least $\min(|C|, 4)$ colors under the extended coloring $\phi$ of $G$. Hence $\phi$ is exactly a 4-acyclic edge coloring of $G$.

Since $\Delta \geq d_c(v) \geq i$, and the edges $vv_2, \ldots, vv_d$ can be colored properly. Since $\bigcup_{i=1}^{d} \phi(vv_i) \cap \bigcup_{i=2}^{d} \Phi(N_C(v_i)) = \emptyset$, every cycle $C$ through $v$ in $G$ must be colored with at least $\min(|C|, 4)$ colors under the extended coloring $\phi$ of $G$. Hence $\phi$ is exactly a 4-acyclic edge coloring of $G$.

Second, we assume that $G$ contains a vertex $v$ that is a pole for $1 \leq k \leq 6$ internally disjoint bunches $B_1, \ldots, B_k$ of length at least 6 and has at most $24 - 4k$ neighbors that do not belong to these bunches. For each bunch $B_i$ incident with $v$, denote the other pole of it by $u_i$ and the end vertices of it by $x_i$ and $y_i$ (here note that $B_i$ may have only one end vertex). Let $S$ be the set of neighbors of $v$ that do not belong to these $k$ bunches. Now consider the bunch $B_1$. Since its length is at least 6, there is at least one strictly interior vertex, say $w$, contained in $B_1$. Without loss of generality, assume that $d(w) = 4$. This implies that $w$ has two interior vertices as its neighbors, say $w_1$ and $w_1$. By the minimality of $G$, $G' = G - w$ has a 4-acyclic edge coloring $\phi$ using $37$ colors. Now we arbitrarily choose four different colors from $A = C \setminus F$ to color the edges $ww, wu_1, wu_1$, and $wu_1$, where

$$F = \begin{cases} \Phi(u_1) \cup \Phi(N_C(w_1)) \cup \Phi(N_C(w_1)) \cup \Phi(S) \cup \bigcup_{i=2}^{k} \Phi(x_i, y_i, u_i)), & \text{if } k \geq 2; \\ \Phi(u_1) \cup \Phi(N_C(w_1)) \cup \Phi(N_C(w_1)) \cup \Phi(S), & \text{if } k = 1. \end{cases}$$

Since $d_C(w_1) \leq 3$, $d_C(w_1) \leq 3$, $|S| \leq 24 - 4k$ and $1 \leq k \leq 6$, we have $|F| \leq \Delta + 3\Delta + 3\Delta + 24 - 4k)\Delta + 3(k - 1)\Delta = (28 - k)\Delta \leq 27\Delta$. Thus, $|A| \geq 37\Delta - 27\Delta > 4$. This implies that the colorings of $ww, wu_1, wu_1$, and $wu_1$, as assigned above do exist. Now we check that the extended coloring of $G$ is still 4-acyclic. Since $A \cap \Phi(N_C(w_1)) \cup \Phi(N_C(w_1)) \cup \Phi(S) = \emptyset$, every $4^k$-cycle through the edge $wu_1$ or $wu_1$, or through the vertex in $S$, and every $4^k$-cycle contained in $B_1$ is colored with at least 4 colors under the final coloring. Moreover, if there is a $4^k$-cycle $C$ that goes through two bunches $B_1$ and $B_2$, then $C$ must walk through either the end vertices of $B_1$ or the other pole of $B_2$. But by the choice of $A$, we have assumed that $A \cap \bigcup_{i=2}^{k} \Phi(x_i, y_i, u_i)) = \emptyset$ when $k \geq 2$. This implies that such a cycle $C$ shall also be colored with at least 4 colors. Thus, every cycle of length at least 4 through $w$ in $G$ is colored with at least 4 colors under the final coloring. At last, it is easy to see that the edges of every 3-cycle in $G$ are colored all distinctly. So we have obtained a 4-acyclic edge coloring of $G$. This contradiction completes the proof of the theorem. \hfill \Box

Now we focus on planar graphs with large girth. Let us begin with a useful structural lemma that was proved by Nešetřil et al. [17].

**Lemma 4.** Every planar graph $G$ with minimum degree at least $2$ and girth at least $5r + 1$ contains an induced path $P = v_1 \cdots v_r$ such that each vertex $v_i$ has degree $2$ in $G$.

The remaining content of this section is dedicated to the following **Theorem 5.** Note that in this theorem the bound $r$ for $\Delta(G)$ cannot be weakened because every $r^+$-cycle in $G$ should be colored with at least $r$ colors under any $r$-acyclic edge coloring of $G$.

**Theorem 5.** Every planar graph $G$ with $g(G) \geq 5r + 1$ and $\Delta(G) \geq r$ has $a'_r(G) = \Delta(G)$ for every $r \geq 4$.

In order to prove the above theorem, we show a slightly stronger result as follows. Indeed, **Theorem 6** is only a technical strengthening of **Theorem 5**, without which we would get complications when considering a subgraph $G' \subset G$ such that $\Delta(G') < \Delta(G)$. Of course, the interesting case of **Theorem 6** is when $M = \Delta(G)$.

![Fig. 1. A bunch of length 8 with one parental edge pq. seven bunch vertices $v_1, v_2, v_3, v_5, v_6, v_7, v_8$, two end vertices $v_1, v_8$, five interior vertices $v_2, v_3, v_5, v_6, v_7$, three strictly interior vertices $v_3, v_5, v_6$.](https://example.com/f1.png)
Theorem 6. Every planar graph $G$ with $g(G) \geq 5r + 1$ and $\Delta(G) \leq M$ has $a'_r(G) \leq M$, where $M \geq r \geq 4$.

Proof. Suppose that $G$ is a counterexample to the theorem with the smallest number of vertices. By a similar argument as in Theorem 3, we have $\delta(G) \geq 2$. So by Lemma 4, $G$ contains an induced path $P = v_1 \cdots v_t$ such that each vertex $v_i$ has degree 2 in $G$. Denote the other neighbor of $v_1$ and $v_t$ by $v_0$ and $v_{t+1}$, respectively. By the minimality of $G$, $G' = G - \{v_2, \ldots, v_{t-2}\}$ has an $r$-acyclic edge coloring $\phi$ using $M$ colors. Let $C$ be the color set of $\phi$ and set $A = C \setminus \{\phi(v_0v_1), \phi(v_{t-1}v_t), \phi(v_tv_{t+1})\}$. If $\{(v_0v_1), \phi(v_0v_1), \phi(v_1v_{t+1})\} \neq \emptyset$, then $|A| = M - 3 \geq r - 3$. Now color the edge $v_{t-1}v_t$ with the color $\phi(v_0v_1)$ and color the remaining $r - 3$ uncolored edges with different colors in $A$. If $\{(v_0v_1), \phi(v_0v_1), \phi(v_1v_{t+1})\} \neq \emptyset$, then $|A| = M - 2 \geq r - 2$ and therefore we can color the $r - 2$ edges deleted from $G$ with different colors in $A$. In each case one can easily verify that every cycle through the path $P$ is colored with at least $r$ colors. Thus an $r$-acyclic edge coloring of $G$ using $M$ colors has been constructed. This contradiction completes the proof of the theorem.

3. Series–parallel graphs

A graph is series–parallel if it contains no subgraphs homeomorphic to $K_4$. Duffin [8] showed that a connected series–parallel graph can be obtained from a $K_4$ by repeatedly applying one of the following operation: insert a vertex into an edge, or duplicate an edge by a path of length 2. With this definition one can see that the connectivity of any series–parallel graph is at most two.

Given a 2-connected series–parallel graph $G$, let $\{u, v\}$ be a 2-vertex cut of $G$ and let $H$ be a component of $G - \{u, v\}$. Set $G[H, u] = G[V(H) \cup \{u\}]$, $G[H, v] = G[V(H) \cup \{v\}]$ and $G[H, u, v] = G[V(H) \cup \{u, v\}]$. We call $H$ a cycle-component of $G - \{u, v\}$ if either $G[H, u]$ or $G[H, v]$ contains a cycle; otherwise we call $H$ a path-component. It is easy to see that if $H$ is a path-component then $G[H, u, v]$ is a $u - v$ chord of $G$, that is, an induced $u - v$ path of $G$ with all of its internal vertices being of degree 2. A cycle component $H$ of $G - \{u, v\}$ is called an $\infty$-component if $G[H, u, v]$ has a unique cut vertex $w$ such that $G[H, u, v]$ consists of $u - w$ chords and $q - v$ chords, where $p \geq 2$ and $q \geq 1$. For our purposes the following structural lemma for 2-connected series–parallel graphs, which was proved by Wu and Wu [20], is a useful starting point.

Lemma 7 ([20]). Every 2-connected series–parallel graph $G$ with $\Delta(G) \geq 3$ contains a 2-vertex cut $\{u, v\}$ such that

1. every component of $G - \{u, v\}$ is path component, or
2. $G - \{u, v\}$ has at least one $\infty$-component and at most one component which is neither path-component nor $\infty$-component.

We now modify the above lemma to obtain the following one.

Lemma 8. If $G$ is a 2-connected series–parallel graph with $\Delta(G) \geq 3$ such that the neighbors of every 2-vertex contained in $G$ are adjacent, then either $G$ is isomorphic to the graph obtained from a complete bipartite graph $K_{p, \Delta(G) - 1}$ by adding an edge between the two vertices with maximum degree (see (a) of Fig. 2) or $G$ contains the configuration (b) of Fig. 2, where $\{u, v\}$ is a 2-vertex cut of $G$, $d(u) = p + q + 2$, $p \geq 2$ and $q \geq 1$.

Proof. By Lemma 7, there is a 2-vertex cut $\{u, v\}$ of $G$ such that either every component of $G - \{u, v\}$ is path component or $G - \{u, v\}$ has at least one $\infty$-component. Since the neighbors of every 2-vertex contained in $G$ are adjacent, $G$ contains no adjacent 2-vertices because otherwise there is a cut vertex in $G$. Thus, every $u - v$ chord contained in $G$ contains at most one 2-vertex. If the former case stated above occurs, then $G$ is isomorphic to the graph (a) of Fig. 2. If the latter case appears, then one can easily find an $\infty$-component as described in (b) of Fig. 2, where $p \geq 2$ and $q \geq 1$. Moreover, we must have $N(u) = \{x_1, \ldots, x_p, y_1, \ldots, y_q, u, v\}$ by the definition of $\infty$-component. Hence, $d(u) = p + q + 2$.

Now we apply Lemma 8 to prove the following theorem.

Theorem 9. The 4-acyclic chromatic index of every series–parallel graph $G$ is at most $\max\{2\Delta(G), 3\Delta(G) - 4\}$.

Proof. Suppose that $G$ is a counterexample to the theorem with the smallest value of $|V(G)| + |E(G)|$. One can easily see that $G$ is 2-connected. In the following, we assume that $\Delta(G) \geq 3$, since the case when $\Delta(G) = 2$ is very easy to verify. Let $u$ be a 2-vertex of $G$ and let $N(u) = \{v, w\}$. If $uw \notin E(G)$, then consider the graph $G' = G - u + uvw$, which has a 4-acyclic edge coloring $\phi$ by the minimality of $G$. Color $uvw$ with $\phi(uvw)$ and color $uvw$ with a color that is different from the used colors incident with $u$ and $w$ (note that there are at most $2\Delta(G) - 1$ used colors). One can check that such an extension of $\phi$ does exist and the resulting coloring of $G$ is still 4-acyclic, a contradiction. So we conclude that the neighbors of every 2-vertex contained in $G$ are adjacent. This implies that either $G$ is isomorphic to the graph (a) in Fig. 2 or $G$ contains the configuration (b) of Fig. 2 by Lemma 8. Note that the 4-acyclic chromatic index of the graph (a) is exactly $2\Delta(G) - 1$ and the maximum degree of the graph (b) is at least $p + q + 2 \geq 5$. We conclude that $a'_4(G) \leq 2\Delta(G)$ if $2 \leq \Delta(G) \leq 4$ and hence we will only consider the latter case that $G$ contains (b). By the minimality of $G$, $G' = G - x_1$ admits a 4-acyclic edge coloring $\phi$ using $3\Delta(G) - 4$ colors. In the following, we use $\Delta$ instead of $\Delta(G)$ for convenience and let $C$ be the color set of $\phi$. For any vertex $u \in V(G')$, define $\Phi(u) = \{\phi(uv) \mid v \in N_C(u)\}$ and for a vertex set $S \subseteq V(G')$ define $\Phi(S) = \bigcup_{u \in S} \Phi(u)$. We extend $\phi$ to $G$ by coloring $ux_1$ with $\phi(ux_1) \in C \setminus F_1$ and coloring $wx_1$ with $\phi(wx_1) \in C \setminus F_2$, where $F_1 = \Phi(u) \cup \Phi(v) \cup \bigcup_{i=1}^{p-3} \Phi(ux_i)$ and $F_2 = \Phi(v) \cup \Phi(w) \cup \bigcup_{i=1}^{q-3} \Phi(wx_i)$. Since $\Delta \geq d(w) = p + q + 2$, we have
This image contains a page from a document discussing graph theory, specifically focusing on 4-acyclic edge colorings and related theorems. The text includes mathematical proofs, theorems, and definitions relevant to the study of graphs. The page references several theorems, including Theorem 9, and introduces a conjecture for the 4-acyclic chromatic index of every series-parallel graph.

**Conjecture 10.** The 4-acyclic chromatic index of every series-parallel graph $G$ is at most $2\Delta(G)$.

**4. Outerplanar graphs**

A graph is outerplanar if it can be drawn in the plane so that all vertices lie on the outside face. In this section, we confirm Conjecture 10 for outerplanar graphs, a subclass of series-parallel graphs. For this, we need the following structural result.

**Lemma 11 ([6]).** Every outerplanar graph with minimum degree at least two contains one of the following configurations:

(a) two adjacent 2-vertices $u$ and $v$;
(b) a 3-cycle $uvw$ with $d(u) = 2$ and $d(v) = 3$;
(c) two intersecting 3-cycles $uvw$ and $xyz$ with $d(u) = d(x) = 2$ and $d(w) = 4$.

**Theorem 12.** The 4-acyclic chromatic index of every outerplanar graph $G$ is at most $2\Delta(G)$.

**Proof.** Suppose that $G$ is a counterexample to the theorem with the smallest number of vertices. It is easy to see that $\delta(G) \geq 2$. Thus by Lemma 11, $G$ contains one of the three configurations among (a)--(c). In the following arguments, we use $\Delta$ instead of $\Delta(G)$ for brevity.

Suppose that $G$ contains (a). Denote the other neighbor of $u$ and $v$ by $w$ and $z$, respectively. By the minimality of $G$, the graph $G' = G - u$ has a 4-acyclic edge coloring $\phi$ using $2\Delta$ colors. For a vertex $x$ in $G'$ define $\Phi(x) = \{\phi(xy) \mid y \in N_G(x)\}$ and for a vertex set $S \subseteq V(G')$ define $\Phi(S) = \bigcup_{x \in S} \Phi(x)$. Let $C$ be the color set of $\phi$ and let $A_1 = C \setminus (\Phi(w) \cup \Phi(z))$. Since $|A_1| \geq 2\Delta - (\Delta - 1 + \Delta) \geq 1$, we can color $uw$ with a color $\phi(uw) \in A_1$. Let $B_1 = C \setminus (\{\phi(uw)\} \cup \Phi(z))$. Since $|B_1| \geq 2\Delta - (1 + \Delta) = \Delta - 1 \geq 1$, we can color $uw$ with a color $\phi(uw) \in B_1$. Since $\phi(uw), \phi(uv) \cap \Phi(z) = \emptyset$, every $4^+$-cycle through $u$ would be colored with at least four colors. Thus a 4-acyclic edge coloring of $G$ has been constructed.

Suppose that $G$ contains (b). Denote the other neighbor of $v$ by $z$. By the minimality of $G$, $G' = G - u$ has a 4-acyclic edge coloring $\phi$ using $2\Delta$ colors. By $C$ we denote the color set of $\phi$. As in the previous case, we first color $uv$ with a color $\phi(uv) \in A_2 = C \setminus (\Phi(u) \cup \Phi(z)) \neq \emptyset$. Let $B_2 = C \setminus (\{\phi(uv), \phi(uw)\} \cup \Phi(z))$. Since $|B_2| \geq 2\Delta - 2 - \Delta = \Delta - 2 \geq 1$ (note that $\Delta \geq 3$), we can color $uv$ with a color $\phi(uv) \in B_2$. Similarly, we can verify that the extended coloring of $G$ is 4-acyclic.

Suppose that $G$ contains (c). We consider the graph $G' = G - u$, which admits a 4-acyclic edge coloring $\phi$ using $2\Delta$ colors by the minimality of $G$. Let $C$ be the color set of $\phi$ and let $A_3 = C \setminus (\Phi(v) \cup \Phi(y))$. As in the previous cases, we can prove that $|A_3| \geq 1$ and thus we can color $uv$ with a color $\phi(uv) \in A_3$. Let $B_3 = C \setminus (\{\phi(uv), \phi(vw), \phi(wx)\} \cup \Phi(y))$. Since $|B_3| \geq 2\Delta - 3 - \Delta = \Delta - 3 \geq 1$ (note that $\Delta \geq 4$), we can color $uv$ with a color $\phi(uv) \in B_3$. Since every $4^+$-cycle through $u$ would go through the vertex $v$ and $\phi(uv), \phi(vw) \cap \Phi(y) = \emptyset$, the resulting coloring of $G$ is still 4-acyclic. This completes the proof of the theorem. □

**5. Concluding remarks**

In this paper we have considered $r$-acyclic edge colorings of planar graphs, series-parallel graphs, and outerplanar graphs, however, here the special case $r = 4$ is the main topic. For these graph classes, we established various upper bounds on the 4-acyclic chromatic index, which are all linear in the maximum degree of the graph. Recalling the proofs of Theorems 3, 9, and 12, we respectively use three “configuration existence” theorems from the literature in combination

![Fig. 2. Two special configurations of a series-parallel graph.](image)
with minimum counterexample arguments. However, these arguments fail to give linear bounds for higher (fixed) values of $r$. Let us discuss why this is so.

Given a graph $G$ and a fixed integer $r$, our task is to establish a linear upper bound (in terms of $\Delta(G)$) for $a'_r(G)$. While trying to prove the validity of such a linear bound, we prefer to use minimum counterexample arguments. Actually, we first find (or prove) a "configuration existence" theorem and then prove that the configurations in that theorem are reducible. Recalling Theorem 2, Lemmas 8 and 11, one can see that the diameter of every configuration involved in those results is bounded by a constant that is independent of $r$. This makes it difficult to extend our results to a higher (fixed) $r$.

Indeed, if we attempt to prove the reducibility of a given configuration that has constant diameter, we would need to (completely or partially) remove this configuration from $G$ and denote the resulting graph by $G'$. By the minimality of $G$, $G'$ has an $r$-acyclic edge coloring $\phi$. The remaining work would then be to extend the coloring $\phi$ to $G$. Let $v_1 \cdots v_m$ be the longest path in the deleted configuration, where $m$ is a constant since the diameter of this configuration is a constant. Now imagine one unlucky case: there is a bicolored path of length $r - m + 1$ in $G'$ from $v_1$ to $v_m$ under the coloring $\phi$. In this case, we would have an $r$-cycle in $G$ that is colored by at most $m + 1$ colors no matter how we color the path $v_1 \cdots v_m$. But every $r$-cycle should be colored with at least $r$ colors if the coloring is $r$-acyclic. Therefore, it would be impossible to extend $\phi$ to an $r$-acyclic edge coloring of $G$. Recalling the proof of Theorem 6, where we have proved the reducibility of an induced path $P = v_1 \cdots v_r$ such that each vertex $v_i$ has degree 2 in $G$. This is possible because the configuration induced by $P$ has a diameter that is bounded in term of $r$.

To end this paper, we leave an interesting problem for further research.

**Problem 13.** For a fixed integer $r \geq 5$, are there linear upper bounds for the $r$-acyclic chromatic index in terms of the maximum degree for planar graph, or for series-parallel graph, or for outerplanar graph?

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