On edge colorings of 1-planar graphs without chordal 5-cycles*

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Abstract

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is proved that every 1-planar graph without chordal 5-cycles and with maximum degree $\Delta \geq 9$ is of class one. Meanwhile, we show that there exist class two 1-planar graphs with maximum degree Δ for each $\Delta \leq 7$.

Keywords: 1-planar graph; edge coloring; cycle; discharging.

1 Introduction

We consider only finite, simple and undirected graphs in this paper. For a plane graph G, we denote by $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$ the set of vertices, the set of edges, the set of faces, the minimum degree and the maximum degree of G, respectively. For an element $x \in V(G) \cup F(G)$, $d_G(x)$ denotes the degree of x in G. Throughout this paper, a k-, k^+- and k^--vertex (resp. face) is a vertex (resp. face) of degree k, at least k and at most k. A k-cycle is a cycle of length k. We call a cycle to be chordal if there is at least one chord contained in this cycle. For other undefined notations, we refer the readers to [5].

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. This notion of 1-planar graphs was introduced by Ringel [7] while trying to simultaneously color the vertices and faces of a planar graph G such that any pair of adjacent/incident elements receive different colors. The coloring problems of 1-planar graphs have been investigated in many papers such as [1, 2, 3, 4, 10, 11].

A graph is k edge-colorable if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The edge chromatic number

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of G, denoted by $\chi'(G)$, is the smallest integer k such that G is k edge-colorable. Vizing's theorem ([5], pp.251) states that the edge chromatic number $\chi'(G)$ of every nonempty graph G is either $\Delta(G)$ or $\Delta(G)+1$. Thus we can divide all graphs into two classes. A graph G is of class one if $\chi'(G) = \Delta(G)$ and is of class two if $\chi'(G) = \Delta(G) + 1$. Naturally, the problem of deciding whether a graph is of class one or class two is a major question in graph edge coloring.

For a 1-planar graph G, it was proved that G is of class one provided $\Delta(G) \geq 10$ [11] or $\Delta(G) \geq 7$ and G containing no 3-cycles [10]. In this paper, we consider 1-planar graph G without chordal 5-cycle by proving that G is of class one provided $\Delta(G) \geq 9$. Moreover, we show by examples that there exists class two 1-planar graph with maximum degree Δ for each $\Delta \leq 7$.

2 1-planar graph without chordal 5-cycles

Throughout this section, for any 1-planar graph G, we always assume that G has been embedded on a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. We call such an embedding of G 1-plane graph. The associated plane graph $G^\times$ of a 1-plane graph G is the plane graph obtained from G by turning all crossings of G into new 4-valent vertices. A vertex in $G^\times$ is called to be false if it is a new added vertex and is called to be true otherwise. We denote by $T(G)$ the set of true vertices and by $C(G)$ the set of false vertices in $G^\times$ If a face f in $G^\times$ is incident with at least one false vertex, then we call f a false face; otherwise we call f a true face. In [11], Zhang and Wu displayed some basic properties on a 1-plane graph G and its associated plane graph $G^\times$.

Lemma 1. (Zhang and Wu [11]) *Let G be a 1-plane graph and $G^\times$ be its associated plane graph. Then the following hold:*
(1) for any two false vertices u and v in $G^\times$, $uv \notin E(G^\times)$;
(2) for any 2-vertex v in G, v is incident with no false 3-faces in $G^\times$;
(3) for any 3-vertex v in G, either v is incident with at most one false 3-face or v is incident with two false 3-faces and one 5^+-face in $G^\times$;
(4) for any 4-vertex v in G, v is incident with at most three false 3-faces in $G^\times$;
(5) for any 5-vertex v in G, v is incident with at most four false 3-faces in $G^\times$.

Now we restrict G to be a 1-plane graph without chordal 5-cycles and prove the following lemma.

Lemma 2. *Let G be a 1-plane graph without chordal 5-cycles and $G^\times$ be its associated plane graph. Then the following hold:*
(1) for any 7-vertex v in G, v is incident with at most six 3-faces in $G^\times$;
(2) for any 8-vertex v in G, v is incident with at most six 3-faces in $G^\times$;
(3) for any 9-vertex v in G, v is incident with at most seven 3-faces in $G^\times$.

Proof. The proofs of the three results stated in this lemma are mutually similar. So we only prove (3) of them and leave another two to the readers. Suppose, to the contrary, that $d_G(v) = 9$ and v is incident with at least eight 3-faces in $G^{\times}$. Without loss of generality, assume $v_1, v_2, \cdots, v_9$ are neighbors of v in $G^{\times}$ lying in a clockwise order with respect to the drawing of G and vv_iv_{i+1} are false 3-faces in $G^{\times}$ for every $1 \le i \le 8$. Let $S = \{v_1, \cdots, v_9\}$. Since the number of false vertices containing in S can not exceed five by Lemma 1(1) (otherwise two adjacent false vertices would be found in $G^{\times}$), there are at least four true vertices in S. Without loss of generality, assume that $\{v_1, v_3, v_5, v_7, v_9\} \subseteq C(G)$ and $\{v_2, v_4, v_6, v_8\} \subseteq T(G)$. It follows that $\{v_2v_4, v_4v_6, v_6v_8, vv_2, vv_4, vv_6, vv_8\} \subseteq E(G)$. At this stage, the four true vertices v_2, v_4, v_6, v_8 along with the vertex v form a chordal 5-cycle in G. This contradiction completes the proof. $\square$

We call a graph G to be Δ-critical if $\Delta(G) = \Delta$, $\chi'(G) = \Delta + 1$ and $\chi'(G - e) = \Delta$ for every $e \in E(G)$. For our purposes the following lemma, which is known as Vizing's Adjacency Lemma, is an useful starting point.

Lemma 3. (Vizing [9]) *Let G be a Δ-critical graph and let v, w be adjacent vertices of G with $d_G(v) = k$. Then*
(1) if $k < \Delta$, then w is adjacent to at least $(\Delta - k + 1)$ Δ-vertices;
(2) if $k = \Delta$, then w is adjacent to at least two Δ-vertices.

Theorem 4. *Let G be a 1-planar graph without chordal 5-cycles. If $\Delta(G) \ge 9$, then $\chi'(G) = \Delta(G)$.*

Proof. By the result of Zhang and Wu [11], we should only prove the case when $\Delta(G) = 9$ here. Suppose, to the contrary, that G is a 9-critical 1-planar graph. Then by Lemma 3(1), the minimum degree of G is at least 2. Let v be a vertex in G. We denote the degree of the neighbors of x in G as $\delta_1(x) \le \delta_2(x) \le \cdots \le \delta_{d_G(x)}(x)$. Our proof of the theorem uses the discharging method. First of all, we assign an initial charge c on $V(G) \cup F(G^{\times})$ by $c(v) = d_G(v) - 4$ for every $v \in V(G)$ and $c(f) = d_{G^{\times}}(f) - 4$ for every $f \in F(G^{\times})$. Then by Euler's formula and the fact that $d_{G^{\times}}(v) = 4$ for every $v \in C(G)$, one can easily deduce that $\sum_{x \in V(G) \cup F(G^{\times})} c(x) = -8$. Let us now discharging along the following rules.

R1. Every 9-vertex sends $\frac{3}{2i-2}$ to each of its adjacent i-vertex in G, where $2 \le i \le 7$.
R2. Every 8-vertex sends $\frac{1}{4}$ to each of its adjacent 4-vertex, $\frac{1}{10}$ to each of its adjacent 5-vertex or 6-vertex in G.
R3. Every 7-vertex sends $\frac{1}{30}$ to each of its adjacent 6-vertex in G.
R4. Every 5^+-face sends $\frac{1}{2}$ to each of its incident 3-vertex in $G^{\times}$.
R5. Every true 3-face receives $\frac{1}{2}$ from each of its incident 6^+-vertex in $G^{\times}$.
R6. Every false 3-face receives $\frac{1}{2}$ from each of its incident true vertex in $G^{\times}$.

In the following, we check that the final charge c' on each vertex and each face is nonnegative. Since our rules only move charge around and do not affect the total charges, this leads to a contradiction that

$$-8 = \sum_{x \in V(G) \cup F(G^{\times})} c(x) = \sum_{x \in V(G) \cup F(G^{\times})} c'(x) \geq 0$$

in final and completes the proof.

By Lemma 3, it is easy to see $d_G(x) + d_G(y) \geq 11$ for every $xy \in E(G)$. This implies that every true 3-face in $G^{\times}$ is incident with at least two 6^+-vertices. On the other hand, by Lemma 1, one can also see that every false 3-face in $G^{\times}$ is incident with at least two true vertices. Thus by R5 and R6, $c'(f) = -1 + 2 \times \frac{1}{2} = 0$ for every 3-face in $G^{\times}$. Since 4-faces are not involved in the rules, $c'(f) = c(f) = 0$ for every 4-face in $G^{\times}$. Let f be a 5^+-face in $G^{\times}$. Then f can be incident with at most $\lfloor \frac{d_{G^{\times}}(f)}{2} \rfloor$ 3-vertices in $G^{\times}$, since any two 3-vertices are not adjacent in $G^{\times}$ by Lemma 3. This implies that $c'(f) \geq d_{G^{\times}}(f) - 4 - \frac{1}{2}\lfloor \frac{d_{G^{\times}}(f)}{2} \rfloor \geq 0$ for $d_{G^{\times}}(f) \geq 5$.

Now we focus on vertices in G. Let v be a k-vertex in G. If $k = 2$, then v must be adjacent to two 9-vertices in G by Lemma 3. Meanwhile, v is not incident with any false 3-face in $G^{\times}$ by Lemma 1(2). So $c'(v) = -2 + 2 \times \frac{3}{2} = 1$ by R1.

If $k = 3$, then by Lemma 3, v is adjacent to at least two 9-vertices in G. If v is incident with at most one false 3-face in $G^{\times}$, then by R1 and R6, $c'(v) \geq -1 - \frac{1}{2} + 2 \times \frac{3}{4} = 0$. If v is incident with at least two false 3-faces in $G^{\times}$, then by Lemma 1(3), v is also incident with one 5^+-face now. So by R1, R4 and R6, it also holds that $c'(v) \geq -1 - 2 \times \frac{1}{2} + 2 \times \frac{3}{4} + \frac{1}{2} = 0$.

If $k = 4$, then by Lemma 1(4), v is incident with at most three false 3-faces in $G^{\times}$. If $\delta_1(v) \geq 8$, then by R1, R2, R6 and Lemma 3, we have $c'(v) \geq 0 - 3 \times \frac{1}{2} + 2 \times \frac{1}{2} + 2 \times \frac{1}{4} = 0$. If $\delta_1(v) \leq 7$, then by Lemma 3, one can see that v must be adjacent to exactly three 9-vertices in G. This follows from R1 and R6 that $c'(v) \geq 0 - 3 \times \frac{1}{2} + 3 \times \frac{1}{2} = 0$.

If $k = 5$, then by Lemma 1(5), v is incident with at most four false 3-faces in $G^{\times}$. If $\delta_1(v) \geq 8$, then by R1, R2, R6 and Lemma 3, we have $c'(v) \geq 1 - 4 \times \frac{1}{2} + 2 \times \frac{3}{8} + 3 \times \frac{1}{10} > 0$. If $\delta_1(v) \leq 7$, then by Lemma 3, one can see that v is adjacent to at least three 9-vertices in G. Thus by R1 and R6, $c'(v) \geq 1 - 4 \times \frac{1}{2} + 3 \times \frac{3}{8} > 0$.

If $k = 6$, then by R5 and R6, v would send $\frac{1}{2}$ to each of its incident 3-faces in $G^{\times}$. If $\delta_1(v) \geq 7$, then by R1, R3, R5 and R6, $c'(v) \geq 2 - 6 \times \frac{1}{2} + \min\{3 \times \frac{3}{10} + 3 \times \frac{1}{30}, 2 \times \frac{3}{10} + 4 \times \frac{1}{10}\} = 0$. If $\delta_1(v) \leq 6$, then by Lemma 3, one can see that v is adjacent to at least four 9-vertices in G. This implies by R1, R5 and R6 that $c'(v) \geq 2 - 6 \times \frac{1}{2} + 4 \times \frac{3}{10} > 0$.

If $k = 7$, then by Lemma 2(1), v is incident with at most six 3-faces. Meanwhile, by Lemma 3, v is adjacent to at least two 9-vertices and is adjacent to at most three 6-vertex in G. So by R1, R3, R5 and R6, $c'(v) \geq 3 - 6 \times \frac{1}{2} - 3 \times \frac{1}{30} + 2 \times \frac{1}{4} > 0$.

If $k = 8$, then by Lemma 2(2), v is incident with at most six 3-faces. So by R2, R5, R6 and Lemma 3, $c'(v) \geq 4 - 6 \times \frac{1}{2} - \max\{2 \times \frac{1}{4}, 4 \times \frac{1}{10}\} > 0$.

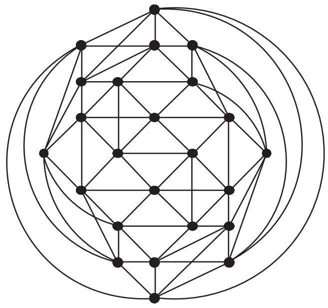

Figure 1: 6-regular 1-planar graph

If $k = 9$, then by Lemma 2(3), v is incident with at most seven 3-faces. Meanwhile, by Lemma 3, v is adjacent to at least $9 - \delta_1(v) + 1 = 10 - \delta_1(v)$ 9-vertices in G. Hence by R1, R5 and R6, we have $c'(v) \geq 5 - 7 \times \frac{1}{2} - \frac{3}{2\delta_1(v)-2}(9 - 10 + \delta_1(v)) = 0$ in final. □

3 1-planar graphs being of class two

This section is devoted to prove the existence of class two 1-planar graphs with small maximum degree. In [8], Vizing presented examples of planar graphs of class two with maximum degree no more than five. It is known that every planar graph is also 1-planar. Therefore, we conclude that there are 1-planar graphs of class two with maximum degree Δ for each $\Delta \leq 5$. In the following, we are going to construct class two 1-planar graphs with maximum degree 6 or 7 based on the following lemma, which is an useful sufficient condition for a graph being of class two.

Lemma 5. ([5], pp.258) *If G is a graph of size m such that $m > \alpha'(G)\Delta(G)$, where $\alpha'(G)$ is the edge independent number of G, then G is of class two.*

Theorem 6. *The graph G derived from a k-regular graph R with even order by adding an new 2-vertex on an arbitrary edge is of class two.*

Proof. Since $|V(G)| = |V(R)| + 1$ is odd, $\alpha'(G) \leq \frac{1}{2}(|V(G)| - 1)$. On the other hand, it is easy to calculate that $|E(G)| = |E(R)| + 1 = \frac{k}{2}|V(R)| + 1 = \frac{1}{2}\Delta(G)(|V(G)| - 1) + 1$. Hence $|E(G)| > \alpha'(G)\Delta(G)$, which implies that G is of class two by Lemma 5. □

One can check that the graph in Figure 1 is a 6-regular 1-planar graph with 24 vertices. In [6], Fabrici and Madaras also presented a 7-regular 1-planar graph with 24 vertices. Hence by Theorem 6, we directly have the following corollary.

Corollary 7. *There exist 1-planar graphs of class two with maximum degree* Δ *for each* $\Delta \leq 7$.

To end this paper, we leave a conjecture to the interested readers.

Conjecture 8. *Every 1-planar graph with maximum degree 8 or 9 is of class one.*

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