List \((d,1)\)-Total Labelling of Graphs
Embedded in Surfaces

YU Yong\(^{1}\)† ZHANG Xin\(^{1}\) LIU Guizhen\(^{1}\)

Abstract The \(\langle d,1\rangle\)-total labelling of graphs was introduced by Havet and Yu. In
this paper, we consider the list version of \(\langle d,1\rangle\)-total labelling of graphs. Let \(G\) be a
graph embedded in a surface with Euler characteristic \(\varepsilon\) whose maximum degree \(\Delta(G)\) is
sufficiently large. We prove that the list \(\langle d,1\rangle\)-total labelling number \(Ch_{d,1}^{T}(G)\) of \(G\) is at
most \(\Delta(G) + 2d\).

Keywords \(\langle d,1\rangle\)-total labelling, list \(\langle d,1\rangle\)-total labelling, list \(\langle d,1\rangle\)-total labelling
number, graphs

Chinese Library Classification O157.5
2010 Mathematics Subject Classification 05C15

关于可嵌入曲面图的列表 \((d, 1)\)- 全标号问题

于 永 \(^{1}\)† 张 欣 1 刘桂真 1

摘要 图的 \((d, 1)\)-全标号问题最初是由 Havet 等人提出的。在本文中，我们考虑了可嵌
入曲面图的列表 \((d, 1)\)-全标号问题，并证明了其列表 \((d, 1)\)-全标号数不超过 \(\Delta(G) + 2d\).

关键词 \((d, 1)\)-全标号，列表 \((d, 1)\)-全标号，列表 \((d, 1)\)-全标号数，图
中图分类号 O157.5
数学分类号 05C15

0 Introduction

In this paper, graph \(G\) is a simple connected graph with a finite vertex set \(V(G)\)
and a finite edge set \(E(G)\). If \(X\) is a set, we usually denote the cardinality of \(X\) by \(|X|\).
Denote the set of vertices adjacent to \(v\) by \(N(v)\). The degree of a vertex \(v\) in \(G\), denoted by
\(d_G(v)\), is the number of edges incident with \(v\). We sometimes write \(V, E, d(v), \Delta, \delta\) instead of
\(V(G), E(G), d_G(v), \Delta(G), \delta(G)\), respectively. Let \(G\) be a plane graph. We always denote
by \(F(G)\) the face set of \(G\). The degree of a face \(f\), denoted by \(d(f)\), is the number of edges
incident with it, where cut edge is counted twice. A \(k-\), \(k^+\)- and \(k^-\)-vertex (or face) in graph
\(G\) is a vertex (or face) of degree \(k\), at least \(k\) and at most \(k\), respectively.

\(^{*}\) Supported by GIIFSDU(yzc11025), NNSF(61070230, 11026184, 10901097) and RDFP (200804220001,
20100131120017) and SRF for ROCS.
1. School of Mathematics, Shandong University, Jinan 250100, China; 山东大学数学学院，济南 250100
\(^{†}\) 通讯作者  Corresponding author
The $(d,1)$-total labelling of graphs was introduced by Havet and Yu\cite{1}. A $k$-$(d,1)$-total labelling of a graph $G$ is a function $c$ from $V(G) \cup E(G)$ to the color set $\{0, 1, \cdots, k\}$ such that $c(u) \neq c(v)$ if $uv \in E(G)$, $c(e) \neq c(e')$ if $e$ and $e'$ are two adjacent edges, and $|c(u) - c(v)| \geq d$ if vertex $u$ is incident to the edge $e$. The minimum $k$ such that $G$ has a $k$-$(d,1)$-total labelling is called the $(d,1)$-total labelling number and denoted by $\lambda_d^T(G)$.

Readers are referred to \cite{2,4-7} for further research. Suppose that $L(x)$ is a list of colors available to choose for each element $x \in V(G) \cup E(G)$. If $G$ has a $(d,1)$-total labelling $c$ such that $c(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that $c$ is an $L$-$(d,1)$-total labelling of $G$, and $G$ is $L$-$(d,1)$-total labelable (sometimes we also say $G$ is list $(d,1)$-total labelable). Furthermore, if $G$ is $L$-$(d,1)$-total labelable for any $L$ with $|L(x)| = k$ for each $x \in V(G) \cup E(G)$, we say that $G$ is $k$-$(d,1)$-total choosable. The list $(d,1)$-total labelling number, denoted by $Ch^T_{d,1}(G)$, is the minimum $k$ such that $G$ is $k$-$(d,1)$-total choosable. Actually, when $d = 1$, the list $(1,1)$-total labelling is the well-known list total coloring of graphs. It is known that for list version of total colorings there is a list total coloring conjecture (LTCC). Therefore, it is natural to conjecture that $Ch^T_{d,1}(G) = \lambda_d^T(G) + 1$. Unfortunately, counterexamples that $Ch^T_{d,1}(G)$ is strictly greater than $\lambda_d^T(G) + 1$ can be found in \cite{9}. Although we can not present a conjecture like LTCC, we conjecture that

$$Ch^T_{d,1}(G) \leq \Delta + 2d$$

for any graph $G$. In \cite{9}, we studied the list $(d,1)$-total labelling of special graphs such as paths, trees, stars and outerplanar graphs which lend positive support to our conjecture.

In this paper, we prove that, for graphs embedded in a surface with Euler characteristic $\varepsilon$, the conjecture is still true when the maximum degree is sufficiently large. Our main results are the following:

**Theorem 0.1** Let $G$ be a graph embedded in a surface of Euler characteristic $\varepsilon \leq 0$ and

$$\Delta(G) \geq \frac{d}{2d - 1} \left(10d - 8 + \sqrt{(10d - 2)^2 - 24(2d - 1)\varepsilon}\right) + 1,$$

where $d \geq 2$. Then

$$Ch^T_{d,1}(G) \leq \Delta(G) + 2d.$$

**Theorem 0.2** Let $G$ be a graph embedded in a surface of Euler characteristic $\varepsilon > 0$. If $\Delta(G) \geq 5d + 2$ where $d \geq 2$, then

$$Ch^T_{d,1}(G) \leq \Delta(G) + 2d.$$

We prove two conclusions which are slightly stronger than the theorems above as follows.

**Theorem 0.3** Let $G$ be a graph embedded in a surface of Euler characteristic $\varepsilon \leq 0$ and let positive integer

$$M \geq \frac{d}{2d - 1} \left(10d - 8 + \sqrt{(10d - 2)^2 - 24(2d - 1)\varepsilon}\right) + 1,$$

where $d \geq 2$. If $\Delta(G) \leq M$, then

$$Ch^T_{d,1}(G) \leq M + 2d.$$
In particular,

\[ Ch_{d,1}^T(G) \leq \Delta(G) + 2d \quad \text{if} \quad \Delta(G) = M. \]

**Theorem 0.4** Let \( G \) be a graph embedded in a surface of Euler characteristic \( \varepsilon > 0 \) and let positive integer \( M \geq 5d + 2 \) where \( d \geq 2 \). If \( \Delta(G) \leq M \), then

\[ Ch_{d,1}^T(G) \leq M + 2d. \]

In particular,

\[ Ch_{d,1}^T(G) \leq \Delta(G) + 2d \quad \text{if} \quad \Delta(G) = M. \]

The interesting cases of Theorem 0.3 and Theorem 0.4 are when \( M = \Delta(G) \). Indeed, Theorem 0.3 and Theorem 0.4 are only technical strengthening of Theorem 0.1 and Theorem 0.2, respectively. But without them we would get complications when a subgraph \( H \subset G \) such that \( \Delta(H) < \Delta(G) \) is considered.

In Section 1, we prove some lemmas. In Section 2, we complete our main proof with discharging method.

## 1 Structural properties

From now on, we will use without distinction the terms *colors* and *labels*. Let \( c \) be a partial list \((d,1)\)-total labelling of \( G \). We denote by \( A(x) \) the set of colors which are still available for coloring element \( x \) of \( G \) with the partial list \((d,1)\)-total labelling \( c \). Let \( G \) be a minimal counterexample in terms of \( |V(G)| + |E(G)| \) to Theorem 0.3 or Theorem 0.4.

**Lemma 1.1** \( G \) is connected.

**Proof** Suppose that \( G \) is not connected. Without loss of generality, let \( G_1 \) be one component of \( G \) and \( G_2 = G \setminus G_1 \). By the minimality of \( G \), \( G_1 \) and \( G_2 \) are both \((M + 2d)-(d,1)\)-total choosable which implies \( G \) is \((M + 2d)-(d,1)\)-total choosable, a contradiction.

**Lemma 1.2** For each edge \( e = uv \in E(G) \),

\[ d(u) + d(v) \geq M - 2d + 4. \]

**Proof** Suppose to the contrary that there exists some edge \( e = uv \in E(G) \) such that

\[ d(u) + d(v) \leq M - 2d + 3. \]

By the minimality of \( G \), \( G - e \) is \((M + 2d)-(d,1)\)-total choosable. We denote this coloring by \( c \). Since

\[ |A(e)| \geq M + 2d - (d(u) + d(v) - 2) - 2(2d - 1) = M + 2d - (M - 2d + 1) - 2(2d - 1) \geq 1 \]

under the coloring \( c \), we can extend \( c \) to \( G \), a contradiction.

**Lemma 1.3** For any edge \( e = uv \in E(G) \) with

\[ \min\{d(u), d(v)\} \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor, \]
we have
\[ d(u) + d(v) \geq M + 3. \]

**Proof** Suppose there is some \( e = uv \in E(G) \) such that
\[ d(u) \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \]
and
\[ d(u) + d(v) \leq M + 2. \]

By the minimality of \( G \), \( G - e \) is \((M + 2d)\)-(d,1)-total choosable. Erase the color of vertex \( u \), and let \( c \) be the partial list \((d,1)\)-total labelling with \(|L| = M + 2d\). Then
\[
|A(e)| \geq M + 2d - (d(u) + d(v) - 2) - (2d - 1) \\
\geq M + 2d - M - (2d - 1) \\
\geq 1,
\]
which implies that \( e \) can be properly colored. Next, for vertex \( u \),
\[
|A(u)| \geq M + 2d - (d(u) + (2d - 1)d(u)) \\
\geq M + 2d - (M + 2d - 1) \\
\geq 1.
\]

Thus we extend the coloring \( c \) to \( G \), a contradiction.

**Lemma 1.4** ([2]) A bipartite graph \( G \) is edge \( f \)-choosable where \( f(uv) = \max\{d(u), d(v)\} \) for any \( uv \in E(G) \).

A \( k \)-alternator for some \( k \) (\( 3 \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \)) is a bipartite subgraph \( B(X, Y) \) of graph \( G \) such that \( d_B(x) = d_G(x) \leq k \) for each \( x \in V(G) \) and \( d_B(y) \geq d_G(y) + k - M - 1 \) for each \( y \in Y \).

The concept of \( k \)-alternator was first introduced by Borodin, Kostochka and Woodall [3] and generalized by Wu and Wang [8].

**Lemma 1.5** There is no \( k \)-alternator \( B(X, Y) \) in \( G \) for any integer \( k \) with \( 3 \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \).

**Proof** Suppose that there exists a \( k \)-alternator \( B(X, Y) \) in \( G \). Obviously, \( X \) is an independent set of vertices in graph \( G \) by Lemma 2.3. By the minimality of \( G \), we can color all elements of subgraph \( G[V(G) \setminus X] \) from their lists of size \( M + 2d \). We denote this partial list \((d,1)\)-total labelling by \( c \). Then for each edge \( e = xy \in B(X, Y) \),
\[
|A(e)| \geq M + 2d - (d_G(y) - d_B(y) + (2d - 1)) \\
\geq M + 2d - (M - d_G(y) + (2d - 1)) \\
\geq d_B(y)
\]
and
\[
|A(e)| \geq M + 2d - (d_G(y) - d_B(y) + (2d - 1))
\]
\[ \geq M + 2d - (M + 2d - k) \]
\[ \geq k \]
because \( B(X,Y) \) is a \( k \)-alternator. Therefore,
\[ |A(e)| \geq \max\{d_B(y), d_B(x)\}. \]

By Lemma 1.4, it follows that \( E(B(X,Y)) \) can be colored properly from their new color lists. Next, for each vertex \( x \in X \),
\[ |A(x)| \geq M + 2d - (d(x) + (2d - 1)d(x)) \]
\[ \geq M + 2d - (M + 2d - 1) \]
\[ \geq 1, \]
because \( d_G(x) \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \). Thus we extend the coloring \( c \) to \( G \), a contradiction.

**Lemma 1.6**

Let
\[ X_k = \{ x \in V(G) \mid d_G(x) \leq k \} \quad \text{and} \quad Y_k = \bigcup_{x \in X_k} N(x) \]
for any integer \( k \) with \( 3 \leq k \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \). If \( X_k \neq \emptyset \), then there exists a bipartite subgraph \( M_k \) of \( G \) with partite sets \( X_k \) and \( Y_k \), such that \( d_{M_k}(x) = 1 \) for each \( x \in X_k \) and \( d_{M_k}(y) \leq k - 2 \) for each \( y \in Y_k \).

**Proof** The proof is omitted here as it is similar with the proof of Lemma 2.4 in [8].

We call \( y \) the \( k \)-master of \( x \) if \( xy \in M_k \) and \( x \in X_k, y \in Y_k \). By Lemma 1.3, if \( uv \in E(G) \) satisfies
\[ d(v) \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \quad \text{and} \quad d(u) = M - i, \]
then
\[ d(v) \geq M + 3 - d(u) \geq i + 3. \]
Together with Lemma 1.6, it follows that each \((M - i)\)-vertex can be a \( j \)-master of at most \( j - 2 \) vertices, where \( 3 \leq i + 3 \leq j \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \). Each \( i \)-vertex has a \( j \)-master by Lemma 1.6, where \( 3 \leq i \leq j \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \).

## 2 Proof of main results

By our Lemmas above, \( G \) has structural properties in the following.

(C1) \( G \) is connected;
(C2) for each \( e = uv \in E(G) \), \( d(u) + d(v) \geq M - 2d + 4 \);
(C3) if \( e = uv \in E(G) \) and \( \min\{d(u), d(v)\} \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \), then \( d(u) + d(v) \geq M + 3 \);
(C4) each \( i \)-vertex (if exists) has one \( j \)-master, where \( 3 \leq i \leq j \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \);
(C5) each \((M - i)\)-vertex (if exists) can be a \( j \)-master of at most \( j - 2 \) vertices, where \( 3 \leq i + 3 \leq j \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \).
Proof of Theorem 0.3 Let $G$ be a minimal counterexample in terms of $|V(G)| + |E(G)|$ to Theorem 0.3. In this theorem,

$$M \geq \frac{d}{2d-1} \left(10d - 8 + \sqrt{(10d - 2)^2 - 24(2d - 1)\varepsilon}\right) + 1$$

\[ \geq 10d + 1. \]

Thus

$$\left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \geq 6.$$

In the following, we apply the discharging method to complete the proof by contradiction. At the very beginning, we assign an initial charge $w(x) = d(x) - 6$ for any $x \in V(G)$. By Euler’s formula

$$|V| - |E| + |F| = \varepsilon,$$

we have

$$\sum_{x \in V} w(x) = \sum_{x \in V} (d(x) - 6)$$

$$= -6\varepsilon - \sum_{x \in F} (2d(x) - 6)$$

$$\leq -6\varepsilon.$$

The discharging rule is as follows.

(R1) each $i$-vertex (if exists) receives charge 1 from each of its $j$-master, where $3 \leq i \leq j \leq 5$.

If $M \geq \Delta + 3$, then $\delta(G) \geq 6$. Otherwise, let $uv \in E(G)$ and $d(u) \leq 5$. Then

$$d(u) + d(v) \leq M - 3 + 5 \leq M + 2$$

and

$$d(u) \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$$

as $\left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \geq 6$,

which is a contradiction to (C3). This obviously contradicts the fact $\delta(G) \leq 5$ for any planar graph. Proof of the theorem is completed. Next, we only consider the case $\Delta \leq M \leq \Delta + 2$.

Claim 1 $\delta \geq M - \Delta + 3$.

Proof If there is some $e = uv \in E(G)$ such that $d(v) \leq M - \Delta + 2$, then

$$d(u) + d(v) \leq \Delta + (M - \Delta + 2) \leq M + 2$$

and

$$d(v) \leq 5 \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$$

as $\left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \geq 6$,

a contradiction to (C3).

Let $v$ be a $k$-vertex of $G$.

(a) If $3 \leq k \leq 5$, then

$$w'(v) = w(v) + \sum_{k \leq i \leq 5} 1 = (k - 6) + (6 - k) = 0$$
by (C4) and rule (R1); 

(b) If $6 \leq k \leq M - 3$, then for all $u \in N(v)$, $d(u) \geq 6$ by (C3). Therefore, $v$ neither receives nor gives any charge by our rule, which implies that $w'(v) = w(v) = k - 6 \geq 0$;

(c) If $M - 2 \leq k \leq \Delta$.

Case 1 $M = \Delta + 2$. Then $\delta \geq 5$ by Claim 1. For $k = \Delta$, $w'(v) \geq w(v) - 3 = \Delta - 9 = M - 11$ by (C5) and rule (R1).

Case 2 $M = \Delta + 1$. Then $\delta \geq 4$ by Claim 1. For $k = \Delta - 1$, $w'(v) \geq w(v) - 3 = \Delta - 1 - 6 - 3 = M - 11$ by (C5) and rule (R1). For $k = \Delta$, $w'(v) \geq w(v) - 3 - 2 = \Delta - 6 - 3 - 2 = M - 12$ by (C5) and rule (R1).

Case 3 $M = \Delta$. Then $\delta(G) \geq 3$ by Claim 1. For $k = \Delta - 2$, $w'(v) \geq w(v) - 3 = \Delta - 2 - 6 - 3 = M - 11$ by (C5) and rule (R1). For $k = \Delta - 1$, $w'(v) \geq w(v) - 3 - 2 = \Delta - 1 - 6 - 3 - 2 = M - 12$ by (C5) and rule (R1). For all cases above, $w'(v) \geq M - 12 > 0$ for any $d(v) \geq \Delta - 2$ as $M \geq 10d + 1 \geq 21$.

Let $X = \{x \in V(G)\mid d_G(x) \leq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor \}$. By (C3), $X$ is an independent set of vertices.

Claim 2 The number of $\left(\frac{M + 2d - 1}{2d}\right) + 1$-vertex of $G$ is at least $M - \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor + 3$. That is, 
\[ |V(G \setminus X)| \geq M - \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor + 3. \]

Proof Otherwise, let $Y = N_{x \in X}(x)$ and $B = B(X,Y)$ be the induced bipartite subgraph. For all $y \in Y$,
\[ d_{G \setminus X}(y) \leq |Y| - 1 \leq M - \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor + 1. \]

Therefore,
\[ d_B(y) = d_G(y) - d_{G \setminus X}(y) \geq d_G(y) + \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor - M - 1, \]

which implies $B$ is a $\left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor$-alternator of $G$, a contradiction to Lemma 2.5.

Since $M \geq 10d + 1$, it follows that
\[ M - 12 > \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor - 5. \]

Thus,
\[ w'(v) \geq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor - 5 \]

when $d_G(v) \geq \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor + 1$. Then
\[
\sum_{x \in V} w(x) = \sum_{x \in V} w'(x) \\
\geq (M - \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor + 3) \left( \left\lfloor \frac{M + 2d - 1}{2d} \right\rfloor - 5 \right) \\
\geq (2d - 1) \left( \left\lfloor \frac{M - 1}{2d} \right\rfloor \right)^2 - (10d - 8) \frac{M - 1}{2d} - 15
\]
\[ M \geq \frac{d}{2d-1} \left(10d - 8 + \sqrt{(10d-2)^2 - 24(2d-1)\varepsilon}\right) + 1.\]

Then this contradiction completes the proof.

**Proof of Theorem 0.4** Let G be a minimal counterexample in terms of \(|V(G)| + |E(G)|\) to Theorem 0.4. In this theorem, \(M \geq 5d + 2\). We define the initial charge function \(w(x) := d(x) - 4\) for all element \(x \in V \cup F\). By Euler’s formula \(|V| - |E| + |F| = \varepsilon\), we have

\[ \sum_{x \in V \cup F} w(x) = \sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (d(f) - 4) = -4\varepsilon < 0.\]

The transition rules are defined as follows.

(R1) Each 3-vertex (if exists) receives charge 1 from its 3-master.

(R2) Each \(k\)-vertex with \(3 \leq k \leq 7\) transfer charge \(\frac{k-3}{\varepsilon}\) to each 3-face that incident with it.

(R3) Each 8\(^{+}\)-vertex transfer charge \(\frac{1}{2}\) to each 3-face that incident with it.

Analogous with Claim 1 in the proof of Theorem 0.3, it is easy to prove that \(\delta(G) \geq 3\) when \(\Delta = M\) and \(\delta(G) \geq 4\) otherwise. Let \(v\) be a \(k\)-vertex of \(G\).

For \(k = 3\), then \(w'(v) = w(v) + 1 = 3 - 4 + 1 = 0\) since it receives 1 from its 3-master;

For \(k = 4\), then \(w'(v) = w(v) = 0\) since we never change the charge by our rules;

For \(3 \leq k \leq 7\), then \(w'(v) \geq w(v) - \frac{k-4}{\varepsilon} = 0\) by (R2);

For \(8 \leq k \leq M - 1\), then \(w'(v) \geq w(v) - \frac{k}{\varepsilon} \geq 0\) by (R3);

If \(M > \Delta\), then \(M - 1 \geq \Delta\). Thus \(w(v) \geq 0\) for all \(v \in V(G)\). Otherwise, \(\Delta = M\). Then for \(k = \Delta\), \(w'(v) \geq w(v) - \frac{1}{2}M - 1 = \frac{M}{2} - 5\) by (C5) and rules (R1), (R3). Since \(M \geq 5d + 2\), we have \(w'(v) \geq \frac{M}{2} - 5 > 0\).

Let \(f\) be a \(k\)-face of \(G\).

If \(k \geq 4\), then \(w'(f) = w(f) \geq 0\) since we never change the charge of them by our rules;

If \(k = 3\), assume that \(f = [v_1, v_2, v_3]\) with \(d(v_1) \leq d(v_2) \leq d(v_3)\). It is easy to see \(w(f) = -1\). Consider the subcases as follows.

(a) Suppose \(d(v_1) = 3\). Then \(M = \Delta = d(v_2) = d(v_3) = \Delta\) by (C3). Thus, \(w'(f) = w(f) + \frac{1}{2} \times 2 = 0\) by (R3);

(b) Suppose \(d(v_1) = 4\). Then \(d(v_3) \geq d(v_2) \geq M - 2d + 4 - d(v_1) \geq 3d + 2 \geq 8\) by (C2). Therefore, \(w'(f) = w(f) + \frac{1}{2} \times 2 = 0\) by (R3);

(c) Suppose \(d(v_1) = 5\). Then \(d(v_3) \geq d(v_2) \geq M - 2d + 4 - d(v_1) \geq 3d + 1 \geq 7\) by (C2). Therefore, \(w'(f) = w(f) + \frac{1}{2} \times 2 + \frac{1}{2} > 0\) by (R2);

(d) Suppose \(d(v_1) = m \geq 6\). Then \(d(v_3) \geq d(v_2) \geq M - 2d + 4 - d(v_1) \geq 3d + 1 \geq 6\) by (C2). Therefore, \(w'(f) = w(f) + 3 \times \min\{\frac{m-4}{m}, \frac{1}{2}\} = 0\) by (R2) and (R3).

Thus, we have \(\sum_{x \in V \cup F} w'(x) \geq 0\) which is a contradiction with

\[ \sum_{x \in V \cup F} w'(x) = \sum_{x \in V \cup F} w(x) < 0.\]
References


