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On $(p, 1)$-total labelling of 1-planar graphs

Research Article

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Abstract: A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, it is proved that the $(p, 1)$-total labelling number of every 1-planar graph G is at most $\Delta(G) + 2p - 2$ provided that $\Delta(G) \geq 8p+4$ or $\Delta(G) \geq 6p+2$ and $g(G) \geq 4$. As a consequence, the well-known $(p, 1)$-total labelling conjecture has been confirmed for some 1-planar graphs.

MSC: 05C10, 05C15

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1. Introduction

In the frequency assignment problems, we need to assign different frequencies to close transmitters so that they could avoid interference and communication link failure. Traditionally, interference information initiates a graph coloring (labelling) problem. An $L(p, q)$-labelling of a graph G is a mapping f from the set of vertices $V(G)$ to the set of integers $\mathbb{Z}_k = \{0, 1, \dots, k\}$ such that $|f(x) - f(y)| \geq p$ if x and y are adjacent and $|f(x) - f(y)| \geq q$ if x and y are at distance 2. $L(p, q)$-labelling problems have been extensively studied in many papers. An interested reader can refer to the surveys by Calamoneri [12] and by Yeh [28].

In 1995, Whittlesey, Georges and Mauro [26] studied the $L(2, 1)$-labelling of the incidence graph $I(G)$ of a graph G, which is obtained from G by inserting one vertex of degree 2 on each edge of G. Such a special labelling of $I(G)$ can be easily translated to another kind of labelling, the so-called $(2, 1)$-total labelling of G, which was introduced by Havet and Yu [16, 17] and generalized to the notion of $(p, 1)$-total labelling.

A k-$(p, 1)$-total labelling of a graph G is a function f from $V(G) \cup E(G)$ to the color set $\{0, 1, \dots, k\}$ such that $|f(u) - f(v)| \geq 1$ if $uv \in E(G)$, $|f(e_1) - f(e_2)| \geq 1$ if e_1 and e_2 are two adjacent edges in G and $|f(u) - f(e)| \geq p$

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if the vertex u is incident to the edge e. The minimum k such that G has a k-$(p,1)$-total labelling, denoted by $\lambda_p^T(G)$, is called the $(p,1)$-*total labelling number* of G. It is easy to see that a $(1,1)$-total labelling is a total coloring as $\lambda_1^T(G) = \chi''(G) - 1$, where $\chi''(G)$ is the total chromatic number of a graph G. We know that $\Delta(G)+1$ is a trivial lower bound for $\chi''(G)$. In fact, this can be easily generalized to a lower bound for the $(p,1)$-total labelling number by looking at the label of a vertex with maximum degree and its incident edges. One can see that $\lambda_p^T(G) \geq \Delta(G) + p - 1$ with $p \geq 1$ and this lower bound is attained at a star when $p < \Delta(G)$. On the other hand, if we are given a graph G, then we can construct a $(p,1)$-total labelling of G by properly coloring its edges with $\chi'(G)$ integers of $[0, \chi'(G)-1]$, and its vertices with $\chi(G)$ integers of $\left[\chi'(G)+p-1, \chi(G)+\chi'(G)+p-2\right]$, where $\chi(G)$ and $\chi'(G)$ denote the vertex chromatic number and the edge chromatic number of G, respectively. Thus we trivially have $\lambda_p^T(G) \leq \chi(G) + \chi'(G) + p - 2$ for every graph G. In view of this, Havet and Yu [17, 18] also remarked that $\lambda_p^T(G) \leq 2\Delta(G) + p - 1$ by Brooks' theorem and Vizing's theorem. However, this upper bound for $\lambda_p^T(G)$ seems to be not tight. As a natural extension of the well-known Total Coloring Conjecture which states that every graph is $(\Delta+2)$-total colorable, Havet and Yu [17, 18] conjectured the following, which is known as the $(p,1)$-Total Labelling Conjecture.

Conjecture 1.1.

Let G be a graph. Then $\lambda_p^T(G) \leq \min\left\{\Delta(G)+2p-1, 2\Delta(G)+p-1\right\}$.

If $p=1$, then this conjecture is nothing else than Total Coloring Conjecture. The total coloring of graphs has been extensively studied in many papers including [5, 9–11, 21, 24]. For $p \geq 2$, the $(2,1)$-Total Labelling Conjecture has already been confirmed for all outerplanar graphs [13, 15] and the $(p,1)$-Total Labelling Conjecture in general has been considered for some other classes such as planar graphs with high girth and high maximum degree [2] and graphs with a given maximum average degree [22]. In particular, Bazzaro, Montassier and Raspaud proved the following theorem for all planar graphs [2].

Theorem 1.2.

Let G be a planar graph with maximum degree Δ. If $\Delta \geq 8p+2$ and $p \geq 2$, then $\lambda_p^T(G) \leq \Delta + 2p - 2$.

In this paper, we focus on 1-planar graphs. A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by at most one other edge. It is easy to see that every planar graph is 1-planar. The notion of 1-planar graphs was introduced by Ringel [23] while trying to simultaneously color the vertices and faces of a planar graph G so that every pair of adjacent/incident elements receives different colors. Note that we can construct a 1-planar graph G' from a planar graph G so that the vertex set of G' is $V(G) \cup F(G)$ and any two vertices in G' are adjacent if and only if their corresponding elements in G are adjacent or incident, moreover, the vertex-face chromatic number of G is equal to the chromatic number of G'. In the abovementioned paper, Ringel proved that the chromatic number of each 1-planar graph is at most 7. This bound was latter improved to 6 (being sharp) by Borodin [4, 6]. The list analogue of vertex coloring of 1-planar graphs was firstly investigated by Albertson and Mohar in [1]. Wang and Lih [25] proved that each 1-planar graph is list 7-colorable. In [8], Borodin et al. proved that each 1-planar graph is acyclically 20-colorable. Recently, Zhang et al. showed that each 1-planar graph G with maximum degree Δ is Δ-edge-colorable provided that $\Delta \geq 10$ [34], or $\Delta \geq 9$ and G contains no chordal 5-cycles [29], or $\Delta \geq 8$ and G contains no chordal 4-cycles [30], or $\Delta \geq 7$ and G contains no 3-cycles [31]. They [35] also proved that each 1-planar graph with maximum degree Δ is list Δ-edge-colorable and list $(\Delta+1)$-total-colorable if $\Delta \geq 21$, is list $(\Delta+1)$-edge-colorable and list $(\Delta+2)$-total-colorable if $\Delta \geq 16$. As far as we know, no other results on the colorability of 1-planar graphs can be found in the literature, although there are many papers concerning the local or global structures of 1-planar graphs (see [7, 14, 19, 20, 32, 33] for details).

The purpose of this paper is to investigate the $(p,1)$-total labellings of 1-planar graphs by proving the following main theorem.

Theorem 1.3.

Let $p \geq 2$ be an integer and let G be a 1-planar graph with maximum degree Δ and girth g. If $\Delta \geq 8p+4$ or $\Delta \geq 6p+2$ and $g \geq 4$, then $\lambda_p^T(G) \leq \Delta + 2p - 2$.

For a 1-planar graph G, as mentioned previously, $\chi(G) \le 6$ and $\chi'(G) = \Delta(G)$ provided that $\Delta(G) \ge 10$ or $\Delta(G) \ge 7$ and $g(G) \ge 4$. Recall that $\lambda_p^T(G) \le \chi(G) + \chi'(G) + p - 2$ for every graph G. So we directly have the following theorem.

Theorem 1.4.

Let $p \ge 1$ be an integer and let G be a 1-planar graph with maximum degree Δ and girth g. If $\Delta \ge 10$ or $\Delta \ge 7$ and $g \ge 4$, then $\lambda_p^T(G) \le \Delta + p + 4$.

Comparing this result to Theorem 1.3, one can find that our upper bound for $\lambda_p^T(G)$ in Theorem 1.3 is better for small p.

In the next section, we will prove Theorem 1.3 by contradiction. In particular, we will prove the following slightly stronger theorem. Indeed, it is only a technical strengthening of Theorem 1.3, without which we would get complications when considering a subgraph $G' \subset G$ such that $\Delta(G') < \Delta(G)$. Of course, the only interesting case is when $M = \Delta$.

Theorem 1.5.

Let M, p be two integers and let G be a 1-planar graph with maximum degree $\Delta \le M$ and girth g. Then $\lambda_d^T(G) \le M + 2p - 2$ with $p \ge 2$ in the following cases:

(1) *$M \ge 8p + 4$;*

(2) *$M \ge 6p + 2$ and $g \ge 4$.*

We close this section by introducing some useful notation. Let f be a $(p, 1)$-total labelling of a given graph G and C be the color set used by f. For a vertex $x \in V(G)$, put $\Theta_f(x) = \{f(xy) : y \in N_G(x)\}$ and $\Phi_f(x) = \{f(x) + i : |i| \le p - 1\} \cap C$. Similarly, for an edge $xy \in E(G)$, put $\Phi_f(xy) = \{f(xy) + i : |i| \le p - 1\} \cap C$. Throughout this paper, a k-, k^+- and k^--*vertex* (resp. *face*) is a vertex (resp. face) of degree k, at least k and at most k. For other undefined standard concepts we refer the readers to [3].

2. Structures of the minimum counterexample to Theorem 1.5

Let G be a counterexample to Theorem 1.5 with $|V(G)| + |E(G)|$ being minimum. First of all, we prove the following two lemmas.

Lemma 2.1.

For any edge $uv \in E(G)$, if

$$\min\{d_G(u), d_G(v)\} \le \left\lfloor \frac{M + 2p - 2}{2p} \right\rfloor,$$

then $d_G(u) + d_G(v) \ge M + 2$.

Proof. Suppose, to the contrary, that there exists an edge $uv \in E(G)$ such that $d_G(u) \le \lfloor (M+2p-2)/(2p) \rfloor$ and $d_G(u) + d_G(v) \le M + 1$. Consider the graph $G' = G - uv$, which has an $(M+2p-2)$-$(p, 1)$-total labelling f by the minimality of G. Let $C = \{0, 1, \ldots, M+2p-2\}$ be the color set involved in f. First of all, we label the edge uv with a color $f(uv) \in C \setminus \big(\Theta_f(u) \cup \Theta_f(v) \cup \Phi_f(v)\big)$. Since

$$\left|C \setminus \big(\Theta_f(u) \cup \Theta_f(v) \cup \Phi_f(v)\big)\right| \ge (M + 2p - 1) - (d_G(u) - 1) - (d_G(v) - 1) - (2p - 1) \ge 1,$$

such a labelling of uv does exist. We denote the labelling at this stage still by f. Then we relabel (if necessary) the vertex u with a color $f(u) \in C \setminus \bigcup_{w \in N_G(u)} (\Phi_f(uw) \cup \{f(w)\})$. One can also check that

$$\left| C \setminus \bigcup_{w \in N_G(u)} (\Phi_f(uw) \cup \{f(w)\}) \right| \ge (M + 2p - 1) - (2p - 1) d_G(u) - d_G(u) \ge (M + 2p - 1) - 2p \left\lfloor \frac{M + 2p - 2}{2p} \right\rfloor \ge 1.$$

Thus we have already constructed an $(M+2p-2)$-$(p,1)$-total labelling of G, a contradiction. □

Lemma 2.2.

For any edge $uv \in E(G)$, it holds that $d_G(u)+d_G(v) \geq M-2p+3$.

The proof of this lemma is similar to that of Lemma 2.1. So we omit it here.

Now we introduce another lemma useful for our proof. To begin with, let us review the following lemma proved by Borodin, Kostochka and Woodall.

Lemma 2.3 ([10]).

A bipartite graph B is edge f-choosable, where $f(uv) = \max\{d_B(u), d_B(v)\}$ for any $uv \in E(B)$.

A bipartite subgraph B with two partite sets X and Y of the graph G is called a *k-alternating subgraph* for some $2 \leq k \leq \lfloor (M+2p-2)/(2p) \rfloor$ if $d_B(x) = d_G(x) \leq k$ for each $x \in X$ and $d_B(y) \geq d_G(y)+k-M$ for each $y \in Y$. This notion, along with the terminology of masters and dependents (which will be mentioned later), was introduced by Borodin, Kostochka and Woodall in [10]. The special case $k=2$ of this notion, under the name of *2-alternating cycle*, was introduced by Borodin in [5] and then used in several dozens of papers on graph colorings.

Lemma 2.4.

There is no k-alternating subgraphs in G for any integer

$$2 \leq k \leq \left\lfloor \frac{M+2p-2}{2p} \right\rfloor. \tag{$\star$}$$

Proof. Suppose that G contains a k-alternating subgraph B with two partite sets X and Y such that $d_B(x) = d_G(x) \leq k$ for each $x \in X$ and $d_B(y) \geq d_G(y)+k-M$ for each $y \in Y$. Consider the graph $G' = G-X$. By the minimality of G, G' has an $(M+2p-2)$-$(p,1)$-total labelling f. Let $C = \{0,1,\dots,M+2p-2\}$ be the color set involved in f. Now we consider the edges between X and Y. Let $xy \in E(B)$ where $x \in X$ and $y \in Y$. By A_{xy} denote the list of available colors to label the edge xy. One can easily see that $A_{xy} = C \setminus (\Theta_f(y) \cup \Phi_f(y))$. It follows that

$$|A_{xy}| \geq (M+2p-2)-(d_G(y)-d_B(y))-(2p-1) = M-d_G(y)+d_B(y) \geq d_B(y),$$
$$|A_{xy}| \geq M-d_G(y)+d_B(y) \geq M-d_G(y)+d_G(y)+k-M = k \geq d_B(x).$$

Thus by Lemma 2.3, each of the edges between X and Y can be properly labelled with a color in C. So we only need to label the vertices in X at the end. Indeed, this can be easily done by a similar argument as that of Lemma 2.1. Hence, G has an $(M+2p-2)$-$(p,1)$-total labelling, a contradiction. □

Lemma 2.5.

For any integer k satisfying ($\star$), let $X_k = \{x \in V(G) : d_G(x) \leq k\}$ and $Y_k = \bigcup_{x \in X_k} N_G(x)$. If $X_k \neq \emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x) = 1$ for every $x \in X_k$ and $d_{M_k}(y) \leq k-1$ for every $y \in Y_k$.

The proof idea of Lemma 2.5 is borrowed from the proof of [10, Theorem 8]. In particular, Wu and Wang [27, Lemma 2.4] proved a total coloring analog of this lemma and their proof can be easily translated to a $(p,1)$-total labelling version. So we omit the detailed proof of Lemma 2.5 here. In fact, if we assume that the required bipartite subgraph M_k in Lemma 2.5 does not appear in G, then we would find a k-alternating subgraph that cannot appear in G by Lemma 2.4. This is why we prove Lemma 2.4 at first.

Following the terms of Borodin, Kostochka and Woodall in [10], in Lemma 2.5 we call y the *k-master* of x if $xy \in M_k$ and $x \in X_k$ and we call x the *k-dependent* of y. From Lemma 2.5, we can deduce the following useful lemma as a corollary.

Lemma 2.6.

Every i-vertex in G has a j-master, where $2 \leq i \leq j \leq \lfloor (M+2p-2)/(2p) \rfloor$, and every vertex in G has at most $k-1$, k-dependents, assuming ($\star$) holds.

The above lemmas are devoted to the structural properties of G which can be considered as a critical graph in terms of labelling. From now on, we investigate some other structural properties of G, which are related to its embedding. Recall that G is a 1-planar graph.

In the following, we always assume that G has been embedded in a plane so that every edge is crossed by at most one other edge and the number of crossings is as small as possible. The *associated plane graph* $G^\times$ of G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is called *false* if it is not a vertex of G and *true* otherwise. By a *false face*, we mean a face f in $G^\times$ that is incident with at least one false vertex; otherwise, we call f *true*. For a true vertex v in $G^\times$, let $\alpha(v)$ and $\beta(v)$ be the number of false and true 3-faces that are incident with v in $G^\times$, respectively. In [34], Zhang and Wu proved the following lemma.

Lemma 2.7.

The following facts hold for G and $G^\times$.

(1) *No two false vertices are adjacent in $G^\times$.*

(2) *If $d_{G^\times}(v) = 2$, then $\alpha(v) = 0$.*

(3) *If $d_{G^\times}(v) = 3$ and $\alpha(v) \geq 2$, then v is incident with a 5^+-face in $G^\times$.*

(4) *If $d_{G^\times}(v) = 4$, then $\alpha(v) \leq 3$.*

(5) *If $d_{G^\times}(v) \geq 5$, then $\alpha(v) \leq 2\lfloor d_{G^\times}(v)/2 \rfloor$.*

In the following we set $t = 7$ while proving (1) of Theorem 1.5 and set $t = 5$ while proving (2) of Theorem 1.5. We call a vertex in G *small* if it is of degree no more than t and *big* otherwise. A false 3-face in $G^\times$ is called *unbalanced* or *balanced* according to whether or not it is incident with a small vertex. For a true vertex v in $G^\times$, let $\alpha_a(v)$ and $\alpha_b(v)$ be the number of unbalanced and balanced false 3-faces that are incident with v in $G^\times$, respectively. By Lemma 2.2, we directly have the following.

Lemma 2.8.

No two small vertices are adjacent in G.

Lemma 2.9.

Let v be a big vertex in G.

(1) *If $\alpha(v)+\beta(v) = d_{G^\times}(v)$, then $\alpha_a(v) \leq \lfloor d_{G^\times}(v)/2 \rfloor$.*

(2) *If $\alpha(v)+\beta(v) = d_{G^\times}(v)-1$, then $\alpha_a(v) \leq \lceil (d_{G^\times}(v)-1)/2 \rceil$.*

(3) *If $\alpha(v)+\beta(v) = d_{G^\times}(v)-2$, then $\alpha_a(v) \leq \lceil (d_{G^\times}(v)-2)/2 \rceil + 1$.*

(4) *If $\alpha(v)+\beta(v) = d_{G^\times}(v)-3$, then $\alpha_a(v) \leq \lceil (d_{G^\times}(v)-3)/2 \rceil + 2$.*

Proof. If any of the three facts does not hold, then there must be three consecutive unbalanced false 3-faces that are incident with v in $G^\times$, say vv_1v_2, vv_2v_3 and vv_3v_4. If v_2 is a false vertex in $G^\times$, then v_1 and v_3 are both small vertices in G and $v_1v_3 \in E(G)$ by the drawing of G. This contradicts Lemma 2.8. So v_2 is a true vertex and v_3 is also true by symmetry. This implies that the 3-face vv_2v_3 in $G^\times$ is true, a contradiction. □

Lemma 2.10.

If $g(G) \geq 4$, then $\alpha(v) \leq \lfloor 2d_{G^\times}(v)/3 \rfloor$ and $\beta(v) = 0$.

Proof. Since $g(G) \geq 4$, we trivially have $\beta(v) = 0$. Suppose that $\alpha(v) > \lfloor 2d_{G^\times}(v)/3 \rfloor$. Then there must be three consecutive false 3-faces that are incident with v in $G^\times$, say vv_1v_2, vv_2v_3 and vv_3v_4. If v_2 or v_3 is a false vertex in $G^\times$, then $v_1v_3 \in E(G)$ or $v_2v_4 \in E(G)$, which implies that vv_1v_3v or vv_2v_4v is a triangle in G, a contradiction. On the other hand, if both v_2 and v_3 are true vertices in $G^\times$, then vv_2v_3v is a triangle in G, a contradiction. □

3. The proof of Theorem 1.5

First of all, we prove (1) of Theorem 1.5. Suppose that G is a minimum counterexample to it. Then it is easy to see that G is connected. Moreover, we have $\delta(G) \geq 2$ by Lemma 2.1. In the following, we will apply the discharging method to the associated plane graph $G^\times$ of G and complete the proof by contradiction. Note that $G^\times$ is also connected.

For every element $x \in V(G^\times) \cup F(G^\times)$, we assign an initial charge $c(x) = d_{G^\times}(x) - 4$ to x. Since $G^\times$ is a planar graph, $\sum_{x \in V(G^\times) \cup F(G^\times)} c(x) = -8$ by the well-known Euler formula. Now we redistribute the initial charges on $V(G^\times) \cup F(G^\times)$ by the discharging rules below, where we use $\tau(x_1 \to x_2)$ to denote the charge moved from x_1 to x_2. Let $c'(x)$ be the final charge of an element $x \in V(G^\times) \cup F(G^\times)$ after discharging. Then we still have $\sum_{x \in V(G^\times) \cup F(G^\times)} c'(x) = -8 < 0$, since our rules only move charge around and do not affect the sum.

R1. Suppose that $f = uvw$ is a true 3-face in $G^\times$. If $d_{G^\times}(u) \geq 8$, then $\tau(u \to f) = 1/2$.

R2. Suppose that $f = uvw$ is a false 3-face in $G^\times$ with a false vertex u.

- R2.1. If $d_{G^\times}(v) \leq 7$, then $\tau(v \to f) = 1/3$ and $\tau(w \to f) = 2/3$.
- R2.2. If $\min\{d_{G^\times}(v), d_{G^\times}(w)\} \geq 8$, then $\tau(v \to f) = \tau(w \to f) = 1/2$.

R3. Suppose that f is a 5^+-face in $G^\times$ and v is a 3^--vertex on the boundary of f. Then $\tau(f \to v) = 1/3$.

R4. Suppose that uv is an edge in G such that u is a k-master of v.

- R4.1. If $k = 2$, then $\tau(u \to v) = 2/3$.
- R4.2. If $k = 3$, then $\tau(u \to v) = 1/3$.
- R4.3. If $k = 4$, then $\tau(u \to v) = 2/3$.
- R4.4. If $k = 5$, then $\tau(u \to v) = 1/3$.

R5. Suppose that uv and xy are two mutually crossed edges in G such that $vx, vy \in E(G)$ and $d_G(u) = 2$. Denote the other neighbor of u in G by w.

- R5.1. If u has a positive charge θ after applying the above four rules, then $\tau(u \to v) = \theta$.
- R5.2. Otherwise, we let $\tau(w \to u) = \tau(u \to v) = 1/3$.

The rule R5.2 means that w will send charge to v through u if u does not have positive charge after applying R1–R4. Under this condition, we claim that v will not send back charge to w through u by R5.2, since v and w are not symmetrical in that rule although they are both M-vertices by Lemma 2.1. The following lemma proves this fact.

Lemma 3.1.
If R5.2 is applied, then the edge uw is not crossed.

Proof. If R5.2 is applied, then by (2) of Lemma 2.7 and by R3 and R4, u is incident with exactly two 4-faces in $G^\times$. This follows that uw must be crossed by xy if uw is a crossed edge in G. However, it is impossible since xy has already been crossed by uv. □

We call the 2-vertex u stated in R5.2 a *special* neighbor of w in G. Note that $d_G(w) = M = \Delta$ by Lemma 2.1. By $s(w)$ denote the number of special 2-vertices that are adjacent to w in G. Then by Lemma 3.1, one can easily prove the following fact.

Lemma 3.2.

Let w be vertex in G. Then $\alpha(w)+\beta(w)+2s(w)\leq d_G(w)$.

In the following, we will check that the final charge $c'(x)$ of each element $x \in V(G^\times)\cup F(G^\times)$ is nonnegative, which implies that $\sum_{x\in V(G^\times)\cup F(G^\times)} c'(x)\geq 0$, a contradiction.

First, suppose that f is a face in $G^\times$. If f is a true 3-face, then by Lemma 2.8, f is incident with at least two 8^+-vertices. This implies that $c'(f)\geq -1+2/2=0$ by R1. If f is a false 3-face, then by R2, f receives exactly 1 from the vertices incident with it and thus $c'(f)=-1+1=0$. If f is a 4-face, then $c'(f)=c(f)=0$ since 4-faces are not involved in the discharging rules. If f is a 5^+-face, then f is incident with at most $\lfloor d_{G^\times}(f)/2\rfloor$, 3^--vertices since no two 3^--vertices are adjacent in G by Lemma 2.8. This implies that

$$c'(f)\geq d_{G^\times}(f)-4-\frac{1}{3}\left\lfloor\frac{d_{G^\times}(f)}{2}\right\rfloor>0.$$

Second, suppose that v is a vertex in $G^\times$. If v is a false vertex, then it is trivial that $c'(v)=c(v)=0$. So, in what follows we always assume that v is a true vertex. Recall that no vertex of degree less than M can be involved in R5.2 by Lemma 2.1.

If $d_{G^\times}(v)=2$, then by (2) of Lemma 2.7, v is incident with no false 3-faces and thus v sends out no charges by R1–R4. On the other hand, v is adjacent to two Δ-vertices in G by Lemma 2.1 and v has one 2-master, one 3-master, one 4-master and one 5-master by Lemma 2.6. Then by R4 and R5, we have $c'(v)\geq -2+2/3+1/3+2/3+1/3=0$.

If $d_{G^\times}(v)=3$, then by Lemma 2.6, v has one 3-master, one 4-master and one 5-master. If $\alpha(v)\leq 1$, then by R2.1 and R4, $c'(v)\geq -1-1/3+1/3+2/3+1/3=0$. If $\alpha(v)\geq 2$, then by (3) of Lemma 2.7, v is incident with one 5^+-face, which also implies that $c'(v)\geq -1-2/3+1/3+2/3+1/3+1/3=0$ by R2.1, R3 and R4.

If $d_{G^\times}(v)=4$, then by Lemma 2.6 and (4) of Lemma 2.7, v has one 4-master, one 5-master and $\alpha(v)\leq 3$. This implies that $c'(v)\geq 0-3/3+2/3+1/3=0$ by R2.1, R4.3 and R4.4.

If $d_{G^\times}(v)=5$, then by Lemma 2.6 and (5) of Lemma 2.7, v has one 5-master and $\alpha(v)\leq 4$. This implies that $c'(v)\geq 1-4/3+1/3=0$ by R2.1 and R4.4.

If $6\leq d_{G^\times}(v)\leq 7$, then $\alpha(v)\leq 6$ by (5) of Lemma 2.7. This implies that $c'(v)\geq d_{G^\times}(v)-4-6/3\geq 0$ by R2.1.

If $8\leq d_{G^\times}(v)\leq 11$, then v is adjacent to no 7^--vertices by Lemma 2.2 and thus $c'(v)\geq d_{G^\times}(v)-4-d_{G^\times}(v)/2\geq 0$ by R1 and R2.2.

If $12\leq d_{G^\times}(v)\leq M-4$, then v is adjacent to no 5^--vertices by Lemma 2.1 and thus $c'(v)\geq d_{G^\times}(v)-4-2d_{G^\times}(v)/3\geq 0$ by R1 and R2.

If $d_{G^\times}(v)=M-3$, then by Lemmas 2.1 and 2.6, v has no i-dependents for $i\leq 4$ but may have at most four 5-dependents. This implies that $c'(v)\geq d_{G^\times}(v)-4-2d_{G^\times}(v)/2-4/3=(M-19)/3>0$ by R1, R2 and R4.4, since $M\geq 8p+4\geq 20$.

If $d_{G^\times}(v)=M-2$, then by Lemmas 2.1 and 2.6, v has no i-dependents for $i\leq 3$ but may have at most three 4-dependents and at most four 5-dependents. If $\alpha(v)+\beta(v)\leq d_{G^\times}(v)-2$, then $c'(v)\geq d_{G^\times}(v)-4-2(d_{G^\times}(v)-2)/3-4/3-3\cdot 2/3=(M-20)/3\geq 0$ by R1, R2, R4.3 and R4.4. If $\alpha(v)+\beta(v)\geq d_{G^\times}(v)-1$, then by (1) and (2) of Lemma 2.9, we have $\alpha_a(v)\leq\lceil(\alpha(v)+\beta(v))/2\rceil$ and thus

$$c'(v)\;\geq\; d_{G^\times}(v)-4-\frac{2}{3}\alpha_a(v)-\frac{1}{2}\big(\alpha(v)+\beta(v)-\alpha_a(v)\big)-4\cdot\frac{1}{3}-3\cdot\frac{2}{3}\;\geq\; M-\frac{7}{6}\left\lceil\frac{M}{2}\right\rceil-\frac{49}{6}>0$$

by R1, R2, R4.3 and R4.4.

If $d_{G^\times}(v)=M-1$, then by Lemmas 2.1 and 2.6, v has no 2-dependents but may have at most two 3-dependents, at most three 4-dependents and at most four 5-dependents. If $\alpha(v)+\beta(v)\leq d_{G^\times}(v)-3$, then $c'(v)\geq d_{G^\times}(v)-4-2(d_{G^\times}(v)-3)/3-$

$4/3 - 3\cdot 2/3 - 2/3 = (M-19)/3 \geq 0$ by R1, R2 and R4. If $d_{G^\times}(v) - 2 \leq \alpha(v)+\beta(v) \leq d_{G^\times}(v)-1$, then by Lemma 2.9, we have $\alpha_a(v) \leq \lceil \alpha(v)+\beta(v))/2\rceil + 1$ and thus

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha_a(v) - \frac{1}{2}(\alpha(v)+\beta(v)-\alpha_a(v)) - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} \geq M - \frac{7}{6}\left\lceil\frac{M}{2}\right\rceil - 8 > 0$$

by R1, R2 and R4. If $\alpha(v)+\beta(v) = d_{G^\times}(v)$, then by (1) of Lemma 2.9, we have $\alpha_a(v) \leq \lfloor(\alpha(v)+\beta(v))/2\rfloor$ and thus

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha_a(v) - \frac{1}{2}(\alpha(v)+\beta(v)-\alpha_a(v)) - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} \geq \frac{M}{2} - \frac{1}{6}\left\lfloor\frac{M-1}{2}\right\rfloor - \frac{17}{2} \geq 0$$

by R1, R2 and R4.

If $d_{G^\times}(v) \geq M$, then we shall assume that $d_{G^\times}(v) = M = \Delta$ since $\Delta \leq M \leq d_{G^\times}(v) \leq \Delta$. By Lemma 2.6, v may have at most one 2-dependent, at most two 3-dependents, at most three 4-dependents and at most four 5-dependents. If $s(v) \geq 2$, then by Lemma 3.2, $\alpha(v)+\beta(v) \leq M - 2s(v)$. It follows that

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}(\alpha(v)+\beta(v)) - \frac{s(v)}{3} - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} - \frac{2}{3} \geq \frac{1}{3}(M - 26 + 3s(v)) \geq \frac{1}{3}(M-20) \geq 0$$

by R1, R2, R4 and R5.2. So we assume that $s(v) \leq 1$. If $\alpha(v)+\beta(v) \leq d_{G^\times}(v) - 4$, then

$$c'(v) \geq d_{G^\times}(v) - 4 - 2\frac{d_{G^\times}(v)-4}{3} - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} - \frac{2}{3} - \frac{1}{3} = \frac{M-19}{3} > 0$$

by R1, R2, R4 and R5.2. If $\alpha(v)+\beta(v) = d_{G^\times}(v) - 3$, then $\alpha_a(v) \leq \lceil(\alpha(v)+\beta(v))/2\rceil + 2$ by Lemma 2.9 and

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha_a(v) - \frac{1}{2}\big(\alpha(v)+\beta(v)-\alpha_a(v)\big) - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} - \frac{2}{3} - \frac{1}{3} \geq \frac{M}{2} - \frac{1}{6}\left\lceil\frac{M-3}{2}\right\rceil - \frac{47}{6} > 0$$

by R1, R2, R4 and R5.2. Similarly, we can also show that $c'(v) \geq 0$ when $d_{G^\times}(v) - 2 \leq \alpha(v)+\beta(v) \leq d_{G^\times}(v)-1$, by Lemma 2.9 and the rules mentioned above. At last, we shall consider the case when $\alpha(v)+\beta(v) = d_{G^\times}(v)$. By (1) of Lemma 2.9 and Lemma 3.2, we have $\alpha_a(v) \leq \lfloor(\alpha(v)+\beta(v))/2\rfloor$ and $s(v) = 0$. If v has no 2-dependents, then by R1, R2 and R4,

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha_a(v) - \frac{1}{2}\big(\alpha(v)+\beta(v)-\alpha_a(v)\big) - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} \geq \frac{5}{12}\Delta - 8 > 0.$$

So we assume that u is a 2-dependent of v. This implies that $uv \in E(G)$ and $d_G(u) = 2$. Since $\alpha(v)+\beta(v) = d_{G^\times}(v)$, uv must be a crossed edge in G. Suppose that the edge xy crosses uv in G at a point z in $G^\times$. By w denote the other neighbor of u in G. If u is incident with at least one 5^+-face f in $G^\times$, then by a similar argument as before, one can easily check that u still has charge $1/3$ after applying the four rules R1–R4 (note that R3 should be applied to u now). So by R5.1, we have $\tau(u\to v) = 1/3$. If u is incident with no 5^+-faces in $G^\times$, then by R5.2 we still have $\tau(w\to u) = \tau(u\to v) = 1/3$ (note that $vx, vy \in E(G)$). Therefore, by R1, R2, R4 and R5 we have

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha_a(v) - \frac{\alpha(v)+\beta(v)-\alpha_a(v)}{2} - 4\cdot\frac{1}{3} - 3\cdot\frac{2}{3} - 2\cdot\frac{1}{3} - \frac{2}{3} + \frac{1}{3} \geq \frac{5}{12}M - \frac{25}{3} \geq 0$$

in each case, since $M \geq 8p+4 \geq 20$. Hence the proof of Theorem 1.5 (1) completes here.

Now we pass to (2) of Theorem 1.5. Assume that G is a minimum counterexample to the theorem. Then G is a connected graph with the minimum degree at least two. Now we apply a discharging procedure to the associated plane graph $G^\times$ of G by assigning an initial charge $c(x) = d_{G^\times}(x) - 4$ to every element $x \in V(G^\times)\cup F(G^\times)$ and defining the discharging rules as follows.

$\tilde{R}1$. Suppose that $f = uvw$ is a false 3-face in $G^\times$ with a false vertex u.

$\tilde{R}1.1$. If $d_{G^\times}(v) \leq 5$, then $\tau(v \to f) = 1/3$ and $\tau(w \to f) = 2/3$.

$\tilde{R}1.2$. If $\min\{d_{G^\times}(v), d_{G^\times}(w)\} \geq 6$, then $\tau(v \to f) = \tau(w \to f) = 1/2$.

$\tilde{R}2$. Suppose that f is a 5^+-face in $G^\times$ and v is a 3-vertex on the boundary of f. Then $\tau(f \to v) = 1/3$.

$\tilde{R}3$. Suppose that uv is an edge in G such that u is an i-master of v for some $i \in \{2,3,4\}$. Then $\tau(u \to v) = 2/3$.

In the following, we will complete the proof of the theorem by showing that the final charge $c'(x)$ of every element $x \in V(G^\times) \cup F(G^\times)$ is nonnegative, a contradiction. First of all, one can check that the final charge of every face and every 4^--vertex in $G^\times$ is nonnegative by a similar proof as that of Theorem 1.5 (1). So in what follows we only need to consider the 5^+-vertex $v \in V(G^\times)$.

If $d_{G^\times}(v) = 5$, then $\alpha(v) \leq 3$ by Lemma 2.10, which implies that $c'(v) \geq 1 - 3 \cdot 1/3 = 0$ by $\tilde{R}1.1$.

If $6 \leq d_{G^\times}(v) \leq 7$, then v is adjacent to no 5^--vertices by Lemma 2.2, which implies that $c'(v) \geq d_{G^\times}(v) - 4 - \alpha(v)/2 \geq d_{G^\times}(v) - 4 - d_{G^\times}(v)/3 \geq 0$ by $\tilde{R}1.2$ and Lemma 2.10.

If $8 \leq d_{G^\times}(v) \leq M-3$, then v is adjacent to no 4^--vertices by Lemma 2.1. This implies that $c'(v) \geq d_{G^\times}(v) - 4 - 2\alpha(v)/3 \geq d_{G^\times}(v) - 4 - 4d_{G^\times}(v)/9 > 0$ by $\tilde{R}1$ and Lemma 2.10.

If $d_{G^\times}(v) = M-2$, then v is not adjacent to 3^--vertices by Lemma 2.1 and may have at most three 4-dependents by Lemma 2.6. This implies that

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha(v) - 3 \cdot \frac{2}{3} \geq d_{G^\times}(v) - 6 - \frac{4}{9} d_{G^\times}(v) \geq \frac{5M-64}{9} > 0$$

by $\tilde{R}1$, $\tilde{R}3$ and Lemma 2.10, since $M \geq 6p+2 \geq 14$.

If $d_{G^\times}(v) = M-1$, then v has no 2-dependents by Lemma 2.1 but may have at most two 3-dependents and at most three 4-dependents by Lemma 2.6. So by $\tilde{R}1$, $\tilde{R}3$ and Lemma 2.10, we have

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha(v) - 3 \cdot \frac{2}{3} - 2 \cdot \frac{2}{3} \geq d_{G^\times}(v) - \frac{22}{3} - \frac{2}{3}\left\lfloor \frac{2}{3} d_{G^\times}(v) \right\rfloor = M - \frac{2}{3}\left\lfloor \frac{2}{3}(M-1) \right\rfloor - \frac{25}{3} > 0.$$

If $d_{G^\times}(v) \geq M$, then we shall again assume that $d_{G^\times}(v) = M = \Delta$, since $\Delta \leq M \leq d_{G^\times}(v) \leq \Delta$. By Lemma 2.6, v may have at most one 2-dependent, at most two 3-dependents and at most three 4-dependents. This implies that

$$c'(v) \geq d_{G^\times}(v) - 4 - \frac{2}{3}\alpha(v) - \frac{2}{3} - 2 \cdot \frac{2}{3} - 3 \cdot \frac{2}{3} \geq M - 8 - \frac{2}{3}\left\lfloor \frac{2}{3} M \right\rfloor \geq 0$$

by $\tilde{R}1$, $\tilde{R}3$ and Lemma 2.10, since $M \geq 6p+2 \geq 14$. Hence we have completed the proof of Theorem 1.5.

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