k-forested coloring of planar graphs with large girth

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(Communicated by Shigefumi MORI, M.J.A., Nov. 12, 2010)

Abstract: A proper vertex coloring of a simple graph G is k-forested if the subgraph induced by the vertices of any two color classes is a forest with maximum degree at most k. The k-forested chromatic number of a graph G, denoted by $\chi^a_k(G)$, is the smallest number of colors in a k-forested coloring of G. In this paper, it is shown that planar graphs with large enough girth do satisfy $\chi^a_k(G) = \lceil\frac{\Delta(G)}{k}\rceil + 1$ for all $\Delta(G) > k \geq 2$, and $\chi^a_k(G) \leq 3$ for all $\Delta(G) \leq k$ with the bound 3 being sharp. Furthermore, a conjecture on k-frugal chromatic number raised in [1] has been partially confirmed.

Key words: Acyclic coloring; frugal coloring; k-forested coloring; planar graphs; girth.

1. Introduction. In this paper, all graphs considered are finite, simple and undirected. For a planar graph G, we use $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the face set, the minimum degree and the maximum degree of a graph G, respectively. If $uv \in E(G)$, then u is said to be the neighbor of v in G. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of v in G. By $d_G(v) = |N_G(v)|$ (or $d(v)$ for simplicity), we denote the degree of a vertex v in G. The degree $d(f)$ of the face f in a planar graph is the number of edges that bound the face, where each cut-edge is counted twice. A k-, $(\geq k)$- and $(\leq k)$-vertex (or face) is a vertex (or face) of degree k, at least k and at most k, respectively. The girth $g(G)$ of a graph G is the length of a shortest cycle of G. The square G^2 of a graph G is the graph with the same vertex set in which two vertices are joined by an edge if their distance in G is at most two. For a real number x, let $\lceil x\rceil$ be the smallest integer not less than x. Any undefined notation follows that of Bondy and Murty [2].

A proper vertex coloring is acyclic if the union of any two color classes forms a forest, and is k-frugal if no color appears more than k times in the neighborhood of any vertex. The acyclic (or k-frugal) chromatic number of G, denoted by $\chi^a(G)$ (or $\chi_k(G)$), is the smallest number of colors in an acyclic coloring (or a k-frugal coloring) of G.

Acyclic coloring problem introduced in [10] has been extensively studied in many papers. In 1979, Borodin [3] proved Grünbaum's conjecture that every planar graph is acyclically 5-colorable and this bound is sharp. Now, acyclic coloring problem has attracted more and more attention since Coleman et al. [7,8] identified acyclic coloring as the model for computing a Hessian via a substitution method.

Frugal vertex coloring was introduced by Hind et al. in [11,12], as a tool towards improving results about the total chromatic number of a graph. It is showed in [11] that a graph with large enough maximum degree Δ has a $(\log^5\Delta)$-frugal coloring using at most $\Delta+1$ colors. By the definition of k-frugal chromatic number of G, it is clearly that $\chi_1(G)$ is the chromatic number of G^2 and $\chi_k(G)$ is the usual chromatic number of G (denoted by $\chi(G)$) while $k \geq \Delta(G)$. Regards the k-frugal chromatic number of a planar graph, in [1], Amini, Esperet and van den Heuvel raised a conjecture as follows:

Conjecture 1. *Planar graphs with large enough girth do satisfy* $\chi_k(G) = \lceil\frac{\Delta(G)}{k}\rceil + 1$ *for all* $k \geq 1$.

However, any non-bipartite planar graph can not be 2-colorable hence Conjecture 1 does not hold for $k \geq \Delta(G)$. Thus a reasonable modification of Conjecture 1 should be

Conjecture 2. *Planar graphs with large enough girth do satisfy*

2010 Mathematics Subject Classification. Primary 05C10; Secondary 05C15.

doi: 10.3792/pjaa.86.169

$$\chi_k(G) = \begin{cases} \lceil \frac{\Delta(G)}{k} \rceil + 1, & \text{if } \Delta(G) > k \geq 1; \\ \chi(G) \leq 3, & \text{if } \Delta(G) \leq k, \end{cases}$$

Moreover, the bound 3 *here is sharp.*

Regards the above conjecture, for the case when $k = 1$, the best known results are given by Borodin et al. [4,5]. They showed that $\chi_1(G) = \Delta(G) + 1$ if G is a planar graph with $\Delta(G) \geq 30$ and $g(G) \geq 7$, or $\Delta(G) \geq 16$ and $g(G) \geq 9$. The other nontrivial case for Conjecture 2 is when $k \geq 2$. In this paper, we want to solve it completely.

Now we start to introduce a concept involving acyclic coloring and k-frugal coloring. Suppose we are given a graph G. We want to properly color the vertices so that the subgraph induced by the union of any two color classes forms a forest with maximum degree at most k. Denote by $\chi_k^a(G)$ the smallest integer t so that a t-coloring of G with the requirements mentioned above is guaranteed to exist. Such an integer $\chi_k^a(G)$ is called the k-forested chromatic number and the corresponding coloring is called k-forested coloring. One can easily see that a k-forested coloring is actually an acyclic k-frugal coloring. Here, let us outline the relationships among all above definitions on many different colorings. The proofs of them are trivial so we omit them here.

Proposition 3. *For any graph* G *and integer* $k \geq 1$*, the following hold*:
(1) $\chi_1(G) = \chi_1^a(G) = \chi(G^2)$;
(2) $\lceil \frac{\Delta(G)}{k} \rceil + 1 \leq \chi_k(G) \leq \chi_k^a(G)$;
(3) $\chi^a(G) \leq \chi_k^a(G)$;
(4) $\chi_{k+1}(G) \leq \chi_k(G)$ *and* $\chi_{k+1}^a(G) \leq \chi_k^a(G)$;
(5) $\chi_k(G) = \chi(G)$ *and* $\chi_k^a(G) = \chi^a(G)$ *if* $k \geq \Delta(G)$.

Now we restrict G to be a planar graph. Regards k-forested chromatic number of G, if $k = 1$, then by Proposition 3(1), $\chi_1^a(G) = \chi(G^2) = \Delta(G) + 1$ if $\Delta(G) \geq 30$ and $g(G) \geq 7$ [4], or $\Delta(G) \geq 16$ and $g(G) \geq 9$ [5]. If $k = 2$, then the parameter $\chi_2^a(G)$ is also called linear chromatic number, and the corresponding coloring is called linear coloring. This special concept was first introduced by Yuster [14], and has been extensively studied in the past (cf: [9,13]). In [13], Raspaud and Wang showed that every planar graph G satisfies $\chi_2^a(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$ if $\Delta(G) \geq 3$ and $g(G) \geq 13$.

In this paper, we aim to estimate the value of k-forested chromatic number of planar graphs with large enough girth when $k \geq 3$. In particular, we show the following main results in the next section.

Theorem 4. *Given any two integers* $M > k \geq 3$*, let* G *be a planar graph with* $g(G) \geq 10$ *and* $\Delta(G) \leq M$*, then* $\chi_k^a(G) \leq \lceil \frac{M}{k} \rceil + 1$.

As a corollary of Propositions 3(2), 3(5), Theorem 4 (setting $M = \Delta(G)$ there) and the following Lemma 5, we deduce Theorem 6 as follows.

Lemma 5 [6]**.** *If* G *is a planar graph with girth* $g(G) \geq 7$*, then* $\chi^a(G) \leq 3$.

Theorem 6. *Let* G *be a planar graph with maximum degree* Δ *and girth* $g(G) \geq 10$*. Then*

$$\chi_k^a(G) = \begin{cases} \lceil \frac{\Delta}{k} \rceil + 1, & \text{if } \Delta > k \geq 3; \\ \chi^a(G) \leq 3, & \text{if } \Delta \leq k. \end{cases}$$

Remark. For the case $\Delta \leq k$ in Theorem 6, the bound 3 there is sharp because any graph that is not a forest admits no acyclic 2-colorings, hence no k-forested 2-colorings.

By the above arguments along with Proposition 3(2) and Theorem 6, we also have the following corollary on k-frugal chromatic number.

Corollary 7. *Let* G *be a planar graph with maximum degree* $\Delta > k$*. Then* $\chi_k(G) = \chi_k^a(G) = \lceil \frac{\Delta}{k} \rceil + 1$ *if* $k \geq 3$ *and* $g(G) \geq 10$*, or* $k = 2$ *and* $g(G) \geq 13$.

Hence, we have confirmed Conjectures 2 for the case when $k \geq 2$. Furthermore, the corresponding k-frugal colorings can also be acyclic for all $k \geq 1$ by Propositions 3(1), 3(2), Theorem 6 and Corollary 7.

2. Proof of Theorem 4. To begin with, we introduce some concepts that will be used frequently in the following proofs. Given a k-forested coloring c of a graph G using the color set C, we use $C_k(v)$ to denote the set of colors that are each used by c on exactly k neighbors of v. An (r, s)-type 2-vertex is a 2-vertex with one neighbor of degree r and the other of degree s. Without loss of generality, we always set $r \geq s$.

In what follows, a graph G with $\Delta(G) \leq M$ and $M > k \geq 3$ is called *critical* if $\chi_k^a(G) > \lceil \frac{M}{k} \rceil + 1$, but for any proper subgraph $H \subset G$, $\chi_k^a(H) \leq \lceil \frac{M}{k} \rceil + 1$. The following many lemmas are dedicated to the structures of the critical graph G. For brevity we will write $q = \lceil \frac{M}{k} \rceil + 1$ in the proofs of these lemmas.

Lemma 8. *There is no* 1*-vertex or* $(\leq 3, 2)$*-type* 2*-vertex in critical graph* G.

Proof. Suppose G contain a 1-vertex v. Let $H = G - v$. Since $|H| < |G|$ and G is critical,

$\chi_k^a(H) \le q$. Let c be a k-forested q-coloring of H using color set C. Now we extend c to v as follows and hence form a contradiction to the fact that $\chi_k^a(G) > q$, which completes the proof of this lemma. Denote the neighbor of v by u. We define a list of available colors for v as follows:

$$L(v) := C\backslash(\{c(u)\} \cup C_k(u)).$$

Since $|C_k(u)| \le \lfloor \frac{d_H(u)}{k} \rfloor = \lfloor \frac{d_G(u)-1}{k} \rfloor \le \lfloor \frac{\Delta-1}{k} \rfloor \le \lfloor \frac{M-1}{k} \rfloor = \lceil \frac{M}{k} \rceil - 1$ and $|C| = \lceil \frac{M}{k} \rceil + 1$, we have $|L(v)| \ge 1$. So we can color v with a color in $L(v)$.

Suppose G contains a 2-vertex v who is adjacent to a 2-vertex u and a (≤ 3)-vertex w. Consider the subgraph $H = G - v$. Since $|H| < |G|$ and G is critical, $\chi_k^a(H) \le q$. Let c be a k-forested q-coloring of H using color set C. Now we extend c to v. Denote another neighbor of u by x. We define a list of available colors for v as follows:

$$L(v) := \begin{cases} C\backslash\{c(u), c(x)\}, & \text{if } c(u) = c(w); \\ C\backslash\{c(u), c(w)\}, & \text{if } c(u) \ne c(w). \end{cases}$$

Since $|C| = \lceil \frac{M}{k} \rceil + 1 \ge \lceil \frac{k+1}{k} \rceil + 1 = 3$, $|L(v)| \ge 1$. So we can color v with a color in $L(v)$. □

Lemma 9. *Let G be a critical graph. If a 4-vertex in G is adjacent to four 2-vertices and three of them are $(4,2)$-type, then the rest one must be $(\ge 5, 4)$-type.*

Proof. Suppose that the lemma is false. Let $d(v) = 4$. Denote x, y, z to be three neighbors of v who are $(4,2)$-type 2-vertices and w to be the fourth neighbor of v who is $(4, \le 4)$-type 2-vertex. Let x_1, y_1, z_1, w_1 be the other neighbor of x, y, z, w, respectively. Then $d(x_1) = d(y_1) = d(z_1) = 2$. Denote the other neighbor of x_1, y_1 and z_1 by x_2, y_2 and z_2 respectively. Since w is $(4, \le 4)$-type 2-vertex, without loss of generality, we may assume $d(w_1) = 4$ and w_2, w_3, w_4 be another three neighbors of w_1. Choose $H = G - \{v, x, w\}$. Since G is critical, $\chi_k^a(H) \le q$. Let c be a k-forested q-coloring of H using color set C. Now we extend c to $\{v, x, w\}$. Suppose $c(y) \ne c(z)$. Without loss of generality, we assume $c(y) = 1$ and $c(z) = 2$. If $c(y_1) \ne 2$, we recolor y by 2. If $c(z_1) \ne 1$, we recolor z by 1. So we assume $c(y_1) = 2$ and $c(z_1) = 1$. Then we recolor both y and z by 3 (it is possible since $\lceil \frac{M}{k} \rceil + 1 \ge \lceil \frac{k+1}{k} \rceil + 1 = 3$). Thus, we can always assume that $c(y) = c(z)$. Without loss of generality, assume both y and z receive the color 3 in c.

Case 1. We can color w by 3.

Without loss of generality, we assume $c(z_1) = 1$. Next, we divide the proof of this case into two subcases.

Subcase 1.1. $c(y_1) = 1$.

Now we color v by 2. Suppose $c(x_1) = 2$ or $c(x_1) = 3$, we can color x by 1. So we assume that $c(x_1) = 1$. Then one of y_2 and z_2 must be colored by 3. For otherwise, we can recolor v by 1 and color x by 2. Without loss of generality, we assume $c(y_2) = 3$. Then we can recolor y and z by 2 and then v by 1. So we can color x by a color in $C\backslash\{c(x_1), c(x_2)\}$ at last (recall that $|C| \ge 3$).

Subcase 1.2. $c(y_1) \ne 1$.

Without loss of generality, we assume $c(w_1) = 1$. Then we color v by 2. Suppose $c(x_1) = 2$ or $c(x_1) = 3$. We can color x by 1. So we assume that $c(x_1) = 1$. By the similar proof as in Subcase 1.1, we have $c(z_2) = 3$. Then we recolor z by 2 and then v by 1. So we can color x by a color in $C\backslash\{c(x_1), c(x_2)\}$ again.

Case 2. We can not color w by 3.

Without loss of generality, we assume $c(w) = 1$. In this case, we can not recolor w by 3. This implies two subcases.

Subcase 2.1. $c(w_1) = 3$.

Suppose $c(y_1) \ne 2$ or $c(z_1) \ne 2$, we can color v by 2. If $c(x_1) = 2$, we can color x by a color in $C\backslash\{c(x_1), c(x_2)\}$. Else, $c(x_1) \ne 2$, we can also color x by a color in $C\backslash\{c(x_1), c(v)\}$. So we assume $c(y_1) = c(z_1) = 2$. Then we have $c(y_2) = c(z_2) = 3$ (for otherwise we can again color v by 2. Then x can be easily colored as before). Now we recolor y by 1. Then we can color v by 2 again. Similarly, we can also color x properly.

Subcase 2.2. $k = 3$ and $c(w_2) = c(w_3) = c(w_4) = 3$.

By the similar proof as in Subcase 2.1, we have $c(y_1) = c(z_1) = 2$ and $c(y_2) = c(z_2) = 3$. Now we recolor y by 1. Then we can color v by 2 again. Similarly as before, we can also color x properly. □

Lemma 10. *Let G be a critical graph. If a 4-vertex in G is adjacent to four 2-vertices and two of them are $(4,2)$-type, then either at least one of another two neighbors is $(\ge 5, 4)$-type or both of them are $(4,4)$-type.*

Proof. Suppose that the lemma is false. Let $d(v) = 4$. Denote w, x to be two neighbors of v who are $(4,2)$-type 2-vertices and y, z to be another two neighbors of v. Without loss of generality, we assume y is $(4,3)$-type 2-vertex and z is $(4,4)$-type 2-vertex (the case when both y and z are $(4,3)$-type 2-vertices can be dealt with similarly but much easierly). Let x_1, y_1, z_1, w_1 be the other neighbor of x, y, z, w, respectively. Then $d(w_1) = 2$, $d(y_1) = 3$, $d(z_1) = 4$. Denote the other neighbor of w_1 by w_2 and another three neighbors of z_1 by z_2, z_3, z_4. Choose $H = G - \{v, w, x\}$. Since G is critical, $\chi_k^a(H) \leq q$. Let c be an k-forested q-coloring of H using color set C. Now we extend c to $\{v, w, x\}$.

Case 1. $c(y) = c(z)$.

Without loss of generality, we assume $c(y) = c(z) = 1$.

Subcase 1.1. $c(w_1) = 1$.

Now we color w by 2. Suppose $c(y_1) \neq 3$ or $c(z_1) \neq 3$. Then we can color v by 3. If $c(x_1) = 3$, we can color x by a color in $C \backslash \{c(x_1), c(x_2)\}$. Else if $c(x_1) \neq 3$, we can color x by a color in $C \backslash \{c(x_1), c(v)\}$. So we assume $c(y_1) = c(z_1) = 3$. We recolor y by 2 (it is possible since $d(y_1) = 3$ and $k \geq 3$) and color v by 3. Then x can be similarly colored as before.

Subcase 1.2. $c(w_1) \neq 1$.

Without loss of generality, we assume $c(w_1) = 2$. Then we color w by 3. By the similar proof as in Subcase 1.1, we must have $c(y_1) = c(z_1) = 2$. Then we recolor y by 3. Suppose $c(w_2) \neq 3$, we can color v by 2. Then x can be easily colored. So we assume $c(w_2) = 3$. Then we recolor w by 1 and color v by 2. At last, x can be also easily colored as before.

Case 2. $c(y) \neq c(z)$.

Without loss of generality, we assume $c(y) = 1$ and $c(z) = 2$. Then we must have $c(y_1) = 2$. For otherwise, we can recolor y by 2 and come back to Case 1. Similarly, we have $c(z_1) = 1$ or $c(z_2) = c(z_3) = c(z_4) = 1$ since for otherwise we can recolor z by 1 and then come back to Case 1 again. In each case, we can color w by a color in $\{1, 2\} \backslash \{c(w_1)\}$ and then color v by 3. Similarly as before, we can color x properly at last. $\square$

Lemma 11. *Let G be a critical graph. If a 3-vertex in G is adjacent to three 2-vertices and one of them is $(3,3)$-type, then at least one of another two neighbors is $(\geq 5, 3)$-type.*

Proof. Suppose that the lemma is false. Let $d(v) = 3$. Denote x, y, z to be neighbors of v of degree 2. Let x_1, y_1, z_1 be the other neighbor of x, y, z, respectively. Suppose y is $(3,3)$-type while x, z are both $(\leq 4, 3)$-type. Without loss of generality, we assume $d(x_1) = d(z_1) = 4$ (that is, x, z are both $(4,3)$-type). Let $N(x_1) = \{x, x_2, x_3, x_4\}$ and $N(z_1) = \{z, z_2, z_3, z_4\}$. Choose $H = G - \{v, x, y\}$. Since G is critical, $\chi_k^a(H) \leq q$. Let c be an k-forested q-coloring of H using color set C. Now we extend c to $\{v, x, y\}$.

Case 1. $c(y_1) \neq c(z)$.

Without loss of generality, we assume $c(y_1) = 1$ and $c(z) = 2$. Then we color y by 2 and then v by 3. Suppose $c(x_1) = 3$. If $k = 3$ and $c(x_2) = c(x_3) = c(x_4) = 1$, we can color x by 2, otherwise we can color x by 1. So $c(x_1) \neq 3$. Suppose $c(x_1) = 1$. If $k = 3$ and $c(x_2) = c(x_3) = c(x_4) = 2$, we recolor y by 3 and then v by 1. Then we color x by 3. Otherwise, we can color x by 2. So $c(x_1) \neq 1$. Similarly, we have $c(x_1) \neq 2$. Thus, $c(x_1) \in C \backslash \{1, 2, 3\}$ (if exists). Since $d(x_1) = 4$, $|C_k(x_1)| \leq 1$. So we can color x by $\{1, 2\} \backslash \{C_k(x_1)\}$.

Case 2. $c(x_1) \neq c(z)$.

Without loss of generality, we assume $c(x_1) = 1$ and $c(z) = 2$.

Subcase 2.1. we can color x by 2.

Now we color v by 3. If $c(y_1) = 3$, we can color y by 1. Else if $c(y_1) \neq 3$, we can color y by a color in $C \backslash \{c(v), c(y_1)\}$.

Subcase 2.2. we can not color x by 2.

This subcase implies that $k = 3$ and $c(x_2) = c(x_3) = c(x_4) = 2$. Then we color x by 3 and v by 1. If $c(y_1) = 1$, we can color y by 3. Else if $c(y_1) \neq 1$, we can color y by a color in $C \backslash \{c(v), c(y_1)\}$.

Case 3. $c(x_1) = c(y_1) = c(z)$.

Without loss of generality, we assume $c(x_1) = c(y_1) = c(z) = 1$. Then we color y by 2 and v by 3. If $k = 3$ and $c(x_2) = c(x_3) = c(x_4) = 2$, we recolor y by 3 and v by 2. Then x can be colored by 3. Otherwise, we can color x by 2. $\square$

In the following, we will complete the proof of Theorem 4.

Proof of Theorem 4. We prove it by contradiction. Suppose that the theorem is false. We choose G to be critical with $g(G) \geq 10$ and use the discharging method on G in the following argument. For a planar graph one can easily deduce the following identity by the well-known Euler's formula

$$\sum_{v \in V(G)} (4d(v) - 10) + \sum_{f \in F(G)} (d(f) - 10) = -20.$$

Let $w(x)$ be the initial charge defined on $x \in V(G) \cup F(G)$. Define $w(v) = 4d(v) - 10$ for each $v \in V(G)$ and $w(f) = d(f) - 10$ for each $f \in F(G)$. Then we have $\Sigma_{x\in V(G)\cup F(G)}w(x) = -20$. Now we state our discharging rules and perform them on vertices and faces of G. Let $w'(x)$ be the charge of $x \in V(G) \cup F(G)$ once the discharging is finished.

R1. Each (≥ 5)-vertex gives 2 to each adjacent 2-vertex.

R2. Each 4-vertex gives 2 to each adjacent $(4,2)$-type 2-vertex, $\frac{4}{3}$ to each adjacent $(4,3)$-type 2-vertex, 1 to each adjacent $(4,4)$-type 2-vertex.

R3. Each 3-vertex gives 1 to each adjacent $(3,3)$-type 2-vertex, $\frac{2}{3}$ to each adjacent $(4,3)$-type 2-vertex.

Let $f \in F(G)$. Since $g(G) \geq 10$, $d(f) \geq 10$. Thus, $w'(f) = w(f) = d(f) - 10 \geq 0$.

Let $v \in E(G)$. Then $d(v) \geq 2$ by Lemma 8. Suppose $d(v) = 2$, we have $w(v) = -2$. If v is $(4,2)$-type, $w'(v) = w(v) + 2 = 0$; If v is $(\geq 5, 2)$-type, $w'(v) = w(v) + 2 = 0$; If v is $(3,3)$-type, $w'(v) = w(v) + 1 \times 2 = 0$; If v is $(4,3)$-type, $w'(v) = w(v) + \frac{4}{3} + \frac{2}{3} = 0$; If v is $(\geq 5, 3)$-type, $w'(v) = w(v) + 2 = 0$; If v is $(4,4)$-type, $w'(v) = w(v) + 1 \times 2 = 0$; If v is $(\geq 5, 4)$-type or $(\geq 5, \geq 5)$-type, $w'(v) \geq w(v) + 2 = 0$.

Suppose $d(v) = 3$. Then $w(v) = 2$. If v is adjacent to at most two 2-vertices, since v gives out at most 1 to each neighbor by R3, then we have $w'(v) \geq w(v) - 1 \times 2 = 0$. If v is adjacent to three 2-vertices but none of them is $(3,3)$-type. Then by R3, v gives out at most $\frac{2}{3}$ to each neighbor. Thus, $w'(v) \geq w(v) - 3 \times \frac{2}{3} = 0$. If v is adjacent to three 2-vertices and at least one of them is $(3,3)$-type, then by Lemma 11 and R3, v gives out charge to at most two neighbors. Thus, $w'(v) \geq w(v) - 2 \times 1 = 0$.

Suppose $d(v) = 4$. Then $w(v) = 6$. If v is adjacent to at most three 2-vertices. Since v gives out at most 2 to each neighbor by R2, $w'(v) \geq w(v) - 3 \times 2 = 0$. If v is adjacent to four 2-vertices and three of them are $(4,2)$-type, then by Lemma 9 and R2, v gives out charge to at most three neighbors. Thus, $w'(v) \geq w(v) - 3 \times 2 = 0$. If v is adjacent to four 2-vertices and two of them are $(4,2)$-type. Then by Lemma 10, either one of another two neighbors of v is $(\geq 5, 4)$-type or both of them are $(4,4)$-type. Thus, $w'(v) \geq \min\{w(v) - 2 \times 2 - 2, w(v) - 2 \times 2 - 1 \times 2\} = 0$. If v is adjacent to four 2-vertices but at most of them are $(4,2)$-type. Then by R2, we have $w'(v) \geq \min\{w(v) - 2 - 3 \times \frac{4}{3}, w(v) - 4 \times \frac{4}{3}\} = 0$.

Suppose $d(v) \geq 5$. By R1, we have $w'(v) \geq w(v) - 2d(v) = 4d(v) - 10 - 2d(v) = 2d(v) - 10 \geq 0$.

Till now we have proved that $w'(x) \geq 0$ for all $x \in V(G) \bigcup F(G)$. So $\Sigma_{x\in V(G)\cup F(G)}w'(x) \geq 0$. But $\Sigma_{x\in V(G)\cup F(G)}w'(x) = \Sigma_{x\in V(G)\cup F(G)}w(x) = -20$ because our rules only move charge around, and do not affect the sum. This contradiction completes the proof of the theorem. □

Acknowledgements. The authors appreciate the anonymous referees sincerely for their helpful comments and suggestions. This work was partially supported by NSFC grants 10871119, 10971121, 61070230 and RFDP grant 200804220001 of China.

References

[1] O. Amini, L. Esperet and J. van den Heuvel, Frugal Colouring of Graphs. `http://arxiv.org/pdf/0705.0422v1`

[2] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[3] O. V. Borodin, On acyclic colorings of planar graphs, Discrete Math. **25** (1979), no. 3, 211–236.

[4] O. V. Borodin et al., Sufficient conditions for planar graphs to be 2-distance $(\Delta + 1)$-colorable, Sib. Èlektron. Mat. Izv. **1** (2004), 129–141.

[5] O. V. Borodin, A. O. Ivanova and T. K. Neustroeva, 2-distance coloring of sparse planar graphs, Sib. Èlektron. Mat. Izv. **1** (2004), 76–90.

[6] O. V. Borodin, A. V. Kostochka and D. R. Woodall, Acyclic colourings of planar graphs with large girth, J. London Math. Soc. (2) **60** (1999), no. 2, 344–352.

[7] T. F. Coleman and J.-Y. Cai, The cyclic coloring problem and estimation of sparse Hessian matrices, SIAM J. Algebraic Discrete Methods **7** (1986), no. 2, 221–235.

[8] T. F. Coleman and J. J. Moré, Estimation of sparse Hessian matrices and graph coloring problems, Math. Programming **28** (1984), no. 3, 243–270.

[9] L. Esperet, M. Montassier and A. Raspaud, Linear choosability of graphs, Discrete Math. **308** (2008), no. 17, 3938–3950.

[10] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math. **14** (1973), 390–408.

[11] H. Hind, M. Molloy and B. Reed, Colouring a graph frugally, Combinatorica **17** (1997), no. 4, 469–482.

[12] H. Hind, M. Molloy and B. Reed, Total coloring with $\Delta + \text{poly}(\log\Delta)$ colors, SIAM J. Comput. **28** (1999), no. 3, 816–821.

[13] A. Raspaud and W. Wang, Linear coloring of planar graphs with large girth, Discrete Math. **309** (2009), no. 18, 5678–5686.

[14] R. Yuster, Linear coloring of graphs, Discrete Math. **185** (1998), no. 1–3, 293–297.