1. Introduction. In this paper, all graphs considered are finite, simple and undirected. For a planar graph $G$, we use $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the face set, the minimum degree and the maximum degree of a graph $G$, respectively. If $uv \in E(G)$, then $u$ is said to be the neighbor of $v$ in $G$. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of neighbors of $v$ in $G$. By $d_G(v) = |N_G(v)|$ (or $d(v)$ for simplicity), we denote the degree of a vertex $v$ in $G$. The degree $d(f)$ of the face $f$ in a planar graph is the number of edges that bound the face, where each cut-edge is counted twice. A $k$-, $(\geq k)$- and $(\leq k)$-vertex (or face) is a vertex (or face) of degree $k$, at least $k$ and at most $k$, respectively. The girth $g(G)$ of a graph $G$ is the length of a shortest cycle of $G$. The square $G^2$ of a graph $G$ is the graph with the same vertex set in which two vertices are joined by an edge if their distance in $G$ is at most two. For a real number $x$, let $\lceil x \rceil$ be the smallest integer not less than $x$. Any undefined notation follows that of Bondy and Murty [2].

A proper vertex coloring is acyclic if the union of any two color classes forms a forest, and is $k$-frugal if no color appears more than $k$ times in the neighborhood of any vertex. The acyclic (or $k$-frugal) chromatic number of $G$, denoted by $\chi^a(G)$ (or $\chi_k(G)$), is the smallest number of colors in an acyclic coloring (or a $k$-frugal coloring) of $G$.

Acyclic coloring problem introduced in [10] has been extensively studied in many papers. In 1979, Borodin [3] proved Grünbaum’s conjecture that every planar graph is acyclically 5-colorable and this bound is sharp. Now, acyclic coloring problem has attracted more and more attention since Coleman et al. [7,8] identified acyclic coloring as the model for computing a Hessian via a substitution method.

Frugal vertex coloring was introduced by Hind et al. in [11,12], as a tool towards improving results about the total chromatic number of a graph. It is showed in [11] that a graph with large enough maximum degree $\Delta$ has a $(\log^2 \Delta)$-frugal coloring using at most $\Delta + 1$ colors. By the definition of $k$-frugal chromatic number of $G$, it is clearly that $\chi_1(G)$ is the chromatic number of $G^2$ and $\chi_k(G)$ is the usual chromatic number of $G$ (denoted by $\chi(G)$) while $k \geq \Delta(G)$. Regards the $k$-frugal chromatic number of a planar graph, in [1], Amini, Esperet and van den Heuvel raised a conjecture as follows:

**Conjecture 1.** Planar graphs with large enough girth do satisfy $\chi_k(G) = \lceil \frac{\Delta(G)+1}{k} \rceil + 1$ for all $k \geq 1$.

However, any non-bipartite planar graph can not be 2-colorable hence Conjecture 1 does not hold for $k \geq \Delta(G)$. Thus a reasonable modification of Conjecture 1 should be

**Conjecture 2.** Planar graphs with large enough girth do satisfy

$k$-forested coloring of planar graphs with large girth

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**Abstract:** A proper vertex coloring of a simple graph $G$ is $k$-forested if the subgraph induced by the vertices of any two color classes is a forest with maximum degree at most $k$. The $k$-forested chromatic number of a graph $G$, denoted by $\chi_k^a(G)$, is the smallest number of colors in a $k$-forested coloring of $G$. In this paper, it is shown that planar graphs with large enough girth do satisfy $\chi_k^a(G) = \lceil \frac{\Delta(G)+1}{k} \rceil + 1$ for all $\Delta(G) > k \geq 2$, and $\chi_k^a(G) \leq 3$ for all $\Delta(G) \leq k$ with the bound 3 being sharp. Furthermore, a conjecture on $k$-frugal chromatic number raised in [1] has been partially confirmed.

**Key words:** Acyclic coloring; frugal coloring; $k$-forested coloring; planar graphs; girth.
\[ \chi_k(G) = \begin{cases} \lceil \frac{\Delta(G)}{k-1} \rceil + 1, & \text{if } \Delta(G) > k \geq 1; \\ \chi(G) \leq 3, & \text{if } \Delta(G) \leq k, \end{cases} \]

Moreover, the bound 3 here is sharp.

Regard the above conjecture, for the case when \( k=1 \), the best known results are given by Borodin et al. [4,5]. They showed that \( \chi_1(G) = \Delta(G) + 1 \) if \( G \) is a planar graph with \( \Delta(G) \geq 30 \) and \( g(G) \geq 7 \), or \( \Delta(G) \geq 16 \) and \( g(G) \geq 9 \). The other nontrivial case for Conjecture 2 is when \( k \geq 2 \). In this paper, we want to solve it completely.

Now we start to introduce a concept involving acyclic coloring and \( k \)-frugal coloring. Suppose we are given a graph \( G \). We want to properly color the vertices so that the subgraph induced by the union of any two color classes forms a forest with maximum degree at most \( k \). Denote by \( \chi^k(G) \) the smallest integer \( t \) so that a \( k \)-coloring of \( G \) with the requirements mentioned above is guaranteed to exist. Such an integer \( \chi^k(G) \) is called the \( k \)-forested chromatic number and the corresponding coloring is called \( k \)-forested coloring. One can easily see that a \( k \)-forested coloring is actually an acyclic \( k \)-frugal coloring. Here, let us outline the relationships among all above definitions on many different colorings. The proofs of them are trivial so we omit them here.

**Proposition 3.** For any graph \( G \) and integer \( k \geq 1 \), the following hold:

1. \( \chi_1(G) = \chi^3(G) = \chi(G^2) \);
2. \( \lceil \frac{\Delta(G)}{k-1} \rceil + 1 \leq \chi_k(G) \leq \chi^k(G) \);
3. \( \chi^k(G) \leq \chi(G) \);
4. \( \chi_{k+1}(G) \leq \chi_k(G) \) and \( \chi_{k+1}^k(G) \leq \chi^k(G) \);
5. \( \chi_k(G) = \chi(G) \) and \( \chi_k^k(G) = \chi^k(G) \) if \( k \geq \Delta(G) \).

Now we restrict \( G \) to be a planar graph. Regards \( k \)-forested chromatic number of \( G \), if \( k=1 \), then by Proposition 3(1), \( \chi_1(G) = \chi(G^2) = \Delta(G) + 1 \) if \( \Delta(G) \geq 30 \) and \( g(G) \geq 7 \) [4], or \( \Delta(G) \geq 16 \) and \( g(G) \geq 9 \) [5]. If \( k=2 \), then the parameter \( \chi^2_k(G) \) is also called linear chromatic number, and the corresponding coloring is called linear coloring. This special concept was first introduced by Yuster [14], and has been extensively studied in the past (cf: [9,13]). In [13], Raspaud and Wang showed that every planar graph \( G \) satisfies \( \chi^2_2(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1 \) if \( \Delta(G) \geq 3 \) and \( g(G) \geq 13 \).

In this paper, we aim to estimate the value of \( k \)-forested chromatic number of planar graphs with large enough girth when \( k \geq 3 \). In particular, we show the following main results in the next section.

**Theorem 4.** Given any two integers \( M > k \geq 3 \), let \( G \) be a planar graph with \( g(G) \geq 10 \) and \( \Delta(G) \leq M \), then \( \chi^k(G) \leq \lceil \frac{M}{k} \rceil + 1 \).

As a corollary of Propositions 3(2), 3(5), Theorem 4 (setting \( M = \Delta(G) \) there) and the following Lemma 5, we deduce Theorem 6 as follows.

**Lemma 5** [6]. If \( G \) is a planar graph with girth \( g(G) \geq 7 \), then \( \chi^k(G) \leq 3 \).

**Theorem 6.** Let \( G \) be a planar graph with maximum degree \( \Delta \) and girth \( g(G) \geq 10 \). Then

\[ \chi^k_2(G) = \begin{cases} \lceil \frac{\Delta}{k} \rceil + 1, & \text{if } \Delta > k \geq 3; \\ \chi^k(G) \leq 3, & \text{if } \Delta \leq k. \end{cases} \]

**Remark.** For the case \( \Delta \leq k \) in Theorem 6, the bound 3 there is sharp because any graph that is not a forest admits no acyclic 2-colorings, hence no \( k \)-forested 2-colorings.

By the above arguments along with Proposition 3(2) and Theorem 6, we also have the following corollary on \( k \)-frugal chromatic number.

**Corollary 7.** Let \( G \) be a planar graph with maximum degree \( \Delta > k \). Then \( \chi^k_2(G) = \chi^k(G) = \lceil \frac{\Delta}{k} \rceil + 1 \) if \( k \geq 3 \) and \( g(G) \geq 10 \), or \( k = 2 \) and \( g(G) \geq 13 \).

Hence, we have confirmed Conjectures 2 for the case when \( k \geq 2 \). Furthermore, the corresponding \( k \)-frugal colorings can also be acyclic for all \( k \geq 1 \) by Propositions 3(1), 3(2), Theorem 6 and Corollary 7.

2. Proof of Theorem 4. To begin with, we introduce some concepts that will be used frequently in the following proofs. Given a \( k \)-forested coloring \( c \) of a graph \( G \) using the color set \( C \), we use \( C_i(v) \) to denote the set of colors that are each used by \( c \) on exactly \( k \) neighbors of \( v \). An \((r,s)\)-type 2-vertex is a 2-vertex with one neighbor of degree \( r \) and the other of degree \( s \). Without loss of generality, we always set \( r \geq s \).

In what follows, a graph \( G \) with \( \Delta(G) \leq M \) and \( M > k \geq 3 \) is called critical if \( \chi^k_2(G) > \lceil \frac{M}{k} \rceil + 1 \), but for any proper subgraph \( H \subset G \), \( \chi^k_2(H) \leq \lceil \frac{M}{k} \rceil + 1 \). The following many lemmas are dedicated to the structures of the critical graph \( G \). For brevity, we will write \( q = \lceil \frac{M}{k} \rceil + 1 \) in the proofs of these lemmas.

**Lemma 8.** There is no 1-vertex or \((\leq 3,2)\)-type 2-vertex in critical graph \( G \).

**Proof.** Suppose \( G \) contain a 1-vertex \( v \). Let \( H = G - v \). Since \( |H| < |G| \) and \( G \) is critical,
\( \chi^*_k(H) \leq q \). Let \( c \) be a \( k \)-forested \( q \)-coloring of \( H \) using color set \( C \). Now we extend \( c \) to \( v \) as follows and hence form a contradiction to the fact that \( \chi^*_k(G) > q \), which completes the proof of this lemma.

Denote the neighbor of \( v \) by \( u \). We define a list of available colors for \( v \) as follows:

\[ L(v) := C \setminus \{\{c(u)\} \cup C_k(u)\} \]

Since \( |C_k(u)| \leq \left\lfloor \frac{q_{u}(u)}{k} \right\rfloor \leq \left\lceil \frac{q_{u}(u) - 1}{k} \right\rceil \leq \left\lceil \frac{\Delta - 1}{k} \right\rceil = \left\lceil \frac{q}{k} \right\rceil - 1 \) and \( |C| = \left\lceil \frac{q}{2} \right\rceil + 1 \), we have \( |L(v)| \geq 1 \). So we can color \( v \) with a color in \( L(v) \).

Suppose \( G \) contains a 2-vertex \( v \) who is adjacent to a 2-vertex \( u \) and a \( (\leq 3) \)-vertex \( w \).

Consider the subgraph \( H = G - v \). Since \( |H| < |G| \) and \( G \) is critical, \( \chi^*_k(H) \leq q \). Let \( c \) be a \( k \)-forested \( q \)-coloring of \( H \) using color set \( C \). Now we extend \( c \) to \( v \).

Denote another neighbor of \( u \) by \( x \). We define a list of available colors for \( v \) as follows:

\[ L(v) := \begin{cases} C \setminus \{c(u), c(x)\}, & \text{if } c(u) = c(w) \; \text{or} \; c(x) \neq c(w) \; \text{and} \; c \in L(v) \end{cases} \]

Since \( |C| = \lceil \frac{q}{2} \rceil + 1 \geq \lceil \frac{\Delta + 1}{3} \rceil + 1 = 3 \), \( |L(v)| \geq 1 \). So we can color \( v \) with a color in \( L(v) \).

**Lemma 9.** Let \( G \) be a critical graph. If a 4-vertex in \( G \) is adjacent to four 2-vertices and three of them are \((4, 2)\)-type, then the rest one must be \((\geq 5, 4)\)-type.

**Proof.** Suppose that the lemma is false. Let \( d(v) = 4 \). Denote \( x, y, z, w \) to be three neighbors of \( v \) who are \((4, 2)\)-type 2-vertices and \( w \) to be the fourth neighbor of \( v \) who is \((4, \leq 4)\)-type 2-vertex.

Let \( x_1, y_1, z_1, w_1 \) be the other neighbors of \( x, y, z, w \), respectively. Then \( d(x_1) = d(y_1) = d(z_1) = 2 \). Denote the other neighbor of \( x_1, y_1 \) and \( z_1 \) by \( x_2, y_2 \) and \( z_2 \), respectively. Since \( w \) is \((4, \leq 4)\)-type 2-vertex, without loss of generality, we may assume \( d(w_1) = 4 \) and \( w_2, w_3, w_4 \) be another three neighbors of \( w_1 \). Choose \( H = G - \{v, x, w\} \). Since \( G \) is critical, \( \chi^*_k(H) \leq q \). Let \( c \) be a \( k \)-forested \( q \)-coloring of \( H \) using color set \( C \). Now we extend \( c \) to \( \{v, x, w\} \).

Suppose \( c(y) \neq c(z) \). Without loss of generality, we assume \( c(y) = 1 \) and \( c(z) = 2 \). If \( c(y_1) \neq 2 \), we recolor \( y \) by 2. If \( c(z_1) \neq 1 \), we recolor \( z \) by 1. So we assume \( c(y_1) = 2 \) and \( c(z_1) = 1 \). Then we recolor both \( y \) and \( z \) by 3 (it is possible since \( \left\lceil \frac{\Delta}{3} \right\rceil + 1 \geq \left\lceil \frac{\Delta + 1}{3} \right\rceil + 1 = 3 \)). Thus, we can always assume that \( c(y) = c(z) \). Without loss of generality, assume both \( y \) and \( z \) receive the color 3 in \( c \).

**Case 1.** We can color \( w \) by 3.

Without loss of generality, we assume \( c(x_1) = 1 \). Next, we divide the proof of this case into two subcases.

Subcase 1.1. \( c(y_1) = 1 \).

Now we color \( v \) by 2. Suppose \( c(x_1) = 2 \) or \( c(x_1) = 3 \), we can color \( x \) by 1. So we assume that \( c(x_1) = 1 \). Then one of \( y_2 \) and \( z_2 \) must be colored by 3. For otherwise, we can recolor \( v \) by 1 and color \( x \) by 2. Without loss of generality, we assume \( c(y_1) = 3 \). Then we can recolor \( y \) and \( z \) by 2 and then \( v \) by 1. So we can color \( x \) by a color in \( C \setminus \{c(x_1), c(x_2)\} \) at last (recall that \(|C| \geq 3\)).

Subcase 1.2. \( c(y_1) \neq 1 \).

Without loss of generality, we assume \( c(w_1) = 1 \). Then we color \( v \) by 2. Suppose \( c(x_1) = 2 \) or \( c(x_1) = 3 \), we can color \( x \) by 1. So we assume that \( c(x_1) = 1 \). By the similar proof as in Subcase 1.1, we have \( c(z_2) = 3 \). Then we recolor \( z \) by 2 and then \( v \) by 1. So we can color \( x \) by a color in \( C \setminus \{c(x_1), c(x_2)\} \) again.

Case 2. We can not color \( w \) by 3.

Without loss of generality, we assume \( c(w) = 1 \). In this case, we can not recolor \( w \) by 3. This implies two subcases.

Subcase 2.1. \( c(w_1) = 3 \).

Suppose \( c(y_1) \neq 2 \) or \( c(z_1) \neq 2 \), we can color \( v \) by 2.

If \( c(x_1) = 2 \), we can color \( x \) by a color in \( C \setminus \{c(x_1) \}, c(x_2)\} \). Else, \( c(x_1) \neq 2 \), we can also color \( x \) by a color in \( C \setminus \{c(x_1)\} \). So we assume \( c(y_1) = c(z_1) = 2 \). Then we have \( c(y_2) = c(z_2) = 3 \) (for otherwise we can again color \( x \) by 2). Then \( x \) can be easily colored as before. Now we recolor \( y \) by 1. Then we can color \( v \) by 2 again. Similarly, we can also color \( x \) properly.

Subcase 2.2. \( k = 3 \) and \( c(w_2) = c(w_3) = c(w_4) = 3 \).

By the similar proof as in Subcase 2.1, we have \( c(y_1) = c(z_1) = 2 \) and \( c(y_2) = c(z_2) = 3 \). Now we recolor \( y \) by 1. Then we can color \( v \) by 2 again. Similarly as before, we can also color \( x \) properly.

**Lemma 10.** Let \( G \) be a critical graph. If a 4-vertex in \( G \) is adjacent to four 2-vertices and two of them are \((4, 2)\)-type, then either at least one of another two neighbors is \((\geq 5, 4)\)-type or both of them are \((4, 4)\)-type.

**Proof.** Suppose that the lemma is false. Let \( d(v) = 4 \). Denote \( w, x \) to be two neighbors of \( v \) who are \((4, 2)\)-type 2-vertices and \( y, z \) to be another two neighbors of \( v \). Without loss of generality, we
assume \( y \) is \((4,3)\)-type 2-vertex and \( z \) is \((4,4)\)-type 2-vertex (the case when both \( y \) and \( z \) are \((4,3)\)-type 2-vertices can be dealt with similarly but much easier). Let \( x_1, y_1, z_1, w_1 \) be the other neighbor of \( x, y, z, w \), respectively. Then \( d(w_1) = 2, d(y_1) = 3, d(z_1) = 4 \). Denote the other neighbor of \( w_1 \) by \( w_2 \) and another three neighbors of \( z_1 \) by \( z_2, z_3, z_4 \). Choose \( H = G - \{v, w, x\} \). Since \( G \) is critical, \( \chi^*(H) \leq q \). Let \( c \) be an \( k \)-forested \( q \)-coloring of \( H \) using color set \( C \). Now we extend \( c \) to \( \{v, w, x\} \).

Case 1. \( c(y_1) \neq c(z) \).

Without loss of generality, we assume \( c(y_1) = c(z) = 1 \).

Subcase 1.1. \( c(w_1) = 1 \).

Now we color \( w \) by 2. Suppose \( c(y_1) \neq 3 \) or \( c(z_1) \neq 3 \). Then we can color \( v \) by 3. If \( c(x_1) = 3 \), we can color \( x \) by a color in \( C \setminus \{c(x_1), c(z_1)\} \). If \( c(x_1) \neq 3 \), we can color \( y \) by a color in \( C \setminus \{c(x_1), c(y_1)\} \). So we assume \( c(y_1) = c(z_1) = 3 \). We recolor \( y \) by 2 (it is possible since \( d(y_1) = 3 \) and \( k \geq 3 \)) and color \( v \) by 3. Then \( x \) can be similarly colored as before.

Subcase 1.2. \( c(w_1) \neq 1 \).

Without loss of generality, we assume \( c(w_1) = 2 \). Then we color \( w \) by 3. By the similar proof as in Subcase 1.1, we must have \( c(y_1) = c(z_1) = 2 \). Then we recolor \( y \) by 3. Suppose \( c(w_2) \neq 3 \), we can color \( v \) by 2. Then \( x \) can be easily colored. So we assume \( c(w_2) = 3 \). Then we recolor \( w \) by 1 and color \( v \) by 2. At last, \( x \) can be also easily colored as before.

Case 2. \( c(y) \neq c(z) \).

Without loss of generality, we assume \( c(y_1) = 1 \) and \( c(z) = 2 \). Then we must have \( c(y_1) = 2 \). For otherwise, we can recolor \( y \) by 2 and come back to Case 1. Similarly, we have \( c(z_1) = 1 \) or \( c(z_2) = c(z_1) = 1 \) since for otherwise we can recolor \( z \) by 1 and then come back to Case 1 again. In each case, we can color \( w \) by a color in \( \{1, 2\} \setminus \{c(w_1)\} \) and then color \( v \) by 3. Similarly as before, we can color \( x \) properly at last. \( \square \)

**Lemma 11.** Let \( G \) be a critical graph. If a 3-vertex in \( G \) is adjacent to three 2-vertices and one of them is \((3,3)\)-type, then at least one of another two neighbors is \((\geq 3,3)\)-type.

**Proof.** Suppose that the lemma is false. Let \( d(v) = 3 \). Denote \( x, y, z \) to be neighbors of \( v \) of degree 2. Let \( x_1, y_1, z_1 \) be the other neighbor of \( x, y, z \), respectively. Suppose \( y \) is \((3,3)\)-type while \( x, z \) are both \((\leq 4,3)\)-type. Without loss of generality, we assume \( d(x_1) = d(z_1) = 4 \) (that is, \( x, z \) are both \((4,3)\)-type). Let \( N(x_1) = \{x, x_2, x_3, x_4\} \) and \( N(z_1) = \{z, z_2, z_3, z_4\} \). Choose \( H = G - \{v, w, x\} \). Since \( G \) is critical, \( \chi^*(H) \leq q \). Let \( c \) be an \( k \)-forested \( q \)-coloring of \( H \) using color set \( C \). Now we extend \( c \) to \( \{v, w, x\} \).

Case 1. \( c(y_1) \neq c(z) \).

Without loss of generality, we assume \( c(y_1) = 1 \) and \( c(z) = 2 \). Then we color \( y \) by 2 and then \( v \) by 3. Suppose \( c(x_1) = 3 \). If \( k = 3 \) and \( c(x_2) = c(x_3) = c(x_4) = 1 \), we can color \( x \) by 2, otherwise we can color \( x \) by 1. So \( c(x_1) \neq 3 \). Suppose \( c(x_1) = 1 \). If \( k = 3 \) and \( c(x_2) = c(x_3) = c(x_4) = 2 \), we recolor \( y \) by 3 and then \( v \) by 1. Then we color \( x \) by 3. Otherwise, we can color \( x \) by 2. So \( c(x_1) \neq 1 \). Similarly, we have \( c(x_1) \neq 2 \). Thus, \( c(x_1) \in C \setminus \{1, 2, 3\} \) (if exists). Since \( d(x_1) = 4 \), \( |C_c(x_1)| \leq 1 \). So we can color \( x \) by \( \{1, 2\} \setminus \{C_c(x_1)\} \).

Case 2. \( c(x_1) \neq c(z) \).

Without loss of generality, we assume \( c(x_1) = 1 \) and \( c(z) = 2 \).

Subcase 2.1. we can color \( x \) by 2.

Now we color \( v \) by 3. If \( c(y_1) = 3 \), we can color \( y \) by 1. Else if \( c(y_1) \neq 3 \), we can color \( y \) by a color in \( C \setminus \{c(v), c(y_1)\} \).

Subcase 2.2. we can not color \( x \) by 2.

This subcase implies that \( k = 3 \) and \( c(x_2) = c(x_3) = c(x_4) = 2 \). Then we color \( x \) by 3 and \( v \) by 1. If \( c(y_1) = 1 \), we can color \( y \) by 3. Else if \( c(y_1) \neq 1 \), we can color \( y \) by a color in \( C \setminus \{c(v), c(y_1)\} \).

Case 3. \( c(x_1) = c(y_1) = c(z) \).

Without loss of generality, we assume \( c(x_1) = c(y_1) = c(z) = 1 \). Then we color \( y \) by 2 and \( v \) by 3. If \( k = 3 \) and \( c(x_2) = c(x_3) = c(x_4) = 2 \), we recolor \( y \) by 3 and \( v \) by 2. Then \( x \) can be colored by 3. Otherwise, we can color \( x \) by 2. \( \square \)

In the following, we will complete the proof of Theorem 4.

**Proof of Theorem 4.** We prove it by contradiction. Suppose that the theorem is false. We choose \( G \) to be critical with \( q(G) \geq 10 \) and use the discharging method on \( G \) in the following argument. For a planar graph one can easily deduce the following identity by the well-known Euler’s formula

\[
\sum_{v \in V(G)} (4d(v) - 10) + \sum_{f \in F(G)} (d(f) - 10) = -20.
\]

Let \( w(x) \) be the initial charge defined on \( x \in V(G) \cup F(G) \). Define \( w(v) = 4d(v) - 10 \) for each \( v \in V(G) \) and \( w(f) = d(f) - 10 \) for each \( f \in F(G) \). Then
we have $\Sigma_{e \in V(G) \cup F(G)} w'(x) = -20$. Now we state our discharging rules and perform them on vertices and faces of $G$. Let $w'(x)$ be the charge of $x \in V(G) \cup F(G)$ once the discharging is finished.

**R1.** Each $(\geq 5)$-vertex gives $2$ to each adjacent $2$-vertex.

**R2.** Each 4-vertex gives 2 to each adjacent $(4,2)$-type 2-vertex, $\frac{1}{2}$ to each adjacent $(4,3)$-type 2-vertex, 1 to each adjacent $(4,4)$-type 2-vertex.

**R3.** Each 3-vertex gives 1 to each adjacent $(3,3)$-type 2-vertex, $\frac{1}{2}$ to each adjacent $(3,4)$-type 2-vertex.

Let $f \in F(G)$. Since $g(G) \geq 10$, $d(f) \geq 10$. Thus, $w'(f) = w(f) = d(f) - 10 \geq 0$.

Let $v \in E(G)$. Then $d(v) \geq 2$ by Lemma 8. Suppose $d(v) = 2$, we have $w(v) = -2$. If $v$ is $(4,2)$-type, $w'(v) = w(v) + 2 = 0$; If $v$ is $(\geq 5,2)$-type, $w'(v) = w(v) + 2 = 0$; If $v$ is $(3,3)$-type, $w'(v) = w(v) + 1 \times 2 = 2 = 0$; If $v$ is $(4,3)$-type, $w'(v) = w(v) + \frac{1}{2} = \frac{1}{2} = 0$; If $v$ is $(\geq 5,3)$-type, $w'(v) = w(v) + 2 = 0$; If $v$ is $(4,4)$-type, $w'(v) = w(v) + 1 \times 2 = 0$; If $v$ is $(\geq 5,4)$-type or $(\geq 5,5)$-type, $w'(v) \geq w(v) + 2 = 0$.

Suppose $d(v) = 3$. Then $w(v) = 2$. If $v$ is adjacent to at most two $2$-vertices, since $v$ gives out at most 1 to each neighbor by R3, then we have $w'(v) \geq w(v) - 1 \times 2 = 0$. If $v$ is adjacent to three $2$-vertices but none of them is $(3,3)$-type. Then by R3, $v$ gives out at most $\frac{2}{3}$ to each neighbor. Thus, $w'(v) \geq w(v) - 3 \times \frac{2}{3} = 0$. If $v$ is adjacent to three $(\geq 2)$-vertices and at least one of them is $(3,3)$-type, then by Lemma 11 and R3, $v$ gives out charge to at most two neighbors. Thus, $w'(v) \geq w(v) - 2 \times 1 = 0$.

Suppose $d(v) = 4$. Then $w(v) = 6$. If $v$ is adjacent to at most three $2$-vertices. Since $v$ gives out at most 2 to each neighbor by R2, $w'(v) \geq w(v) - 3 \times 2 = 0$. If $v$ is adjacent to four $2$-vertices and three of them are $(4,2)$-type, then by Lemma 9 and R2, $v$ gives out charge to at most three neighbors. Thus, $w'(v) \geq w(v) - 3 \times 2 = 0$. If $v$ is adjacent to four $2$-vertices and two of them are $(4,2)$-type. Then by Lemma 10, either one of another two neighbors of $v$ is $(\geq 5,4)$-type or both of them are $(4,4)$-type. Thus, $w'(v) \geq \min\{w(v) - 2 \times 2 - 2, w(v) - 2 \times 2 - 1 \times 2\} = 0$. If $v$ is adjacent to four $2$-vertices but at most of them are $(4,2)$-type. Then by R2, we have $w'(v) \geq \min\{w(v) - 2 \times 3 \times \frac{1}{3}, w(v) - 4 \times \frac{3}{4}\} = 0$.

Suppose $d(v) \geq 5$. By R1, we have $w'(v) \geq w(v) - 2d(v) = 4d(v) - 10 - 2d(v) = 2d(v) - 10 \geq 0$.

Till now we have proved that $w'(x) \geq 0$ for all $x \in V(G) \cup F(G)$. So $\Sigma_{e \in V(G) \cup F(G)} w'(x) \geq 0$. But $\Sigma_{e \in V(G) \cup F(G)} w'(x) = \Sigma_{e \in V(G) \cup F(G)} w(x) = -20$ because our rules only move charge around, and do not affect the sum. This contradiction completes the proof of the theorem.

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**References**


