Algebra is interesting and useful

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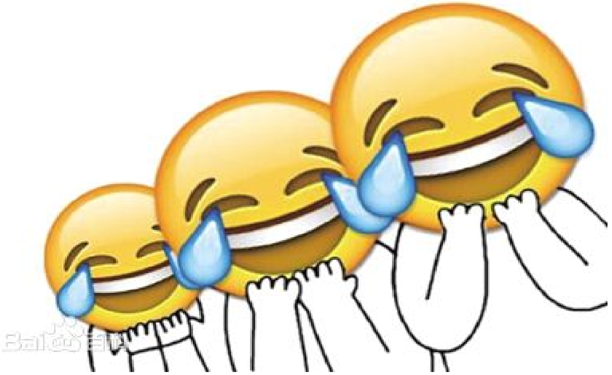
2018.07.06



Topic 1: Nonlinear programming

$$\max \quad 2x_1x_2 + 2x_2x_3 + 2x_3x_4 + 2x_4x_1$$
$$s.t. \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

Topic 2: Real symmetric matrix

If A is an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then

$$\lambda_1 = \max_{||X||=1} X'AX = \max_{||X||\neq 0} \frac{X'AX}{X'X}$$

$$\lambda_n = \min_{||X||=1} X'AX = \min_{||X||\neq 0} \frac{X'AX}{X'X}$$

Topic 2: Real symmetric matrix

Def. $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the <u>spectrum</u> of A

Def.

$$\rho(A) = \max_i |\lambda_i|$$

is the <u>spectral radius</u> of A

Topic 2: Real symmetric matrix

If A is an $n \times n$ positive semidefinite matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then

$$\rho(A) = \min_i |\lambda_i| = \max_i \lambda_i > 0$$

max

Topic 3: Applications of the Jordan normal form

Do you remember the Jordan normal form?

How to use it?

What is the most classical theorem on JNF?

Note

JNF will appear on your final examination paper!

Topic 3: Applications of the Jordan normal form

If A is an $n \times n$ complex valued matrix, then

$$\lim_{k \to +\infty} A^k = O \text{ if and only if}$$

$$\rho(A) < 1$$

Hints: (1) $A \sim \mathrm{JNF}$; (2) how can we say about the *k*-th power of a Jordan block?

Topic 3: Applications of the Jordan normal form

$$J_{m_i}^k(\lambda_i) = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \binom{k}{m_i-1}\lambda_i^{k-m_i+1} \\ 0 & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \cdots & \binom{k}{m_i-2}\lambda_i^{k-m_i+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} \\ 0 & 0 & \cdots & 0 & \lambda_i^k \end{bmatrix}$$

Topic 4: Algebraic games

First, a trick:

Use ***three 2***

and

some algebraic operations

to represent 2, 3, 4, 5, ……

Topic 4: Algebraic games

 ¥125.00 containing at most 5 notes

 ¥120.00 containing at most 5 notes

 ¥100.00 containing at most 5 notes

Choose one note from each red envelope so that the chosen three notes have different denominations! Can you finish it?

Topic 4: Algebraic games

Alon's Theorem in 1999

(Combinatorial Nullstellensatz). Let $f \in F[x_1, x_2, \ldots, x_n]$ be a polynomial of degree $t_1 + \cdots + t_n$. If $S_1, S_2, \ldots, S_n$ are nonempty subsets of F such that $|S_i| \geq t_i + 1$ for all i, then there exists $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ for which

$$f(s_1, s_2, \ldots, s_n) \neq 0$$

as long as the coefficient of $x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}$ is nonzero.

To prove it, the division algorithm and mathematical induction on $t_1 + t_2 + \cdots + t_n$ is enough!