## Section 5.2

# Colouring vertices

A *vertex colouring* of a graph $G = (V, E)$ is a map $c: V \to S$ such that $c(v) \neq c(w)$ whenever $v$ and $w$ are adjacent. The elements of the set $S$ are called the available *colours*. All that interests us about $S$ is its size: typically, we shall be asking for the smallest integer $k$ such that $G$ has a *k-colouring*, a vertex colouring $c: V \to \{1, \ldots, k\}$. This $k$ is the (*vertex*-) *chromatic number* of $G$; it is denoted by $\chi(G)$. A graph $G$ with $\chi(G) = k$ is called *k-chromatic*; if $\chi(G) \leqslant k$, we call $G$ *k-colourable*.

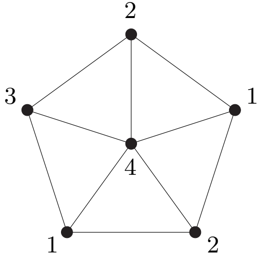


*Fig. 5.0.1.* A vertex colouring $V \to \{1, \ldots, 4\}$

Note that a $k$-colouring is nothing but a vertex partition into $k$ independent sets, now called *colour classes*; the non-trivial 2-colourable graphs, for example,are precisely the bipartite graphs.

How do we determine the chromatic number of a given graph? How can we *find* a vertex-colouring with as few colours as possible? How does the chromatic number relate to other graph invariants, such as average degree, connectivity or girth?

Straight from the definition of the chromatic number we may derive the following upper bound:

**Proposition 5.2.1.** *Every graph $G$ with $m$ edges satisfies*

$$\chi(G) \leqslant \tfrac{1}{2} + \sqrt{2m + \tfrac{1}{4}}\,.$$

*Proof*. Let $c$ be a vertex colouring of $G$ with $k = \chi(G)$ colours. Then $G$ has at least one edge between any two colour classes: if not, we could have used the same colour for both classes. Thus, $m \geqslant \frac{1}{2}k(k-1)$. Solving this inequality for $k$, we obtain the assertion claimed. □

*greedy algorithm*

One obvious way to colour a graph $G$ with not too many colours is the following *greedy algorithm*: starting from a fixed vertex enumeration $v_1, \ldots, v_n$ of $G$, we consider the vertices in turn and colour each $v_i$ with the first available colour—e.g., with the smallest positive integer not already used to colour any neighbour of $v_i$ among $v_1, \ldots, v_{i-1}$. In this way, we never use more than $\Delta(G) + 1$ colours, even for unfavourable choices of the enumeration $v_1, \ldots, v_n$. If $G$ is complete or an odd cycle, then this is even best possible.

In general, though, this upper bound of $\Delta + 1$ is rather generous, even for greedy colourings. Indeed, when we come to colour the vertex $v_i$ in the above algorithm, we only need a supply of $d_{G[\,v_1,\ldots,v_i\,]}(v_i) + 1$ rather than $d_G(v_i) + 1$ colours to proceed; recall that, at this stage, the algorithm ignores any neighbours $v_j$ of $v_i$ with $j > i$. Hence in most graphs, there will be scope for an improvement of the $\Delta + 1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbours are ignored) and vertices of small degree last. Locally, the number $d_{G[\,v_1,\ldots,v_i\,]}(v_i) + 1$ of colours required will be smallest if $v_i$ has minimum degree in $G[\,v_1, \ldots, v_i\,]$. But this is easily achieved: we just choose $v_n$ first, with $d(v_n) = \delta(G)$, then choose as $v_{n-1}$ a vertex of minimum degree in $G - v_n$, and so on.

*colouring number* col($G$)

The least number $k$ such that $G$ has a vertex enumeration in which each vertex is preceded by fewer than $k$ of its neighbours is called the *colouring number* col($G$) of $G$. The enumeration we just discussed shows that $\mathrm{col}(G) \leqslant \max_{H \subseteq G} \delta(H) + 1$. But for $H \subseteq G$ clearly also $\mathrm{col}(G) \geqslant \mathrm{col}(H)$ and $\mathrm{col}(H) \geqslant \delta(H) + 1$, since the 'back-degree' of the last vertex in any enumeration of $H$ is just its ordinary degree in $H$, which is at least $\delta(H)$. So we have proved the following:

**Proposition 5.2.2.** *Every graph $G$ satisfies*

$$\chi(G) \leqslant \operatorname{col}(G) = \max\{\,\delta(H) \mid H \subseteq G\,\} + 1\,.$$

□

**Corollary 5.2.3.** *Every graph $G$ has a subgraph of minimum degree at least $\chi(G) - 1$.* □

As we have seen, every graph $G$ satisfies $\chi(G) \leqslant \Delta(G) + 1$, with equality for complete graphs and odd cycles. In all other cases, this general bound can be improved a little:

**Theorem 5.2.4.** (Brooks 1941)
*Let $G$ be a connected graph. If $G$ is neither complete nor an odd cycle, then*

$$\chi(G) \leqslant \Delta(G)\,.$$

*Proof*. We apply induction on $|G|$. If $\Delta(G) \leqslant 2$, then $G$ is a path or a cycle, and the assertion is trivial. We therefore assume that $\Delta := \Delta(G) \geqslant 3$, and that the assertion holds for graphs of smaller order. Suppose that $\chi(G) > \Delta$.

Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leqslant \Delta$: by induction, every component $H'$ of $H$ satisfies $\chi(H') \leqslant \Delta(H') \leqslant \Delta$ unless $H'$ is complete or an odd cycle, in which case $\chi(H') = \Delta(H') + 1 \leqslant \Delta$ as every vertex of $H'$ has maximum degree in $H'$ and one such vertex is also adjacent to $v$ in $G$.

Since $H$ can be $\Delta$-coloured but $G$ cannot, we have the following:

*Every $\Delta$-colouring of $H$ uses all the colours $1, \ldots, \Delta$ on the neighbours of $v$; in particular, $d(v) = \Delta$.* (1)

Given any $\Delta$-colouring of $H$, let us denote the neighbour of $v$ coloured $i$ by $v_i$, $i = 1, \ldots, \Delta$. For all $i \neq j$, let $H_{i,j}$ denote the subgraph of $H$ spanned by all the vertices coloured $i$ or $j$.

*For all $i \neq j$, the vertices $v_i$ and $v_j$ lie in a common component $C_{i,j}$ of $H_{i,j}$.* (2)

Otherwise we could interchange the colours $i$ and $j$ in one of those components; then $v_i$ and $v_j$ would be coloured the same, contrary to (1).

*$C_{i,j}$ is always a $v_i$–$v_j$ path.* (3)

Indeed, let $P$ be a $v_i$–$v_j$ path in $C_{i,j}$. As $d_H(v_i) \leqslant \Delta - 1$, the neighbours of $v_i$ have pairwise different colours: otherwise we could recolour $v_i$, contrary to (1). Hence the neighbour of $v_i$ on $P$ is its only neighbour in $C_{i,j}$, and similarly for $v_j$. Thus if $C_{i,j} \neq P$, then $P$ has an inner vertex with three identically coloured neighbours in $H$; let $u$ be the first such vertex on $P$ (Fig. 5.2.1). Since at most $\Delta - 2$ colours are used on the neighbours of $u$, we may recolour $u$. But this makes $P\mathring{u}$ into a component of $H_{i,j}$, contradicting (2).

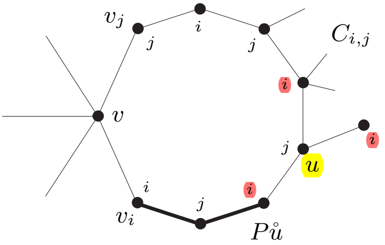


*Fig. 5.2.1.* The proof of (3) in Brooks's theorem

*For distinct $i, j, k$, the paths $C_{i,j}$ and $C_{i,k}$ meet only in $v_i$.* (4)

For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then $u$ has two neighbours coloured $j$ and two coloured $k$, so we may recolour $u$. In the new colouring, $v_i$ and $v_j$ lie in different components of $H_{i,j}$, contrary to (2).

The proof of the theorem now follows easily. If the neighbours of $v$ are pairwise adjacent, then each has $\Delta$ neighbours in $N(v) \cup \{ v \}$ already, so $G = G[N(v) \cup \{ v \}] = K^{\Delta+1}$. As $G$ is complete, there is nothing to show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \dots, v_\Delta$ derive their names from some fixed $\Delta$-colouring $c$ of $H$. Let $u \neq v_2$ be the neighbour of $v_1$ on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colours 1 and 3 in $C_{1,3}$, we obtain a new colouring $c'$ of $H$; let $v'_i$, $H'_{i,j}$, $C'_{i,j}$ etc. be defined with respect to $c'$ in the obvious way. As a neighbour of $v_1 = v'_3$, our vertex $u$ now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for $c$, however, the path $\mathring{v}_1 C_{1,2}$ retained its original colouring, so $u \in \mathring{v}_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for $c'$. $\square$

# Exercises

**1** Show that every graph $G$ has a vertex ordering for which the greedy algorithm uses only $\chi(G)$ colours.

**2** A $k$-chromatic graph is called *critically $k$-chromatic*, or just *critical*, if $\chi(G - v) < k$ for every $v \in V(G)$. Show that every $k$-chromatic graph has a critical $k$-chromatic induced subgraph, and that any such subgraph has minimum degree at least $k - 1$.

**3** An $n \times n$-matrix with entries from $\{1, \ldots, n\}$ is called a *Latin square* if every element of $\{1, \ldots, n\}$ appears exactly once in each column and exactly once in each row. Recast the problem of constructing Latin squares as a colouring problem.