Chapter 3B

The Frequency Response of an LTI Discrete-Time System

Part B: Frequency Response

1.1 Definition

- An LTI discrete-time system is completely characterized in the time-domain by its impulse response sequence $\{h(n)\}$.
- Thus, the *transform-domain representation* of a discrete-time *signal* can also be equally applied to the *transform-domain representation* of *an LTI discrete-time system*.

1.1 Definition

- Such transform-domain representations provide additional *insights* into the behavior of such systems.
- It is *easier to design and implement* these systems in the transformed-domain for certain applications.
- We consider now the use of the DTFT in developing the transform domain representations of an LTI system.

1.1 Definition

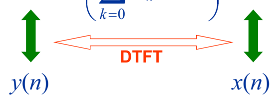
- In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:

$$\sum_{k=0}^{N} d_k y(n-k) = \sum_{k=0}^{M} p_k x(n-k)$$

1.1 Definition

- Applying the DTFT to the difference equation and making use of the linearity and the time-invariance properties, we arrive at the input-output relation in the transform-domain as

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k}\right) X(e^{j\omega})$$



1.1 Definition

- Most discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite number of sinusoidal discrete-time signals of different angular frequencies.
- Thus, knowing the response of the LTI system to *a single sinusoidal signal*, we can determine its response to more complicated signals by making use of the superposition property.

1.1 Definition

- An important property of an LTI system is that for certain types of input signals, called eigen functions, the output signal is the input signal multiplied by a complex constant.
- We consider one such eigen function as the input.
- Consider the following LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

1.1 Definition

- Its I-O relationship in the time domain is given by the convolution sum.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

- If the input is of the form

$$x(n) = e^{j\omega n} \quad -\infty < n < \infty$$

then

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = \underbrace{\left(\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right)}_{H(e^{j\omega})} e^{j\omega n}$$

1.1 Definition

- Then we can write

$$y(n) = H(e^{j\omega})e^{j\omega n}$$

- Thus for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$
- Thus $e^{j\omega n}$ is an eigen function of the system

1.1 Definition

Definition

- The DTFT of the impulse response of an LTI system is called the Frequency Response of this system

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi)}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})$$

$$= \underset{\substack{\Downarrow\\ \text{magnitude}\\ \text{response}}}{\left|H(e^{j\omega})\right|}\, \underset{\substack{\Downarrow\\ \text{phase}\\ \text{response}}}{e^{j\arg\{H(e^{j\omega})\}}}$$

1.1 Definition

- In some cases, the magnitude function is specified in decibels as

$$\mathcal{G}(\omega) = 20\log_{10}\left|H(e^{j\omega})\right| \text{ dB}$$

where $\mathcal{G}(\omega)$ is called the gain function
- The negative of the gain function

$$\mathcal{A}(\omega) = -\mathcal{G}(\omega)$$

is called the attenuation or loss function

1.2 Frequency-Domain Characterization of the LTI Discrete-Time System

- The convolution sum description of the LTI discrete-time system is given by

$$y(n)=\sum_{k=-\infty}^{\infty}h(k)x(n-k)$$

- Taking the DTFT of both sides we obtain

$$Y(e^{j\omega})=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}h(k)x(n-k)\right)e^{-j\omega n}$$

1.2 Frequency-Domain Characterization of the LTI Discrete-Time System

- Interchanging the summation signs on the right-hand side and rearranging we arrive at

$$\begin{aligned}Y(e^{j\omega})&=\sum_{k=-\infty}^{\infty}h(k)\left(\sum_{n=-\infty}^{\infty}x(n-k)e^{-j\omega n}\right)\\&\overset{l=n-k}{=}\sum_{k=-\infty}^{\infty}h(k)\left(\sum_{l=-\infty}^{\infty}x(l)e^{-j\omega(l+k)}\right)\\&=\sum_{k=-\infty}^{\infty}h(k)\left(\sum_{l=-\infty}^{\infty}x(l)e^{-j\omega l}\right)e^{-j\omega k}\\&=H(e^{j\omega})X(e^{j\omega})\end{aligned}$$

1.2 Frequency-Domain Characterization of the LTI Discrete-Time System

- It follows from the previous equation

$$H(e^{j\omega})=Y(e^{j\omega})/X(e^{j\omega})$$

- For an LTI system described by *a linear constant coefficient difference equation* of the form we have

$$H(e^{j\omega})=\frac{\sum_{k=0}^{M}p_k e^{-j\omega k}}{\sum_{k=0}^{N}d_k e^{-j\omega k}}$$

1.3 Frequency Response Computation using Matlab

- The function `freqz(h,w)` can be used to determine the values of the frequency response vector ***h*** at a set of given frequency points `w`
- From ***h***, the real and imaginary parts can be computed using the functions `real` and `imag`, and the magnitude and phase functions using the functions `abs` and `angle`

1.3 Frequency Response Computation using Matlab

Example

- Consider a *moving-average* filter

$$h(n)=\begin{cases}1/M, & 0\le n\le M-1\\ 0, & \text{otherwise}\end{cases}$$

- Program 3_2 can be used to generate the magnitude and gain responses of an M-point moving average filter as shown in the next slide

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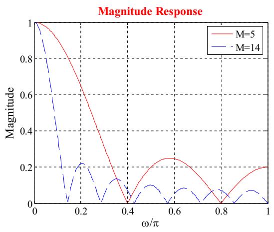
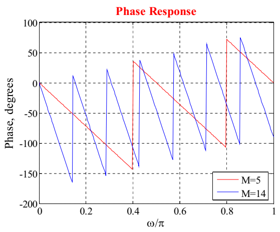
1.3 Frequency Response Computation using Matlab





1.4 The Concept of Filtering

- One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components.
- Such systems are called *digital filters* and one of the main subjects of discussion in this course.

1.4 The Concept of Filtering

- The key to the filtering process is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences.

1.4 The Concept of Filtering

- By appropriately choosing the values of the magnitude function $\left|H(e^{j\omega})\right|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others.

1.4 The Concept of Filtering

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function.

$$\left|H(e^{j\omega})\right| \cong \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

1.4 The Concept of Filtering

- We apply an input

$$x(n) = A\cos\omega_1 n + B\cos\omega_2 n, \quad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

to this system

- Because of *linearity*, the output of this system is of the form

$$y(n) = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1)) + B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

Eigen-function, conjugate-symmetric for real *h*(*n*)

1.4 The Concept of Filtering

- As

$$\left|H(e^{j\omega_1})\right| \cong 1 \qquad \left|H(e^{j\omega_2})\right| \cong 0$$

the output reduces to

$$y(n) \cong A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

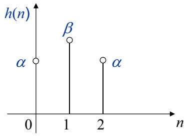
- Thus, the system acts like a lowpass filter
- In the following example, we consider the design of a very simple digital filter.

1.4 The Concept of Filtering

Example

- Design of a high pass digital filter
- The input $x(n)=[\cos(0.1n)+\cos(0.4n)]\cdot u(n)$ which consists of two frequency components $0.1\ rad/sample$ and $0.4\ rad/sample$.
- For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:

1.4 The Concept of Filtering



- Note that $h(n)$ is a **linear phase FIR filter** which will be discussed in the latter chapters
- The frequency response of this filter is given by

$$H(e^{j\omega})=h(0)+h(1)e^{-j\omega}+h(2)e^{-2j\omega}$$
$$=(2\alpha\cos\omega+\beta)e^{-j\omega}$$

1.4 The Concept of Filtering

- The magnitude and phase functions are

$$\left|H(e^{j\omega})\right|=\left|2\alpha\cos\omega+\beta\right| \qquad \theta(\omega)=-\omega$$

- In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero
- Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

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1.4 The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$2\alpha\cos 0.1+\beta=0 \qquad 2\alpha\cos 0.4+\beta=1$$

- Solving the above two equations we get

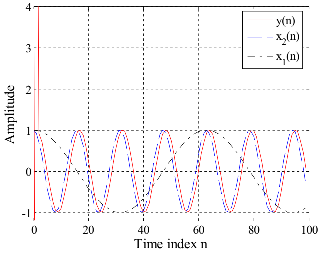
$$\alpha=-6.76195 \qquad \beta=13.456335$$

- Thus the output-input relation of the FIR filter is given by

$$y(n)=-6.76195x(n)+13.456335x(n-1)$$
$$-6.76195x(n-2)$$

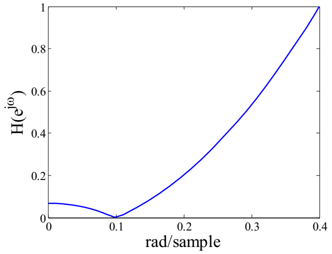
1.4 The Concept of Filtering

- Figure below shows the plots generated by running program 3_3



1.4 The Concept of Filtering

- Figure below shows the frequency response of this highpass filter



2.1 Definition of Phase and Group delays

- The output $h(n)$ of a frequency-selective LTI discrete-time system with a frequency response $\left|H(e^{j\omega})\right|$ exhibits some *delay* relative to the input caused by the *nonzero phase response* of the system

$$\theta(\omega) = \arg\left\{H(e^{j\omega})\right\}$$

- For an input

$$x(n) = A\cos(\omega_0 n + \phi) \quad -\infty < n < \infty$$

2.1 Definition of Phase and Group delays

The output is

$$y(n) = A\left|H(e^{j\omega_0})\right|\cos(\omega_0 n + \theta(\omega_0) + \phi)$$

- Thus, the output *lags in phase* by $\theta(\omega_0)$ radians
- Rewriting the above equation we get

$$y(n) = A\left|H(e^{j\omega})\right|\cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \phi\right)$$

2.1 Definition of Phase and Group delays

- Denote phase delay $\tau_p(\omega_0) = -\theta(\omega_0)/\omega_0$
- Now consider the case when the input signal contains many sinusoidal components with different frequencies that are not harmonically related
- In this case, each component of the input will go through different phase delays when processed by a frequency-selective LTI discrete-time system

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2.1 Definition of Phase and Group delays

- To develop the necessary expression, consider a discrete-time signal $x(n)$ obtained by a double-sideband suppressed carrier (DSB-SC) modulation with a carrier frequency ω_c of a low-frequency sinusoidal signal of frequency ω_0

$$x(n) = A\cos(\omega_0 n)\cos(\omega_c n)$$

$$= \frac{A}{2}\cos(\omega_l n) + \frac{A}{2}\cos(\omega_u n)$$

$$\omega_l = \omega_c - \omega_0 \quad \omega_u = \omega_c + \omega_0$$

2.1 Definition of Phase and Group delays

- Let the above input be processed by an LTI discrete-time system with a frequency response $H(e^{j\omega})$ satisfying the condition

$$\left|H(e^{j\omega})\right| \cong 1 \quad for \;\; \omega_l \le |\omega| \le \omega_u$$

- The output $y(n)$ is then given by

$$y(n) = \frac{A}{2}\cos\left(\omega_l n + \theta(\omega_l)\right) + \frac{A}{2}\cos\left(\omega_u n + \theta(\omega_u)\right)$$

$$= A\cos\left(\omega_c n + \frac{\theta(\omega_u) + \theta(\omega_l)}{2}\right)\cos\left(\omega_0 n + \frac{\theta(\omega_u) - \theta(\omega_l)}{2}\right)$$

2.1 Definition of Phase and Group delays

- Note: The output is also in the form of a modulated carrier signal with the same carrier frequency ω_c and the same modulation frequency ω_0 as the input.
- However, the two components have different phase lags relative to their corresponding components in the input

2.1 Definition of Phase and Group delays

- Now consider the case when the modulated input is a narrow band signal with the frequencies ω_l and ω_u very close to the carrier frequency ω_c, i.e. ω_0 is very small
- In the neighborhood of ω_c we can express the phase response $\theta(\omega)$ as

$$\theta(\omega) \cong \theta(\omega_c) + \left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c} \cdot (\omega - \omega_c)$$

2.1 Definition of Phase and Group delays

by making a Taylor's series expansion and keeping only the first two terms

- Using the above formula, we now evaluate the time delays of the carrier and the modulating components

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2.1 Definition of Phase and Group delays

- In the case of the carrier signal we have

$$-\frac{\theta(\omega_u)+\theta(\omega_l)}{2\omega_c} \cong -\frac{\theta(\omega_c)}{\omega_c}$$

which is seen to be the same as the phase delay if only the carrier signal is passed through the system

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2.1 Definition of Phase and Group delays

- In the case of the modulating component we have

$$-\frac{\theta(\omega_u)-\theta(\omega_l)}{2\omega_0} = -\frac{\theta(\omega_u)-\theta(\omega_l)}{\omega_u-\omega_l} \cong -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}$$

- The parameter $\tau_g(\omega_c) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_c}$

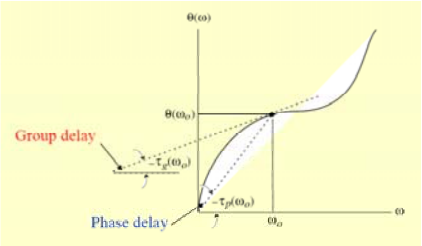
is called the group delay or envelope delay caused by the system at $\omega = \omega_c$

2.1 Definition of Phase and Group delays

- The group delay is a measure of *the linearity of the phase function* as a function of the frequency
- It is the time delay between the waveforms of underlying continuous-time signals whose sampled versions, sampled at $t = nT$, are precisely the input and the output discrete-time signals
- If the phase function and the angular frequency ω are in *radians per second*, then the group delay is in *seconds*

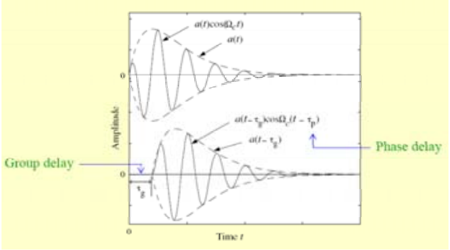
2.1 Definition of Phase and Group delays

- Figure below illustrates the evaluation of the phase delay and the group delay



2.1 Definition of Phase and Group delays

- Figure below shows the waveform of an amplitude-modulated input and the output generated by an LTI system



2.1 Definition of Phase and Group delays

- The carrier component at the output is delayed by the phase delay and the envelope of the output is delayed by the group delay relative to the waveform of the continuous-time input signal in the previous slide
- The waveform of the underlying continuous time output shows distortion when the group delay of the LTI system is not constant over the bandwidth of the modulated signal

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2.1 Definition of Phase and Group delays

- If the distortion is unacceptable, a delay equalizer is usually cascaded with the LTI system so that the overall group delay of the cascade is approximately linear over the band of interest.
- To keep the magnitude response of the parent LTI system unchanged, the equalizer must have a constant magnitude response at all frequencies

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2.1 Definition of Phase and Group delays

Example

- The phase function of the FIR Filter

$$y(n)=\alpha x(n)+\beta x(n-1)+\alpha x(n-2)$$

is $\theta(\omega)=-\omega$

- Hence its group delay is given by $\tau_g(\omega)=1$

2.2 Phase and Group delay Computation Using Matlab

- Phase delay and group delay can be computed using the function `phasedelay, grpdelay` respectively
- Figures in the next slide shows the phase delay and group delay of the DTFT

$$H(e^{j\omega})=\frac{0.1367(1-e^{-j2\omega})}{1-0.5335e^{-j\omega}+0.7265e^{-j2\omega}}$$

2.2 Phase and Group delay Computation Using Matlab