Chapter 3

Discrete-Time Fourier Transform (DTFT)

Chapter 3

Two major topics of this chapter:

- ***Discrete-Time Fourier Transform***
- ***Frequency Response of an LTI Discrete-Time Systems (DTFT of the impulse response)***

Chapter 3A

Discrete-Time Fourier Transform

Part A: DTFT

1.1 Definition of CTFT

Definition

- The CTFT of a continuous-time signal $x_a(t)$ is given by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as the Fourier Spectrum or simply the Spectrum of the continuous-time signal

1.1 Definition of CTFT

Definition

- The inverse CTFT of a Fourier Transform $X_a(j\Omega)$ is given by

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

- Often referred to as the Fourier integral
- A CTFT pair will be denoted as

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

1.1 Definition of CTFT

- Ω is real and denotes the continuous-time angular frequency variable in radians
- In general, the CTFT is a complex function of Ω in the range $-\infty<\Omega<\infty$
- It can be expressed in the polar form as

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)}$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

1.1 Definition of CTFT

- The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum
- Both spectrums are real functions of Ω
- In general, the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet Conditions (狄利克雷条件) given on the next slide

1.1 Definition of CTFT

Dirichlet Conditions

(a) The signal $x_a(t)$ has a finite number of discontinuities and a finite number of maxima and minima in any finite interval

(b) The signal is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |x_a(t)|\, dt < \infty$$

1.1 Definition of CTFT

- If the Dirichlet Conditions are satisfied, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

converges to $x_a(t)$ at values of t except at values of t where $x_a(t)$ has discontinuities

- It can be shown that if $x_a(t)$ is absolutely integrable, then $|X_a(j\Omega)| < \infty$ proving the existence of the CTFT

1.2 Energy Density Spectrum

- The total energy $\mathcal{E}_x$ of a finite energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt$$

- The above expression can be rewritten as

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt$$

1.2 Energy Density Spectrum

- Interchanging the order of the integration, we get

$$\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega) \left[\int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt \right] d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega) X_a(j\Omega) d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

1.2 Energy Density Spectrum

- Hence

$$\int_{-\infty}^{\infty} |x_a(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega$$

- The above relation is more commonly known as the Parseval's relation for finite energy continuous-time signals

1.2 Energy Density Spectrum

- The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and usually denoted as

$$S_{xx}(\Omega) = |X_a(j\Omega)|^2$$

- The energy over a specified range of frequencies $\Omega_a \le \Omega \le \Omega_b$ can be computed using

$$\mathcal{E}_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) d\Omega$$

1.3 Band-limited Continuous-Time Signals

- A full-band, finite-energy, continuous-time signal has a spectrum occupying the whole frequency range $-\infty < \Omega < \infty$
- A band-limited continuous-time signal has a spectrum that is limited to a portion of the frequency range $-\infty < \Omega < \infty$

1.3 Band-limited Continuous-Time Signals

- An ideal band-limited signal has a spectrum that is zero outside a finite frequency range $\Omega_a \le |\Omega| \le \Omega_b$, that is

$$X_a(j\Omega) = \begin{cases} 0, & 0 \le |\Omega| \le \Omega_a \\ 0, & \Omega_b \le |\Omega| \le \infty \end{cases}$$

- However, an ideal band-limited signal cannot be generated in practice

1.3 Band-limited Continuous-Time Signals

- Band-limited signals are classified according to the frequency range where most of the signal's is concentrated
- A lowpass, continuous-time signal has a spectrum occupying the frequency range $|\Omega| \le \Omega_p < \infty$ where Ω_p is called the bandwidth of the signal

1.3 Band-limited Continuous-Time Signals

- A highpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_p \le |\Omega| < \infty$ where the bandwidth of the signal is from Ω_p to ∞
- A bandpass, continuous-time signal has a spectrum occupying the frequency range $0 < \Omega_L \le |\Omega| \le \Omega_H < \infty$ where $\Omega_H - \Omega_L$ is the bandwidth

2.1 Definition of DTFT

Definition

- The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x(n)$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

2.1 Definition of DTFT

- $|X(e^{j\omega})|$ is called the magnitude function and $\theta(\omega)$ is called the phase function
- In many applications, the DTFT is called the Fourier spectrum
- Likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called the magnitude and phase spectra
- It should be noted that DTFT is a continuous function of ω

2.1 Definition of DTFT

- For a real sequence $x(n)$, $|X(e^{j\omega})|$ and $\mathrm{Re}[X(e^{j\omega})]$ are even functions of ω, whereas, $\theta(\omega)$ and $\mathrm{Im}[X(e^{j\omega})]$ are odd functions of ω
- Note that, for any integer k

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$
$$= \left|X(e^{j\omega})\right| e^{j(\theta(\omega)+2k\pi)}$$

2.1 Definition of DTFT

- The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT
- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \le \theta(\omega) \le \pi$$

called the principal value

2.1 Definition of DTFT

Example

- The DTFT of the unit sample sequence $\{\delta(n)\}$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n)e^{-j\omega n} = \delta(0) = 1$$

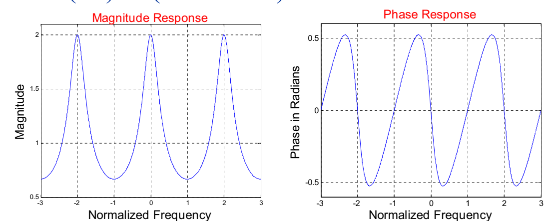
- Consider the causal sequence $x(n)=a^n u(n)$ $|a|<1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}$$

2.1 Definition of DTFT

Simulation Results

- The magnitude and phase of the DTFT $X(e^{j\omega})=1/(1-0.5e^{-j\omega})$ are shown below



2.1 Definition of DTFT

- The DTFT of a sequence $x(n)$, is a continuous function of ω. It is also a periodic function of ω with a period 2π
- The Inverse discrete-time Fourier transform (IDTFT) of $X(e^{j\omega})$ is given by

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Proof

2.2 Convergence Condition

- If $x(n)$ is an absolutely summable sequence, i.e., if

$$\sum_{n=\infty}^{\infty}|x(n)| < \infty$$

Then

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right| < \sum_{n=-\infty}^{\infty}|x(n)| < \infty$$

- Thus, the absolute summability of $x(n)$ is a *sufficient condition* for the existence of the DTFT

2.3 DTFT Properties

- Linearity
- Shifting (in time and in frequency domain)
- Differentiation

$$nx(n) \Leftrightarrow j\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

- Convolution (in time and in frequency domain)

$$x(n) * y(n) \Leftrightarrow X\left(e^{j\omega}\right)\cdot Y\left(e^{j\omega}\right)$$

$$x(n)y(n) \Leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$$

2.3 DTFT Properties

- Area Theorem (simple but useful)

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)d\omega \quad X\left(e^{j0}\right) = \sum_{n=-\infty}^{\infty} x(n)$$

- Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$

Corollary——Energy is preserved

$$\sum_{n=-\infty}^{\infty}\left|x(n)^2\right| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

2.3 DTFT Properties

- Symmetry Relations (DTFT pairs)

$$x_r(n) \Leftrightarrow X_{cs}(e^{j\omega}) \qquad jx_i(n) \Leftrightarrow X_{ca}(e^{j\omega})$$

$$x_{cs}(n) \Leftrightarrow X_R(e^{j\omega}) \qquad x_{ca}(n) \Leftrightarrow jX_I(e^{j\omega})$$

- For an arbitrary real sequence

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

Corollary

$$X_R(e^{j\omega}) \quad X_I(e^{j\omega}) \quad \left|X(e^{j\omega})\right| \quad \arg\left[X_R(e^{j\omega})\right]$$

2.3 DTFT Properties

Some Common Discrete-Time Fourier Transform Pairs

Sequence	Transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1(\forall n)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
$\frac{\sin(\omega_0 n}{\pi n}$	$X(e^{j\omega}) = \begin{cases}1, & \|\omega\| < \omega_0 \\ 0, \omega_0 < \|\omega\| < \pi\end{cases}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
$\cos(\omega_0 n+\phi)$	$\pi\sum_{k=-\infty}^{\infty}\left[e^{j\phi}\delta(\omega-\omega_0+2\pi k)+e^{-j\phi}\delta(\omega+\omega_0+2\pi k)\right]$
$x[n] = \begin{cases}1, & 0 \le n \le M \\ 0, & \text{Otherwise}\end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$

2.3 DTFT Properties

Example

Determine the DFT $Y(e^{j\omega})$ of $y(n)=(n+1)a^n u(n)$ ($|a|<1$)

Step 1: Let $x(n)=a^n u(n)$. Therefore

$$y(n)=nx(n)+x(n)$$

Step 2: Calculate the DTFT $X(e^{j\omega})$

$$X(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$$

2.3 DTFT Properties

Step 3: Calculate the DTFT of $nx(n)$

$$j\frac{dX(e^{j\omega})}{d\omega}=j\frac{-aje^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2}=\frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2}$$

Step 4: Calculate the DTFT $Y(e^{j\omega})$ of $y(n)$

$$Y\left(e^{j\omega}\right)=\frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2}+\frac{1}{1-ae^{-j\omega}}=\frac{1}{\left(1-ae^{-j\omega}\right)^2}$$

2.4 Energy Density Spectrum

- The total energy of a finite-energy sequence $g(n)$ is given by

$$\mathcal{E}_g=\sum_{n=-\infty}^{\infty}|g(n)|^2$$

- From Parseval's relation we observe that

$$\mathcal{E}_g=\sum_{n=-\infty}^{\infty}|g(n)|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|G\left(e^{j\omega}\right)\right|^2 d\omega$$

2.4 Energy Density Spectrum

- The quantity

$$S_{gg}\left(e^{j\omega}\right)=\left|G\left(e^{j\omega}\right)\right|^2$$

  is called the energy density spectrum

- The area under this curve in the range $-\pi\le\omega\le\pi$ divided by 2π is the energy of the sequence

2.4 Energy Density Spectrum

- Recall that the autocorrelation sequence $r_{gg}(l)$ of $g(n)$ can be expressed as

$$r_{gg}(l)=\sum_{n=-\infty}^{\infty}g(n)g(-(l-n))=g(l)*g(-l)$$

- As we know that the DTFT of $g(-l)$ is $G(e^{-j\omega})$, therefore, the DTFT of $g(l)*g(-l)$ is given by $|G(e^{j\omega})|^2$, where we have used the fact that for a real sequence $g(n)$, $G(e^{-j\omega})=G^*(e^{j\omega})$

2.4 Energy Density Spectrum

- As a result, the energy density spectrum $S_{gg}(e^{j\omega})$ of a real sequence $g(n)$ can be computed by taking the DTFT of its autocorrelation sequence $r_{gg}(l)$, i.e.,

$$S_{gg}\left(e^{j\omega}\right)=\sum_{l=-\infty}^{\infty}r_{gg}(l)e^{-j\omega l}$$

2.4 Energy Density Spectrum

Example

- Compute the energy of the sequence
- Here $$h_{LP}(n) = \frac{\sin \omega_c n}{n\pi}, \quad -\infty \le n \le \infty$$

$$\sum_{n=-\infty}^{\infty} |h_{LP}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LP}\left(e^{j\omega}\right)\right|^2 d\omega$$

where $H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$

2.4 Energy Density Spectrum

- Therefore, Compute the energy of the sequence

$$\sum_{n=-\infty}^{\infty} |h_{LP}(n)|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}(n)$ is a finite-energy sequence

2.5 DTFT Computation Using MATLAB

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of

$$X\left(e^{j\omega}\right) = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$$

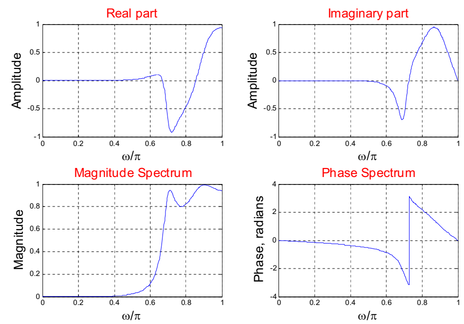
at a prescribed set of discrete frequency points $\omega = \omega_l$.

2.5 DTFT Computation Using MATLAB

- For example, the statement ***H*= *freqz*(*p*,*d*,*ω*)** returns the frequency response values as a vector ***H*** of a DTFT defined in terms of the vectors ***p*** and ***d*** containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector ω.
- For example

p=[0.008 −0.033 0.05 −0.033 0.008]
d=[1 2.37 2.7 1.6 0.41]

2.5 DTFT Computation Using MATLAB



2.6 Linear Convolution Using DTFT

- According to the convolution theorem

$$y(n) = x(n) * h(n) \Leftrightarrow Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

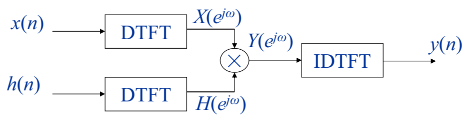
- An implication of this result is that the linear convolution $y(n)$ of the sequences $x(n)$ and $h(n)$ can be performed as follows:

2.6 Linear Convolution Using DTFT

Step 1: Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x(n)$ and $h(n)$, respectively.

Step 2: Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

Step 3: Compute the IDTFT $y(n)$ of $Y(e^{j\omega})$



2.7 DTFT for Special Sequence

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable.
- Examples of such sequences are the unit step sequence $u(n)$, the sinusoidal sequence $\cos(\omega_0 n + \phi)$ and the exponential sequence $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$

2.7 DTFT for Special Sequence

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero satisfying

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} p_\Delta(\omega) d\omega = \int_{-\infty}^{\infty} \delta(\omega) d\omega$$

2.7 DTFT for Special Sequence