Chapter 2B

Discrete-Time Signals and Systems in the Time-Domain

Part B

Discrete-Time Systems

Part B: Discrete-Time Systems

- **Linear System**
- **Shift (Time)-Invariant System**
- **Linear Time-Invariant System**
- **Causal System**
- **Stable System**
- **Passive and Lossless Systems**

1. Linear System

Definition

If $y_1(n)$ is the output due to an input $x_1(n)$ and $y_2(n)$ is the output due to an input $x_2(n)$ then for an input

$$x(n) = \alpha x_1(n) + \beta x_2(n)$$

the output is given by

$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

Above property must hold for any arbitrary constants α and β and for all possible inputs $x_1(n)$ and $x_2(n)$

1. Linear System

- In other words, the system considered satisfies the *principle of Superposition*.
- Recall that the input $x(n)$ can be written as

$$x(n) = \sum_{n=-\infty}^{\infty} x(k)\delta(n-k)$$

Hence, the output $y(n)$ can be computed as follows according to the principle of superposition $y(n) = \sum_{n=-\infty}^{\infty} x(k)h(n,k)$

where $h(n,k)$ is the output due to the input $\delta(n-k)$

2. Shift Invariant System

Definition

For a shift-invariant system, if $y_1(n)$ is the response to an input $x_1(n)$, then the response to an input $x(n) = x_1(n-n_0)$ is simply $y(n) = y_1(n-n_0)$ where n_0 is any positive or negative integer

- The above relation must hold for any arbitrary input and its corresponding output

2. Shift Invariant System

- Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied

Question

Consider the linearity and time-invariance properties of the following systems

(1) $y(n)=x(n-n_0)$ (2) $y(n)=ax(n)+b$

3. Linear Time-Invariant Systems

- LTI is the abbreviation of "Linear Time-Invariant"
- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design
- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades

3. Linear Time-Invariant Systems

- For a linear system, the output $y(n)$ due to the input $x(n)$ can be written as

$$y(n)=\sum_{n=-\infty}^{\infty} x(k)h(n,k)$$

- If this system is also time-invariant, the time-varying function $h(n,k)$ becomes a time-invariant function $h(n-k)$, and the output is now given by

$$y(n)=\sum_{n=-\infty}^{\infty} x(k)h(n-k)$$

- The above operation is called a linear convolution sum.

4. Causal System

Definition

- In a causal system, the n_0-th output sample $y(n_0)$ depends only on input samples $x(n)$ for $n \leq n_0$ and does not depend on input samples for $n>n_0$
- Let $y_1(n)$ and $y_2(n)$ be the responses of a causal discrete-time system to the inputs $x_1(n)$ and $x_2(n)$, respectively

4. Causal System

- Then $x_1(n)=x_2(n)$, for $n < N$

  implies also that $y_1(n)=y_2(n)$, for $n < N$
- For a causal system, changes in output samples do not precede changes in the input samples

Question

- Consider the Causality of the following systems

  (1) $y(n)= ax(n)+b$ (2) $y(n)= nx(n-n_0)$

5. Stable System

Definition

- There are various definitions of stability. We consider here the *bounded-input, bounded-output* (*BIBO*) stability
- If $y(n)$ is the response to an input $x(n)$ and if $x(n)$ is bounded, i.e.

  $|x(n)|<B_x$ for all values of n

  then $y(n)$ is bounded, i.e

  $|y(n)|<B_y$ for all values of n

6. Passive and Lossless Systems

Definition

- A discrete-time system is defined to be passive, if for every finite-energy input $x(n)$, the output $y(n)$ has, at most, the same energy, i.e.

$$\sum_{n=-\infty}^{\infty} |y(n)|^2 \le \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

- For a lossless system, the above inequality is satisfied with an equal sign for every input

Chapter 2C

Discrete-Time Signals and Systems in the Time-Domain

Part C

Time-Domain Characterization of LTI Discrete-Time Systems

Time-Domain Characterization of LTI Discrete-Time Systems

- **Input-Output Relationship**
- **Simple interconnection schemes**
- **Stability condition in terms of the Impulse response**
- **Causality condition in terms of the Impulse response**

1. Input-Output Relationship

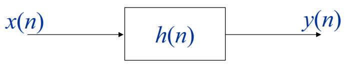
- Unit impulse response $\{h(n)\}$ is defined by the response of a digital filter to a unit sample sequence $\{\delta(n)\}$
- Unit sample response $\{s(n)\}$ is defined by the response of a discrete system to a unit step sequence $\{u(n)\}$

1. Input-Output Relationship

- A consequence of the linear, time invariance property is that a LTI discrete time system is completely characterized by its impulse response
- ➡ Knowing the impulse response one can compute the output of the system for any arbitrary input

1. Input-Output Relationship

- Let $h(n)$ denote the impulse response of an LTI discrete-time system



- Recall that $x(n)$ can be written as

$$x(n) = \sum_{n=-\infty}^{\infty} x(k)\delta(n-k)$$

1. Input-Output Relationship

- Since $h(n)$ is the response of input $\delta(n)$ and the system is time invariant, we have

$$\delta(n-k) \rightarrow h(n-k)$$

- Likewise, as the system is linear

$$x(k)\ \delta(n-k) \rightarrow x(k)h(n-k)$$

- Note that, $x(k)$ is considered as a constant in this case

1. Input-Output Relationship

- Taking advantage of the property of linear, we have

$$\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \rightarrow \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- Eventually, the I-O relationship of an LTI system can be written as follows

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

1. Input-Output Relationship

- This operation is called a linear convolution sum and can be represented compactly as

$$y(n) = x(n) * h(n)$$

- The linear convolution sum has

  Commutative property

  Associative property

  Distributive property

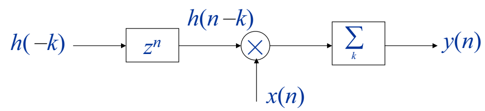
1. Input-Output Relationship

Solution

- Time-reverse $h(k)$ to form $h(-k)$
- Shift $h(-k)$ to the right by n sampling periods if $n > 0$ or shift $h(-k)$ to the left by n sampling periods if $n < 0$ to form $h(n-k)$
- Form the product $v(k) = x(k)h(n-k)$
- Sum all samples of $v(k)$ to develop the n-th sample of $y(n)$ of the convolution sum

1. Input-Output Relationship

Schematic Representation



- If the lengths of $x(n)$ and $h(n)$ are N_1 and N_2 respectively, the length of the convolution sum $y(n)$ will be N_1+N_2-1

1. Input-Output Relationship

Example

Calculate the convolution sum: $y(n) = R_3(n) * R_4(n)$

Method 1: analytical method

Method 1: graphically method

Application

Using Convolution to calculate the Correlations of sequence

1. Input-Output Relationship

Correlation of Signals

Definition

A measure of similarity between a pair of energy signals, $x(n)$ and $y(n)$, is given by the cross-correlation sequence $r_{xy}(l)$ defined by

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

1. Input-Output Relationship

- If $x(n)=y(n)$, we obtain the definition of the autocorrelation of $x(n)$

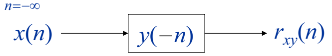
$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

- Note, $r_{xx}(0) = \sum_{n=-\infty}^{\infty} x^2(n)$ is the energy of the signal $x(n)$

1. Input-Output Relationship

Properties

- $r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l) = \sum_{m=-\infty}^{\infty} y(m+l)x(m) = r_{xy}(-l)$
- It follows that $r_{xx}(l)=r_{xx}(-l)$ implying that $r_{xx}(l)$ is an even function for real $x(n)$
- $r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y\big(-(l-n)\big) = x(l) * y(-l)$



2. Simple Interconnection

Cascade Connection

$$h(n) = h_1(n) * h_2(n) = h_2(n) * h_1(n)$$

- An application is in the development of an inverse system, if the cascade connection satisfies the relation

$$h_1(n) * h_2(n) = \delta(n)$$

- Then the LTI system $h_1(n)$ is said to be the inverse of $h_2(n)$ and vice-versa

2. Simple Interconnection

- An application of the inverse system concept is in the recovery of a signal $x(n)$ from its distorted version $\hat{x}(n)$ appearing at the output of a transmission channel
- If the impulse response of the channel is known, then $x(n)$ can be recovered by designing an inverse system of the channel

2. Simple Interconnection

Parallel Connection

$$h(n) = h_1(n) + h_2(n)$$

- The parallel connection of two stable systems is stable
- However, the parallel connection of two passive systems may or may or may not be passive

3. Stability Condition

BIBO Stability Condition --

- A discrete-time system is BIBO stable if the output sequence $\{y(n)\}$ remains bounded for all bounded input sequence $\{x(n)\}$
- An LTI discrete-time system is BIBO stable if and only if its impulse response sequence $\{h(n)\}$ is absolutely summable, i.e.

$$S = \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

4. Causality Condition

Causality Condition --

- An LTI discrete-time system is causal if and only if its impulse response $\{h(n)\}$ is a causal sequence
- A non-causal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay

5. Differential Equations

- An LTI discrete-Time system can also be described by a linear constant coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

If $a_N \neq 0$, then the difference equation is of order N

If $N=0$, we call this system an FIR filter

If $N \neq 0$, we call this system an IIR filter