# Chapter 2A

**Discrete-Time Signals and Systems in the Time-Domain**

## Chapter 2

- ***Part A*** **-- Discrete-Time Signals**
- ***Part*** **B -- Discrete-Time Systems**
- ***Part C*** **-- Time-Domain Characterization of LTI Discrete-Time Systems**



# Part A

**Discrete-Time Signals**

## Part A: Discrete-Time Signals

- **Time-Domain Representation**
- **Operations on Sequences**
- **Classification of Sequences**
- **Typical Sequences**
- **The Sample Process**



## 1. Time-Domain Representation

- Signals represented as sequences of numbers, called samples （采样、样本）
- Sample value of a typical signal or sequence denoted as $x(n)$ with $n$ being an integer in the range $-\infty \leqslant n \leqslant \infty$
- $x(n)$ defined only for integer values of $n$ and undefined for non-integer values of $n$
- Discrete-time signal represented by $\{x(n)\}$



## 1. Time-Domain Representation

- Discrete-time signal may also be written as a sequence of numbers inside braces:

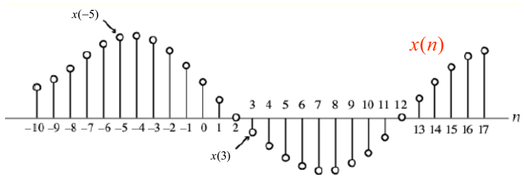
$$\{x(n)\}=\{\ldots, -0.2, \underset{\uparrow}{2.2}, 1.1, 0.2, -3, 7, 2.9, \ldots\}$$

In the above, $x(-1)= -0.2$, $x(0)=2.2$, $x(3)=-3$ etc.

- The arrow is placed under the sample at time index $n = 0$

## 1. Time-Domain Representation

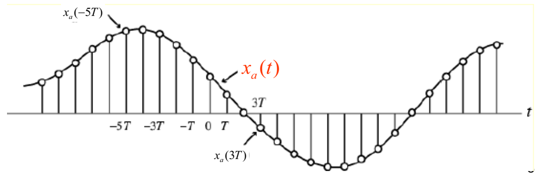
- Graphical representation of a discrete-time signal with real-valued samples is as shown below:






## 1. Time-Domain Representation

- In some applications, a discrete-time sequence $\{x(n)\}$ may be generated by periodically sampling a continuous-time signal at uniform intervals of time $x_a(t)$




## 1. Time-Domain Representation

- Here, $n$-th sample is given by

  $x(n)=x_a(t)|_{t=nT}=x_a(nT),\ n=\ldots,\ -2,\ -1,0,\ -1,\ldots$

- The spacing $T$ between two consecutive samples is called the sampling interval or sampling period
- Reciprocal of sampling interval $T$, denoted as , is called the sampling frequency:

  $F_T=1/T$



## 1. Time-Domain Representation

- Unit of sampling frequency is cycles per second, or hertz (Hz), if $T$ is in seconds
- Whether or not the sequence $\{x(n)\}$ has been obtained by sampling, the quantity $x(n)$ is called the $n$-th sample of the sequence
- $\{x(n)\}$ is a real sequence, if the $n$-th sample $x(n)$ is real for all values of $n$
- Otherwise, $\{x(n)\}$ is a complex sequence



## 1. Time-Domain Representation

- Two types of discrete-time signals:
  - -- Sampled-data signals
  - -- Digital signals
- Signals in a practical digital signal processing system are digital signals obtained by quantizing the sample values either by rounding （取整） or truncation （舍位）



## 1. Time-Domain Representation

- A discrete-time signal may be a finite-length or an infinite-length sequence
- Finite-length (also called finite-duration or finite-extent) sequence is defined only for a finite time interval $N_1 \leqslant n \leqslant N_2$,

  where $-\infty < N_1$ and $N_2 < \infty$ with $N_1 < N_2$

- Length or duration of the above sequence is

  $N = N_2 - N_1 + 1$

## 1. Time-Domain Representation

- The length of a finite-length sequence can be increased by zero-padding, i.e., by appending it with zeros
- Infinite-length sequences can be classified as following

$x(n)=0$ for $n<N_1$ right-sided sequence

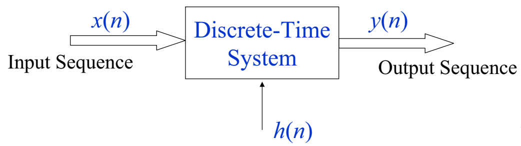
$x(n)=0$ for $n>N_2$ left-sided sequence

$x(n)\neq 0$ for $\infty \leqslant n \leqslant \infty$ double-sided sequence



## 2. Operations on Sequences

- A single-input, single-output discrete-time system operates on a sequence, called the input sequence, according to some prescribed rules and develops another sequence, called the output sequence, with more desirable properties






## 2. Operations on Sequences

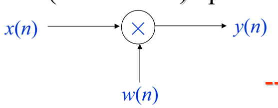
**Basic Operations**

- Product
- Addition
- Multiplication
- Time-Shifting
- Time-Reverse (folding)
- Branching



## 2. Operations on Sequences

- Product (modulation) operation:



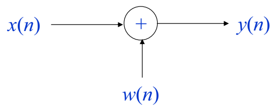

--Modulator

- An application is in forming a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length sequence called an window sequence. The process is called windowing（加窗）



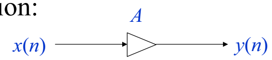
## 2. Operations on Sequences

- Addition operation:




--Adder $y(n)=x(n)+w(n)$

- Multiplication operation:




--Multiplier $y(n)=Ax(n)$

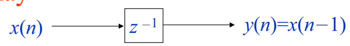


## 2. Operations on Sequences

- Time-shifting operation: $y(n)=x(n-N)$

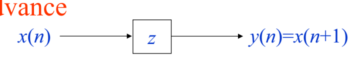
If $N>0$, it is delaying operation

--Unit delay



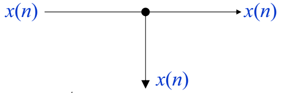

If $N<0$, it is an advance operation

--Unit advance



## 2. Operations on Sequences

- Time-reversal (folding) operation: $y(n)=x(-n)$
- Branching operation: Used to provide multiple copies of a sequence




- Operations on two or more sequences can be carried out if all sequences involved are of same length and defined for the same range of the time index $n$



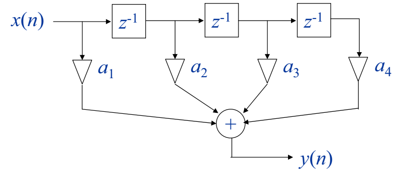
## 2. Operations on Sequences

- However if the sequences are not of same length, in some situations, this problem can be circumvented by appending zero-valued samples to the sequence(s) of smaller lengths to make all sequences have the same range of the time index
- The combination of basic operations can realize desirable functions



## 2. Operations on Sequences

**An Example**






## 2. Operations on Sequences

### Sampling Rate Alteration

- Employed to generate a new sequence $y(n)$ with a sampling rate $F_T^{'}$ higher or lower than that of the sampling rate $F_T$ of a given sequence $x(n)$
- Sampling rate alteration ratio is $R = F_T^{'} / F_T$

  If $R > 1$, the process called interpolation (内插)

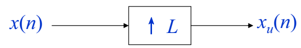
  If $R < 1$, the process called decimation (抽取)



## 2. Operations on Sequences

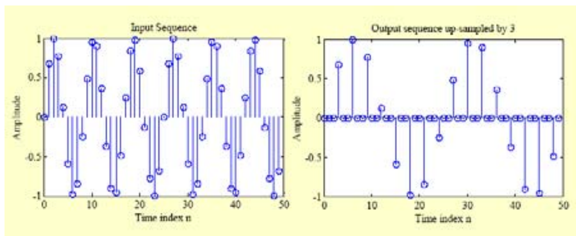
- In up-sampling (升采样) by an integer factor $L > 1$, equidistant zero-valued samples are inserted by the up-sampler between each two consecutive samples of the input sequence $x(n)$:

$$x_u(n) = \begin{cases} x(n/L), & n = 0, \pm L, \pm 2L, \ldots \\ 0, & otherwise \end{cases}$$





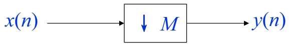

## 2. Operations on Sequences



## 2. Operations on Sequences

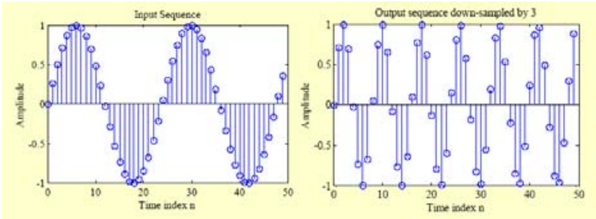
- In down-sampling (降采样) by an integer factor $M > 1$, every $M$-th samples of the input sequence are kept and $M-1$ in-between samples are removed:

$$y(n) = x(nM)$$






## 2. Operations on Sequences






## 3. Classification of sequences

- A discrete-time signal can be classified in various ways, such as length, symmetry, summability, energy and power.
- Conjugate-symmetric sequence: $x(n)=x^*(-n)$

  If $x(n)$ is real, then it is an even sequence
- Conjugate-antisymmetric sequence: $x(n)=-x^*(-n)$, If $x(n)$ is real, then it is an odd sequence



## 3. Classification of sequences

- It follows from the definition that for a conjugate-symmetric sequence $\{x(n)\}$, $x(0)$ must be a real number
- Likewise, it follows from the definition that for a conjugate anti-symmetric sequence $\{y(n)\}$, $y(0)$ must be an imaginary number
- From the above, it also follows that for an odd sequence $\{w(n)\}$, $w(0) = 0$



## 3. Classification of sequences

- Any complex sequence can be expressed as a sum of its conjugate-symmetric part and its conjugate anti-symmetric part:

$$x(n) = x_{cs}(n) + x_{ca}(n)$$

where

$$x_{cs}(n) = (1/2)[x(n)+x^*(-n)]$$
$$x_{ca}(n) = (1/2)[x(n)-x^*(-n)]$$



## 3. Classification of sequences

- For a length-$N$ sequence defined for $0 \leqslant n \leqslant N-1$, it has a different definition as follows

$$x(n) = x_{pcs}(n) + x_{pca}(n) \quad 0 \leqslant n \leqslant N-1$$

where

$$x_{pcs}(n) = (1/2)[x(n)+x^*(N-n)] \quad 0 \leqslant n \leqslant N-1$$

is the periodic conjugate-symmetric part, and

$$x_{pca}(n) = (1/2)[x(n)-x^*(N-n)] \quad 0 \leqslant n \leqslant N-1$$

is the periodic conjugate-antisymmetric part

## 3. Classification of sequences

- A length-$N$ sequence $x(n)$ is called a periodic conjugate-symmetric sequence if

$$x(n) = x^*(\langle -n \rangle_N) = x^*(N-n) \quad 0 \le n \le N-1$$

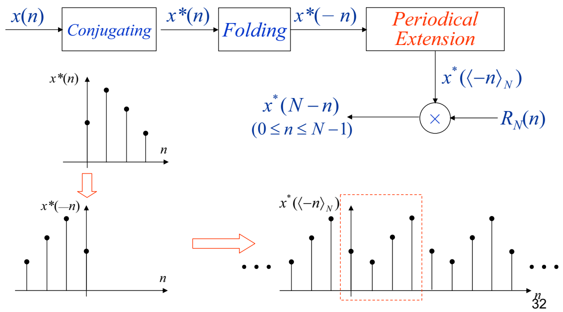
and is called a periodic conjugate-anti-symmetric sequence if

$$x(n) = -x^*(\langle -n \rangle_N) = -x^*(N-n) \quad 0 \le n \le N-1$$

***Q***: How to get $x(-n)$ in the interval $0 \leqslant n \leqslant N-1$



## 3. Classification of sequences






## 3. Classification of sequences

- A sequence $\tilde{x}(n)$ satisfying

$$\tilde{x}(n) = \tilde{x}(n+kN) \quad \text{for all } n$$

is called a periodic sequence with a period $N$, where $N$ is a positive integer and $k$ is any integer
- Smallest value of $N$ satisfying $\tilde{x}(n) = \tilde{x}(n+kN)$ is called the fundamental period
- A sequence not satisfying the periodicity condition is called an aperiodic sequence



## 3. Classification of sequences

- Total energy of a sequence $x(n)$ is defined by

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- An infinite length sequence with finite sample values may or may not have finite energy
- A finite length sequence with finite sample values has finite energy



## 3. Classification of sequences

- The average power of a periodic sequence with a period $N$ is given by

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |\tilde{x}(n)|^2$$

- The average power of an aperiodic sequence is defined by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1}\sum_{n=-K}^{K} |x(n)|^2$$



## 3. Classification of sequences

- A sequence $x(n)$ is said to be bounded if

$$|x(n)| \le B_x < \infty$$

- A sequence $x(n)$ is said to be absolutely summable if

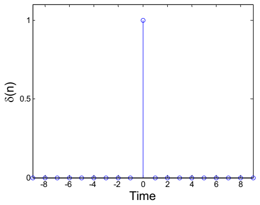
$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

- A sequence $x(n)$ is said to be square summable if

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

## 4. Typical Sequences

- Unit Sample Sequence $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



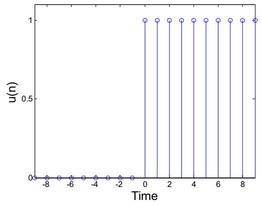

**$\delta(n)$ vs. $\delta(t)$**

**Their energies are both equal to 1. $\delta(n)$ is of engineering value, but $\delta(t)$ only has meaning in theory**



## 4. Typical Sequences

- Unit Step Sequence $\mu(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n \leq 0 \end{cases}$




**$\delta(n)$ vs. $u(n)$**

$$\mu(n) = \sum_{k=0}^{\infty} \delta(n-k) = \sum_{k=-\infty}^{n} \delta(k)$$

$$\delta(n) = \mu(n) - \mu(n-1)$$



## 4. Typical Sequences

- Real sinusoidal sequence-

$$x(n) = A\cos(\omega_0 n + \phi)$$

where $A$ is the amplitude, $\omega_0$ is the angular frequency, and $\phi$ is the phase of $x(n)$

- Complex exponential sequence-

$$x(n) = A\alpha^n, \quad -\infty \leq n \leq \infty$$

where $A$ and $\alpha$ are real or complex numbers. In general, $\alpha = e^{(\sigma_0 + j\omega_0)}$ and $A = |A| e^{j\phi}$



## 4. Typical Sequences

- An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its delayed (advanced) versions

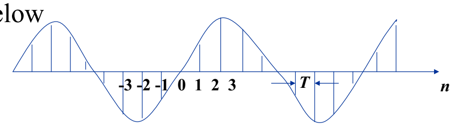
$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

- Another interpretation is that the above equation can be viewed as a convolution of $x(n)$ and $\delta(n)$



## 5. The Sampling Process

- Often, a discrete-time sequence $x(n)$ is developed by uniformly sampling a continuous-time signal $x_a(n)$ as indicated below




- The relation between the two signals is

$$x(n) = x_a(t)|_{t=nT} = x_a(nT),\ n = \ldots, -2, -1, 0, 1, 2, \ldots$$



## 5. The Sampling Process

- Time variable $t$ of $x_a(t)$ is related to the time variable $n$ of $x(n)$ only at discrete-time $t_n$ instants given by

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

with $F_T = 1/T$ denoting the sampling frequency and $\Omega_T = 2\pi F_T$ denoting the sampling angular frequency

## 5. The Sampling Process

- Consider the continuous-time signal

$$x_a(t) = A\cos(2\pi f_0 t + \phi) = A\cos(\Omega_0 t + \phi)$$

- The corresponding discrete-time signal is

$$x(n) = A\cos(\Omega_0 nT + \phi) = A\cos\left(\Omega_0 n \frac{1}{F_T} + \phi\right)$$

$$= A\cos\left(\frac{2\pi\Omega_0}{\Omega_T} n + \phi\right) = A\cos(\omega_0 n + \phi)$$



## 5. The Sampling Process

where $\omega_0 = \dfrac{2\pi\Omega_0}{\Omega_T}$

is the normalized digital angular frequency of $x(n)$

- If the unit of sampling period $T$ is in seconds, the unit of normalized digital angular frequency $\omega_o$ is radians/sample while the unit of normalized analog angular frequency $\Omega_0$ is radians/second



## 5. The Sampling Process

An Example

Consider the three continuous-time signals

$g_1(t)=\cos(6\pi t)$, $g_2(t)=\cos(14\pi t)$, $g_3(t)=\cos(26\pi t)$

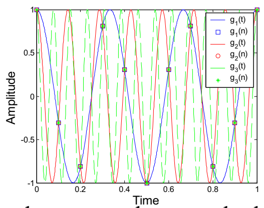
of frequencies 3 Hz, 7 Hz, and 13 Hz, are sampled at a sampling rate of 10 Hz, i.e. with $T = 0.1$ sec. generating the three sequences

$g_1(n)=\cos(0.6\pi n)$, $g_2(n)=\cos(1.4\pi n)$, $g_3(n)=\cos(2.6\pi n)$



## 5. The Sampling Process

Plots of these sequences and their parent time functions are shown below:




Note that each sequence has exactly the same sample value for any given $n$



## 5. The Sampling Process

This fact can also be verified by observing that

$$g_2(n) = \cos(1.4\pi n) = \cos((2\pi - 0.6\pi)n) = \cos(0.6\pi n)$$

$$g_3(n) = \cos(2.6n) = \cos((2\pi + 0.6\pi)n) = \cos(0.6\pi n)$$

As a result, all three sequences are identical and it is difficult to associate a unique continuous-time function with each of these sequences



## 5. The Sampling Process

- The above phenomenon of a continuous-time signal of higher frequency acquiring the identity of a sinusoidal sequence of lower frequency after sampling is called aliasing
- Therefore, additional conditions need to imposed so that the sequence $x(n)$ can uniquely represent the parent continuous-time signal $x_a(t)$

## 5. The Sampling Process

Another Example

Determine the discrete-time signal $v(n)$ obtained by uniformly sampling at a sampling rate of 200 Hz the continuous-time signal

$v_a(t)$=6cos(60 π $t$)+3sin(300 π $t$)+2cos(340 π $t$)

+4cos(500 π $t$)+10sin(660 π $t$)

Note: It is composed of 5 sinusoidal signals of frequencies 30 Hz, 150 Hz, 170Hz, 250 Hz and 330 Hz



## 5. The Sampling Process

- The sampling period is $T$=1/200=0.005sec
- The generated discrete-time signal $v(n)$ is thus given by

$$
\begin{aligned}
v(n) &= 6\cos(0.3\pi n) + 3\sin(1.5\pi n) \\
&\quad + 2\cos(1.7\pi n) + 4\cos(2.5\pi n) + 10\sin(3.3\pi n) \\
&= 6\cos(0.3\pi n) + 3\sin((2-0.5)\pi n) 2\cos((2-0.3)\pi n) \\
&\quad + 4\cos((2+0.5)\pi n) + 10\sin((4-0.7)\pi n) \\
&= 6\cos(0.3\pi n) - 3\sin(0.5\pi n) + 2\cos(0.3\pi n) \\
&\quad + 4\cos(0.5\pi n) - 10\sin(0.7\pi n) \\
&= 8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n)
\end{aligned}
$$



## 5. The Sampling Process

- Note: $v(n)$ is composed of 3 discrete-time sinusoidal signals of normalized angular frequencies: 0.3 π , 0.5 π , and 0.7 π
- Note: An identical discrete-time signal is also generated by uniformly sampling at a 200-Hz sampling rate the following continuous-time signals:

$$w_a(t) = 8\cos(60\pi t) + 5\cos(100\pi t + 0.6435) - 10\sin(140\pi t)$$

$$
\begin{aligned}
u_a(t) &= 2\cos(60\pi t) + 4\cos(100\pi t) + 10\sin(260\pi t) \\
&\quad + 6\cos(460\pi t) + 3\sin(700\pi t)
\end{aligned}
$$



## 5. The Sampling Process

- Recall $$\omega_0 = \frac{2\pi\Omega_0}{\Omega_T}$$

Thus if $\Omega_T > 2\Omega_0$, then the corresponding normalized digital angular frequency of the discrete-time signal obtained by sampling the parent continuous-time sinusoidal signal will be in the range $-\pi < \omega < \pi$

- ***No aliasing***



## 5. The Sampling Process

- On the other hand, if $\Omega_T < 2\Omega_0$, the normalized digital angular frequency will foldover into a lower digital frequency $\omega_0 = \langle 2\pi\Omega_0 / \Omega_T \rangle_{2\pi}$ in the range $-\pi < \omega < \pi$ because of aliasing
- Hence, to prevent aliasing, the sampling frequency $\Omega_T$ should be greater than 2 times the frequency $\Omega_0$ of the sinusoidal signal being sampled
- From the above analysis, we state the Sample Theorem as follows



## Sampling Theorem

- Consider an arbitrary continuous-time signal $x_a(t)$ composed of a weighted sum of a number of sinusoidal signals
- $x_a(t)$ can be represented uniquely by its sampled version $\{x(n)\}$ if the sampling frequency $\Omega_T$ is chosen to be greater than 2 times the highest frequency contained in $x_a(t)$
- This theorem can be proofed via Fourier Transform in chapter 5