Chapter 10

FIR Digital Filter Design

FIR Digital Filter Design

The transfer function is a polynomial in z^{-1}.
Basic approaches in designing FIR filters

- Truncating the Fourier series representation of the desired frequency response => Window method
- Computer-aided design based on optimization

1. Truncating the Impulse Response

- Let $H_d(e^{j\omega})$ denote the desired frequency response function. $H_d(e^{j\omega})$ is periodic function of ω with period 2π and can be expressed as a Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega n}$$

- The Fourier coefficients $\{h_d(n)\}$ are the impulse response samples

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega, \quad -\infty \le n \le \infty$$

1. Truncating the Impulse Response

- Thus, given $H_d(e^{j\omega})$ we can compute $h_d(n)$ and the corresponding $H_d(z)$
- Usually, $H_d(e^{j\omega})$ is piecewise constant with ideal (or sharp) transitions between bands => $\{h_d(n)\}$ sequence is of infinite length and noncausal
- The objective is to find a finite-duration impulse response $\{h_t(n)\}$ of length $2M+1$ whose DTFT $H_t(e^{j\omega})$ approximates the desired DTFT $H_d(e^{j\omega})$

1. Truncating the Impulse Response

- Minimizing the integral squared error

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_t(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

where $H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t(n)e^{-j\omega n}$

- Using the Parseval's relation

$$\Phi = \sum_{n=-\infty}^{\infty}\left|h_t(n) - h_d(n)\right|^2$$

$$= \sum_{n=-M}^{M}\left|h_t(n) - h_d(n)\right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2(n) + \sum_{n=M+1}^{\infty} h_d^2(n)$$

Φ is minimum when $h_t(n) = h_d(n)$ for $-M \leqslant n \leqslant M$,

constant term

1. Truncating the Impulse Response

- The best finite-length approximation is obtained by truncating the impulse response
- A causal impulse response $h(n)$ can be obtained from $h_t(n)$ by delaying it with M samples

$$h(n) = h_t(n - M)$$

- $h(n)$ has the same magnitude response as $h_t(n)$ but its phase response has a linear phase shift of ωM radians

1. Truncating the Impulse Response

- The group delay of $h(n)$ is M samples

$$\tau(\omega) = -\frac{d}{d\omega}(-\omega M) = M$$

where the linear phase response is $-\omega M$

2. Impulse Response of Ideal Lowpass Filters

- The ideal lowpass filter has a zero-phase frequency response

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi_c \end{cases}$$

- The corresponding impulse response coefficients

$$h_{LP}(n) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty \le n \le \infty$$

is doubly infinite, not absolutely summable, and therefore *unrealizable*

2. Impulse Response of Ideal Lowpass Filters

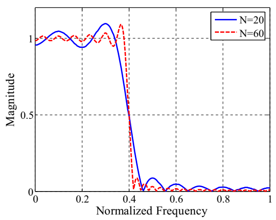
- Truncating to range $-M \leqslant n \leqslant M$ and delaying with M samples yields the causal FIR lowpass filter

$$\hat{h}_{LP}(n) = \begin{cases} \frac{\sin(\omega_c (n-M))}{\pi(n-M)}, & 0 \le n \le 2M \\ 0, & \text{otherwise} \end{cases}$$

- The truncation of the impulse response coefficients of the ideal filters exhibit an oscillatory behavior in the respective magnitude responses

3. Gibbs Phenomenon

- *Gibbs phenomenon* - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



Impact of the length of the window function

(1) Narrower transition band

(2) More ripples

(3) Smaller ripple width

(4) Same largest peak ripple

The performance is better.

How to reduce the highest ripple?

3. Gibbs Phenomenon

- As can be seen, as the length of the lowpass filter is increased, the number of ripples in both passband and stopband increases, with a corresponding decrease in the ripple widths
- Height of the largest ripples remain the same independent of length
- Similar oscillatory behavior observed in the magnitude responses of the truncated versions of other types of ideal filters

3. Gibbs Phenomenon

- Truncation of $h_d(n)$ can be expressed by *windowing operation*, i.e., by multiplying the $h_d(n)$ sequence with a finite-length sequence $w(n)$

$$h_t(n) = h_d(n) \cdot w(n)$$

where $w(n)$ is a window function

3. Gibbs Phenomenon

- For a rectangular window

$$w_R(n) = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- The Gibbs phenomenon can be explained in the frequency domain by the convolution theorem

3. Gibbs Phenomenon

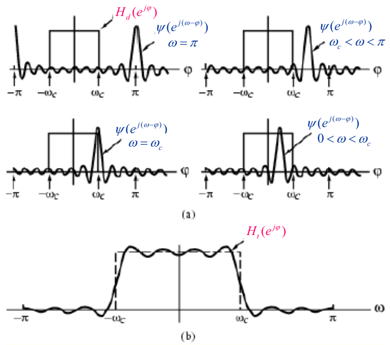
- Multiplication in the time domain corresponds to convolution in the frequency domain

$$H_t(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\varphi})\Psi(e^{j(\omega-\varphi)})d\varphi$$

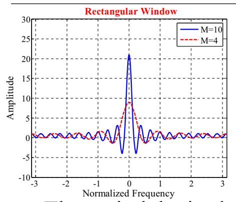
where $H_d(e^{j\omega}) = F\{h_d(n)\}$ $\quad \Psi(e^{j\omega}) = F\{w(n)\}$

- $H_t(e^{j\omega})$ is obtained by a periodic continuous convolution of the frequency response $H_d(e^{j\omega})$ with the Fourier transform $\Psi(e^{j\omega})$ of the window

3. Gibbs Phenomenon



3. Gibbs Phenomenon



- The frequency response $\Psi(e^{j\omega})$ has a narrow *mainlobe* centered at $\omega=0$
- All the other ripples in the frequency response are called *sidelobes*
- The main lobe is characterized by its width $4\pi/(2M+1)$ defined by the first zero crossings on both sides of $\omega=0$
- As M increases the width of the main lobe decreases
- The area under each lobe remains constant, while the width of each lobe decreases with increasing M

3. Gibbs Phenomenon

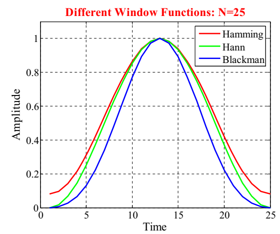
- Rectangular window has an abrupt transition to zero outside the range $-M \le n \le M$, which results in Gibbs phenomenon in $H_t(e^{j\omega})$
- Gibbs phenomenon can be reduced either:

  (a) Using a window that tapers smoothly to zero at each end, or

  (b) Providing a smooth transition from passband to stopband in the magnitude specifications

4. Fixed Window Functions

- Symmetric window functions are used in FIR filter design in order to guarantee the linear phase response
- Smoother behavior cutoff frequency is obtained by using different cosine-type functions instead of the rectangular window



4. Fixed Window Functions

- Various window functions: (rised cosine)

Hann:

$$w(n)=\frac{1}{2}\left[1+\cos\left(\frac{2\pi n}{2M+1}\right)\right],\ -M\le n\le M$$

Hamming:

$$w(n)=0.54+0.46\cos\left(\frac{2\pi n}{2M+1}\right),\ -M\le n\le M$$

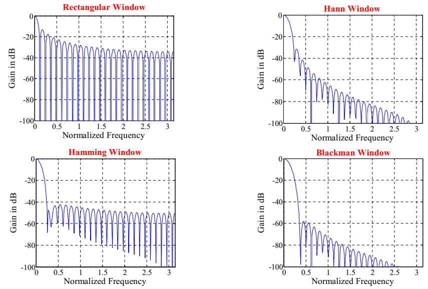
Blackman:

$$w(n)=0.42+0.5\cos\left(\frac{2\pi n}{2M+1}\right)+0.08\cos\left(\frac{4\pi n}{2M+1}\right)$$

$$-M\le n\le M$$

4. Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M=25$ are shown below:



4. Fixed Window Functions

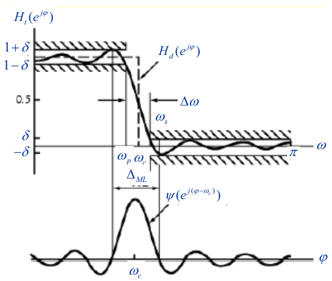
- Magnitude spectrum of each window characterized by a main lobe centered at $\omega=0$ followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are:
  1) Main lobe width
  2) Relative sidelobe level

4. Fixed Window Functions

- Main lobe width - Δ_{ML} given by the distance between zero crossings on both sides of main lobe
- Relative sidelobe level - A_{sl} given by the difference in dB between amplitudes of largest sidelobe and main lobe

4. Fixed Window Functions

Lowpass Filter Design by Windowing



4. Fixed Window Functions

- Observe $H_t(e^{j(\omega_c+\Delta\omega)})+H_t(e^{j(\omega_c-\Delta\omega)})\cong 1$
- Thus $H_t(e^{j\omega_c})\cong 0.5$
- Passband and stopband ripples are the same
- Distance between the locations of the maximum passband deviation and minimum stopband value $\approx\Delta_{ML}$
- Width of transition band $\Delta\omega=\omega_s-\omega_p<\Delta_{ML}$

4. Fixed Window Functions

- To ensure a fast transition from passband to stopband, window should have a very small main-lobe width
- To reduce the passband and stopband ripple δ, the area under the sidelobes should be very small
- Unfortunately, these two requirements are **contradictory**

4. Fixed Window Functions

- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency ω_c, and is essentially constant
- In addition, $\Delta\omega \approx c/M$

  where c is a constant for most practical purposes

4. Fixed Window Functions

Table 10.2: Properties of fixed window functions

- **Rectangular window** - $\Delta_{ML} = 4\pi/(2M+1)$
  $A_{s\ell} = 13.3$ dB, $\alpha_s = 20.9$ dB, $\Delta\omega = 0.92\pi/M$
- **Hann window** - $\Delta_{ML} = 8\pi/(2M+1)$
  $A_{s\ell} = 31.5$ dB, $\alpha_s = 43.9$ dB, $\Delta\omega = 3.11\pi/M$
- **Hamming window** - $\Delta_{ML} = 8\pi/(2M+1)$
  $A_{s\ell} = 42.7$ dB, $\alpha_s = 54.5$ dB, $\Delta\omega = 3.32\pi/M$
- **Blackman window** - $\Delta_{ML} = 12\pi/(2M+1)$
  $A_{s\ell} = 58.1$ dB, $\alpha_s = 75.3$ dB, $\Delta\omega = 5.56\pi/M$

4. Fixed Window Functions

Filter Design Steps -

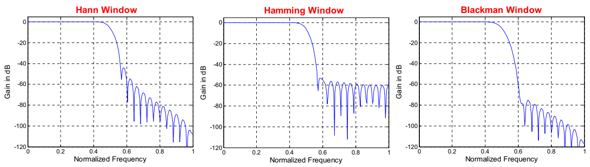
① Set $\omega_c = (\omega_p + \omega_s)/2$

② Choose window based on specified α_s

③ Estimate M using $\Delta\omega \approx c/M$

4. Fixed Window Functions

- Lowpass filter of length 51 and $\omega_c = \pi/2$



- An increase in the main lobe width is associated with an increase in the width of the transition band
- A decrease in the sidelobe amplitude results in an increase in the stopband attenuation

5. Adjustable Window Functions

- Dolph-Chebyshev Window –

$$w(n) = \frac{1}{2M+1}\left[\frac{1}{\gamma} + 2\sum_{k=1}^{M} T_k\left(\beta\cos\frac{k\pi}{2M+1}\right)\cos\left(\frac{2nk\pi}{2M+1}\right)\right] \qquad -M \le n \le M$$

where
$$\gamma = \frac{\text{amplitude of sidelobe}}{\text{main lobe amplitude}}$$
$$\beta = \cosh\left(\frac{1}{2M}\cosh^{-1}\frac{1}{\gamma}\right)$$

and
$$T_l(x) = \begin{cases} \cos(l\cos^{-1}x), & \text{for } |x| \le 1 \\ \cosh(l\cos^{-1}x), & \text{for } |x| > 1 \end{cases}$$

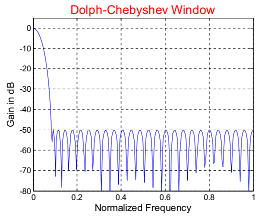
5. Adjustable Window Functions

- Dolph-Chebyshev window can be designed with any specified relative sidelobe level while the main lobe width adjusted by choosing length appropriately
- Filter order is estimated using $N = \dfrac{2.056\alpha_s - 16.4}{2.285(\Delta\omega)}$

  where $\Delta\omega$ is the normalized transition bandwidth, e.g, for a lowpass filter $\Delta\omega = \omega_s - \omega_p$

5. Adjustable Window Functions

- Gain response of a Dolph-Chebyshev window of length 51 and relative sidelobe level of 50 dB is shown below



5. Adjustable Window Functions

Properties of Dolph-Chebyshev window:

- All sidelobes are of equal height
- Stopband approximation error of filters designed have essentially equiripple behavior
- For a given window length, it has the smallest main lobe width compared to other windows resulting in filters with the smallest transition band

5. Adjustable Window Functions

- Kaiser Window -

$$w(n) = \frac{I_0\left\{\beta\sqrt{1-(n/M)^2}\right\}}{I_0(\beta)}, \quad -M \le n \le M$$

where β is an adjustable parameter and $I_0(u)$ is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty}\left[\frac{(u/2)^r}{r!}\right]^2$$

- Note $I_0(u)>0$ for u being real

5. Adjustable Window Functions

- In practice

$$I_0(u) = 1 + \sum_{r=1}^{20}\left[\frac{(u/2)^r}{r!}\right]^2$$

- β controls the minimum stopband attenuation of the windowed filter response
- β is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5824(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

5. Adjustable Window Functions

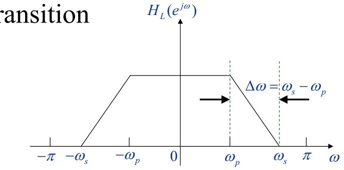
- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth

6. Impulse Responses of FIR Filters with a Smooth Transition

- First-order spline passband-to-stopband transition



$$\omega_c = (\omega_s + \omega_p)/2$$
$$\Delta\omega = \omega_s - \omega_p$$

$$h_{LP}(n) = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \cdot \dfrac{\sin(\omega_c n)}{\pi n}, & |n| > 0 \end{cases}$$

6. Impulse Responses of FIR Filters with a Smooth Transition

- *P*th-order spline passband-to-stopband transition

$$h_{LP}(n) = \begin{cases} \omega_c/\pi, & n = 0 \\ \left(\dfrac{2\sin(\Delta\omega n/2P)}{\Delta\omega n}\right)^P \cdot \dfrac{\sin(\omega_c n)}{\pi n}, & |n| > 0 \end{cases}$$

6. Impulse Responses of FIR Filters with a Smooth Transition

Example- $\omega_p = 0.35\pi$ $\omega_s = 0.45\pi$