Chapter 9B

IIR Digital Filter Design

Part B

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- Impulse Invariance Method
- Bilinear Transform Method
- Spectral Transformations of IIR Filters
  - Lowpass-to-Lowpass Transformation

1. Impulse Invariance Method

Definition –

The impulse response of the digital filter is identical to the impulse response of an analog prototype filter at sampling instants

- Analog transfer function: $H_a(s)$

$$h_a(t) = \mathrm{ST}^{-1}\{H_a(s)\}$$

- The impulse response of the digital filter is:

$$h(n) = h_a(nT), \quad t = 0, 1, 2, \ldots$$

1. Impulse Invariance Method

The relation between ZT and ST

$$H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{-st} dt$$

$$\hat{h}_a(t) = \sum_{n=-\infty}^{\infty} h_a(nT)\delta(t - nT)$$

$$\hat{H}_a(s) = \int_{-\infty}^{\infty} \hat{h}_a(t) e^{-st} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h_a(nT)\delta(t - nT) e^{-st} dt$$

$$= \sum_{n=-\infty}^{\infty} h_a(nT) \int_{-\infty}^{\infty} \delta(t - nT) e^{-st} dt$$

$$= \sum_{n=-\infty}^{\infty} h_a(nT) e^{-nsT}$$

1. Impulse Invariance Method

The relation between ZT and ST

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \Longleftrightarrow \hat{H}_a(s) = \sum_{n=-\infty}^{\infty} h_a(nT) e^{-nsT}$$

$$h(n) = h_a(nT), \quad t = 0, 1, 2, \ldots$$

$$H(z)\big|_{z=e^{sT}} = H(e^{sT}) = \hat{H}_a(s)$$

$$z = e^{sT}, \quad s = \frac{1}{T} \ln z$$

1. Impulse Invariance Method

The relation between ZT and ST

$$\hat{H}_a(j\Omega) = \hat{H}_a(s)\Big|_{s=j\Omega}$$

$$\hat{H}_a(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a(j\Omega - kj\Omega_s) \qquad \Omega_s = \frac{2\pi}{T}$$

$$\hat{H}_a(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - kj\frac{2\pi}{T}\right)$$

$$\boxed{H(z)\Big|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - kj\frac{2\pi}{T}\right)}$$

1. Impulse Invariance Method

- The digital filter transfer function $H(z)$ is:

$$H(z) = \mathrm{ZT}(h(n)) = \mathrm{ZT}(h_a(nT))$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right)\Bigg|_{s=\frac{1}{T}\ln z}$$

- The frequency responses are obtained by substituting $z=e^{j\omega}$ and $s=j\Omega$:

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(j\Omega - j\frac{2\pi k}{T}\right)$$

1. Impulse Invariance Method

- According to the sampling theorem $H(e^{j\omega})$ is a periodic version of $H_a(j\Omega)$
- Transformation from ***s***-plane to ***z***-plane: $z = e^{sT}$
- For $s=\sigma_0+j\Omega_0$: $z = re^{j\omega} = e^{\sigma_0 T} e^{j\Omega_0 T}, \quad |z| = r = e^{\sigma_0 T}$
- Mapping relations

I $\quad r = e^{\sigma_0 T}$

II $\quad e^{j\omega} = e^{j\Omega_0 T}$ ⟶ $\omega = \Omega_0 T + 2k\pi = T\left\{\Omega_0 + \frac{2k\pi}{T}\right\}$

1. Impulse Invariance Method

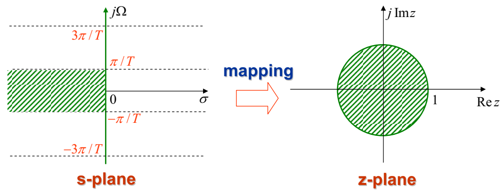
- Mapping I: $r = e^{\sigma_0 T}$ means
  - A point on the frequency axis in the s-plane ($\sigma_0=0$) is mapped to a point on the unit circle in the z-plane
  - A point on the left-half s-plane with $\sigma_0<0$ is mapped to z-plane with $|z|<1$, i.e., the left-half s-plane is mapped inside the unit circle
  - Similarly, A point on the right-half s-plane with $\sigma_0>0$ is mapped to z-plane with $|z|>1$, i.e., the right-half s-plane is mapped outside the unit circle

1. Impulse Invariance Method

- Thus, the impulse invariance mapping has the desired properties:
  - Frequency axis $j\Omega$ corresponds to unit circle
  - Stability is preserved

1. Impulse Invariance Method

- Mapping II: $\omega = \Omega T + 2k\pi = T\left\{\Omega + \frac{2k\pi}{T}\right\}$



1. Impulse Invariance Method

- Due to sampling the mapping is *many-to-one*
- The strips of length $2\pi/T$ are all mapped onto the unit circle
- Only if $h_a(t)$ is a band-limited signal, no alias will occur
- Hence, this method is not suitable for *highpass* and *bandstop* filters design

1. Impulse Invariance Method

- Assume that $H_a(s)$ has the form of

$$H_a(s) = \frac{A}{s+\alpha}$$

- The corresponding signal in time-domain is

$$h_a(t) = \mathrm{ST}^{-1}\{H_a(s)\} = Ae^{-\alpha t}u(t)$$

- By sampling $h_a(t)$

$$h(n) = h_a(nT) = Ae^{-\alpha nT}u(nT)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} = A\sum_{n=0}^{\infty} e^{-\alpha nT}z^{-n} = \frac{A}{1-e^{-\alpha T}z^{-1}}$$

1. Impulse Invariance Method

- $H(z)$ converges if $\left|e^{-\alpha T}\right| < 1$ or $\alpha > 0$, indicating that $H_a(s)$ is stable
- Generalizing to higher order (N) analog transfer functions

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s+\alpha_k}$$

$$h_a(t) = \sum_{k=1}^{N} A_k e^{-\alpha_k t}u(t)$$

$$\Rightarrow H(z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{-\alpha T}z^{-1}}$$

1. Impulse Invariance Method

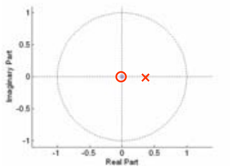
Example

- **First Order Butterworth Filter Designed Using the Impulse Invariant Method (T=1)**

$$H_a(s) = \frac{1}{s+1} \longrightarrow h_a(t) = e^{-t}u(t) \longrightarrow H(z) = \frac{1}{1-e^{-1}z^{-1}}$$

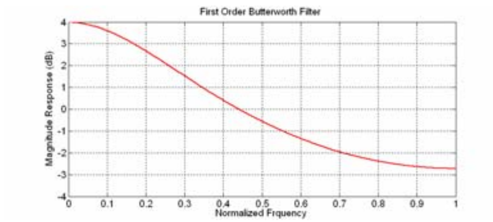
zero at $z=0$

pole at $z=1/e$



1. Impulse Invariance Method

- **Magnitude Response**



2. Bilinear Transform Method

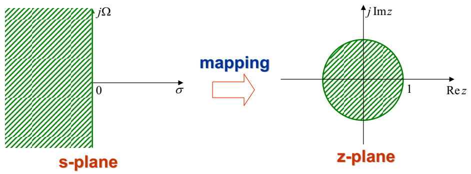
Definition –

- To avoid aliasing, the mapping from ***s***-plane to ***z***-plane should be one-to-one, i.e., a single point in the *s*-plane should be mapped to a unique point in the ***z***-plane and vice versa

1) The entire $j\Omega$-axis should be mapped onto the unit circle

2) The entire left-half s-plane should be mapped inside the unit circle

2. Bilinear Transform Method



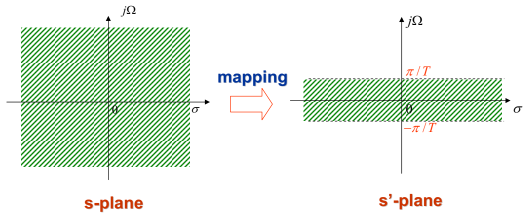
Derivation of the bilinear transform:

1) One-to-one mapping from s to s' which compresses the entire s-plane into the strip

$$-\pi/T < \mathrm{Im}(s') < \pi/T$$

2. Bilinear Transform Method

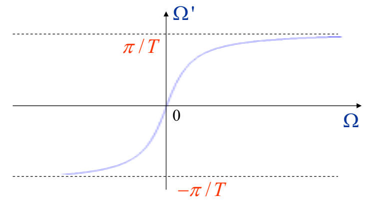
2) Employ impulse invariance method to s'-plane with $z=e^{s'T}$



2. Bilinear Transform Method

- One-to-one mapping from s to s'

$$\Omega' = \frac{2}{T}\tan^{-1}\left(\frac{\Omega T}{2}\right)$$



2. Bilinear Transform Method

- The normalized frequency ω now corresponds to $\Omega' T$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

- Thus, the entire $j\Omega$-axis is compressed to the interval $(-\pi, \pi)$ for ω in a one-to-one manner
- The mapping is highly nonlinear
- However, for small $\omega=\Omega' T$ it is approximately linear

2. Bilinear Transform Method

- The desired transformation from s to z (via s')

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \qquad \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

- As we know

$$j\tan x = \frac{e^{jx}-e^{-jx}}{e^{jx}+e^{-jx}} = \frac{1-e^{-2jx}}{1+e^{-2jx}}$$

- Hence

$$j\Omega = j\frac{2}{T}\tan\left(\frac{\omega}{2}\right) = \frac{2}{T}\frac{1-e^{-j\omega}}{1+e^{-j\omega}}$$

- Let $s=j\Omega$ and $z=e^{j\omega}$,we can arrive at $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$

The bilinear transform

2. Bilinear Transform Method

The bilinear transform: $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$

- The s-plane transfer function $H_a(s)$ gives a z-plane transfer function

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

- Solving z gives:

$$z = \left(1+\frac{T}{2}s\right)\Big/\left(1-\frac{T}{2}s\right)$$

2. Bilinear Transform Method

① $j\Omega$-axis, Re(s)=0; this gives $|z|=1$

The frequency axis from s-plane is mapped onto the unit circle

② Left-half s-plane, Re(s)<0; $|1+(T/2)s| < |1-(T/2)s|$ or $|z|<1$

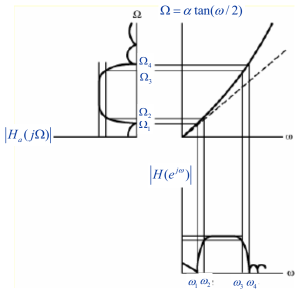
Left-half s-plane is mapped inside the unit circle

③ Right-half s-plane, Re(s)>0; $|1+(T/2)s| > |1-(T/2)s|$ or $|z|>1$

Right-half s-plane is mapped outside the unit circle

2. Bilinear Transform Method

Frequency Warping



Distortion due to nonlinearity of the mapping

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

2. Bilinear Transform Method

To design a digital filter meeting the desired (digital) specifications we have to:

① Prewarp the critical bandedge frequencies (ω_p and ω_s) to analog frequencies (Ω_p and Ω_s)

② Design an analog prototype filter $H_a(s)$ using the prewarped critical frequencies

③ Transform $H_a(s)$ to $G(z)$ using the bilinear transformation

2. Bilinear Transform Method

Example

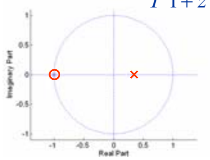
- **First Order Butterworth Filter Designed by the Bilinear Transformation**

$$H_a(s) = \frac{1}{s+1} \longrightarrow H(z) = \left.\frac{1}{s+1}\right|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+1}$$

$$\Rightarrow H(z)\big|_{T=1} = \frac{1+z^{-1}}{3-z^{-1}}$$

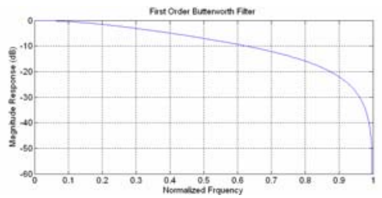
zero at $z=-1$

pole at $z=1/3$



2. Bilinear Transform Method

- **Magnitude Response**



- The entire frequency axis from the s-plane is mapped onto the unit circle in the z-plane one-to-one *NO ALIASING !*

3. Spectral Transformations of IIR Filters

- Transformation of a given digital IIR lowpass transfer function $G_L(z)$ to another digital transfer function $G_D(z)$
- Prototype lowpass $G_L(z)$; variable z^{-1} Transformed filter $G_D(z')$; variable z'^{-1}
- Transformation from z-domain to z'-domain: $z=F(z')$
- Now, $G_L(z)$ is transformed to $G_D(z')$ through

$$G_D(z') = G_L(F(z'))$$

3. Spectral Transformations of IIR Filters

- To transform a rational $G_L(z)$ into a rational $G_D(z')$, $F(z')$ must be a rational function in z'
- The inside of the z-plane should be mapped into the inside of z'-plane
- In order to map a lowpass magnitude response to one of the four basic types of magnitude responses, points on the unit circle in z-plane should be mapped onto the unit circle in z'-plane

3. Spectral Transformations of IIR Filters

- The requirements

$$|F(z')| = \begin{cases} >1, & \text{if } |z|>1 \\ =1, & \text{if } |z|=1 \\ <1, & \text{if } |z|<1 \end{cases}$$

- $F^{-1}(z')$ must be a stable allpass function
- The most general form of $F^{-1}(z')$ with real coefficients is given by

$$F^{-1}(z') = \pm\prod_{l=1}^{L}\left(\frac{1-\alpha_l^* z'}{z'-\alpha_l}\right)$$

4. Lowpass-to-Lowpass Transformation

- $G_L(z)$ with cutoff frequency ω_c is transformed to another lowpass filter $G_D(z')$ with ω'_c

$$z^{-1} = F^{-1}(z') = \frac{1-\alpha z'}{z'-\alpha}$$

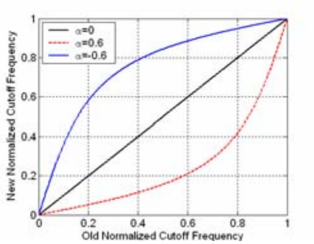
with α real

$$e^{-j\omega} = \frac{e^{-j\omega'}-\alpha}{1-\alpha e^{-j\omega'}}$$

$$\tan\left(\frac{\omega}{2}\right) = \frac{1+\alpha}{1-\alpha}\tan\left(\frac{\omega'}{2}\right)$$

4. Lowpass-to-Lowpass Transformation

$$\alpha = \frac{\sin\left(\dfrac{\omega_c-\omega_c'}{2}\right)}{\sin\left(\dfrac{\omega_c+\omega_c'}{2}\right)}$$



5. Computer-Aided Design of IIR Digital Filters

- The IIR and FIR filter design techniques discussed so far can be easily implemented on a computer
- In addition, there are a number of filter design algorithms that rely on some type of optimization techniques that are used to minimize the error between the desired frequency response and that of the computer generated filter

5. Computer-Aided Design of IIR Digital Filters

- Basic idea behind the computer-based is iterative technique
- Let $H(e^{j\omega})$ denote the frequency response of the digital filter $H(z)$ to be designed approximating the desired frequency response $D(e^{j\omega})$, given as a piecewise linear function of ω, in some sense

5. Computer-Aided Design of IIR Digital Filters

Objective -- Determine iteratively the coefficients of $H(z)$ so that the difference between $D(e^{j\omega})$ and $H(e^{j\omega})$ over closed subintervals of $0 \le \omega \le \pi$ is minimized

- This difference usually specified as a weighted error function

$$E(\omega) = W(e^{j\omega})\left[H(e^{j\omega}) - D(e^{j\omega})\right]$$

where $W(e^{j\omega})$ is some user-specified weighting function

5. Computer-Aided Design of IIR Digital Filters

Chebyshev or minimax criterion

- Minimizes the peak absolute value of the weighted error:

$$\varepsilon = \max_{\omega \in R} |E(\omega)|$$

where R is the set of disjoint frequency bands in the range $0 \le \omega \le \pi$, on which $D(e^{j\omega})$ is defined

- For example, for a lowpass filter design, R is the disjoint union of $(0, \omega_p)$ and (ω_s, π)

5. Computer-Aided Design of IIR Digital Filters

Least-p Criterion

- Minimizes

$$\varepsilon = \int_{\omega \in R} \left| W(e^{j\omega})\left[H(e^{j\omega}) - D(e^{j\omega})\right]\right|^{P} d\omega$$

over the specified frequency range R with p a positive integer

- $p=2$ yields the least-squares criterion
- As $p \to \infty$, the least p-th solution approaches the minimax solution

5. Computer-Aided Design of IIR Digital Filters

- In practice, the p-th power error measure is approximated as

$$\varepsilon = \sum_{i=1}^{K} \left\{ W(e^{j\omega_i})\left[H(e^{j\omega_i}) - D(e^{j\omega_i})\right]\right\}^{P}$$

where ω_i , $1 \le i \le K$, is a suitably chosen dense grid of digital angular frequencies

- For linear-phase FIR filter design, $H(e^{j\omega})$ and $D(e^{j\omega})$ are zero-phase frequency responses
- For IIR filter design, $H(e^{j\omega})$ and $D(e^{j\omega})$ are magnitude functions