Chapter 8

Digital Filter Structures

Digital Filter Structures

- **Block Diagram Representation**
- **Equivalent Structures**
- **Basic FIR Digital Filter Structures**
- **Basic IIR Digital Filter Structures**

1. Block Diagram Representation

- Input-output relation of an LTI system can be realized using different computational algorithms
- Basic realization forms of FIR and IIR digital filters are considered
- Mitra's book covers also various more sophisticated realizations of digital filters, e.g. lattice structures, allpass sections, and state space structures, not discussed in this course

1. Block Diagram Representation

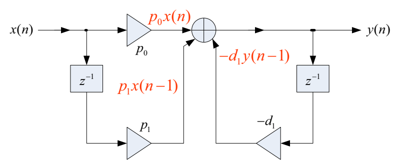
- The convolution sum description of an LTI discrete-time system can, in principle, be used to implement the system.
- For an IIR finite-dimensional system, this approach is not practical as here the impulse response is of infinite length.
- However, a direct implementation of the IIR finite-dimensional system is practical

1. Block Diagram Representation

- In the time domain, the input-output relations of an LTI digital filter is given by the convolution sum or, by the linear constant coefficient difference equation
- For the implementation of an LTI digital filter, the input-output relationship must be described by a valid computational algorithm.

1. Block Diagram Representation

- To illustrate what we mean by a computational algorithm, consider the causal first-order LTI digital filter shown below



$$y(n) = -d_1 y(n-1) + p_0 x(n) + p_1 x(n-1)$$

1. Block Diagram Representation

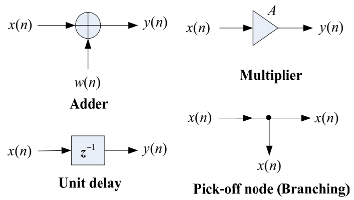
- Using the above equation we can compute $y(n)$ for $n \geqslant 0$ knowing the initial condition $y(-1)$ and the input $x(n)$ for $n \geqslant -1$

$$y(0) = -d_1 y(-1) + p_0 x(0) + p_1 x(-1)$$
$$y(1) = -d_1 y(0) + p_0 x(1) + p_1 x(0)$$
$$y(2) = -d_1 y(1) + p_0 x(2) + p_1 x(1)$$

- We can continue this calculation for any value of n we desire (by iterative computation)

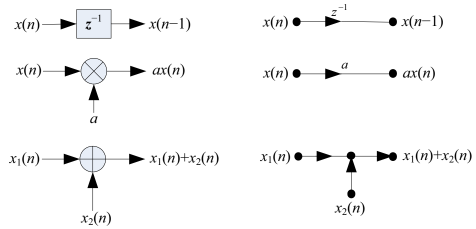
1.1 Basic Building Blocks

- The computational algorithm of an LTI digital filter can be conveniently represented in block diagram form using the basic building blocks shown below



1.1 Basic Building Blocks

- The corresponding signal flow charts are shown on the right-hand side



1.1 Basic Building Blocks

Advantages of block diagram/signal flow chart representation

① Easy to write down the computational algorithm by inspection.

② Easy to analyze the block diagram to determine the explicit relation between the output and input.

1.1 Basic Building Blocks

③ Easy to manipulate a block diagram to derive other "equivalent" block diagrams yielding different computational algorithms.

④ Easy to determine the hardware requirements.

⑤ Easier to develop block diagram representations from the transfer function directly.

1.2 Analysis of Block Diagrams

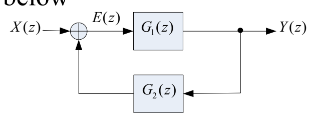
Steps of Analyzing Block Diagrams

- Carried out by writing down the expressions for the output signals of each adder as a sum of its input signals, and developing a set of equations relating the filter input and output signals in terms of all internal signals
- Eliminating the unwanted internal variables then results in the expression for the output signal as a function of the input signal and the filter parameters that are the multiplier coefficients

1.2 Analysis of Block Diagrams

Example

- Consider the single-loop feedback structure shown below



The output $E(z)$ of the adder is

$$E(z) = X(z) + G_2(z)Y(z)$$

But from the figure $Y(z) = G_1(z)E(z)$

1.2 Analysis of Block Diagrams

- Eliminating $E(z)$ from the previous two equations we arrive at

$$[1 - G_1(z)G_2(z)]Y(z) = G_1(z)X(z)$$

which leads to

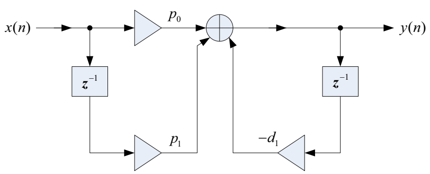
$$H(z) = \frac{Y(z)}{X(z)} = \frac{G_1(z)}{1 - G_1(z)G_2(z)}$$

1.3 Canonic and Noncanonic Structures

- A digital filter structure is said to be canonic if the number of delays in the block diagram representation is equal to the order of the transfer function
- Otherwise, it is a noncanonic structure
- The structure shown in the next slide is noncanonic as it employs two delays to realize a first-order difference equation

1.3 Canonic and Noncanonic Structures

$$y(n) = -d_1 y(n-1) + p_0 x(n) + p_1 x(n-1)$$



2. Equivalent Structures

- Two digital filter structures are defined to be equivalent if they have the same transfer function
- We describe next a number of methods for the generation of equivalent structures
- However, a fairly simple way to generate an equivalent structure from a given realization is via the transpose operation

2. Equivalent Structures

Transpose Operation

(1) Reverse all paths

(2) Replace pick-off nodes by adders, and vice versa

(3) Interchange the input and output nodes

- All other methods for developing equivalent structures are based on a specific algorithm for each structure

2. Equivalent Structures

- There are literally an infinite number of equivalent structures realizing the same transfer function
- It is thus impossible to develop all equivalent realizations
- In this course we restrict our attention to a discussion of some commonly used structures

2. Equivalent Structures

- Under infinite precision arithmetic any given realization of a digital filter behaves identically to any other equivalent structure
- However, in practice, due to the finite wordlength limitations, a specific realization behaves totally differently from its other equivalent realizations

2. Equivalent Structures

- Hence, it is important to choose a structure that has the least quantization effects when implemented using finite precision arithmetic
- One way to arrive at such a structure is to determine a large number of equivalent structures, analyze the finite wordlength effects in each case, and select the one showing the least effects

2. Equivalent Structures

- In certain cases, it is possible to develop a structure that by construction has the least quantization effects
- We defer the review of these structures after a discussion of the analysis of quantization effects (not included in Kuo's revised book)
- Here, we review some simple realizations that in many applications are quite adequate

3. FIR Digital Filter Structures

- **Direct Form**
- **Cascade Form**
- **Linear-phase Structure**

3. FIR Digital Filter Structures

- A causal FIR filter of order $N-1$ is characterized by a transfer function $H(z)$ given by

$$H(z) = \sum_{k=0}^{N-1} h(k) z^{-k}$$

which is a polynomial in z^{-1}

- In the time-domain the input-output relation of the above FIR filter is given by

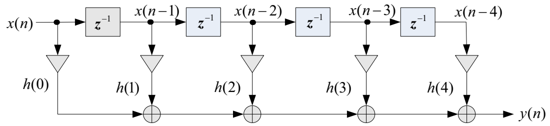
$$y(n) = \sum_{k=0}^{N-1} h(k) x(n-k)$$

3.1 Direct Form FIR Digital Filter Structures

- An FIR filter of order $N-1$ is characterized by N coefficients and, in general, require N multipliers and $N-1$ two-input adders
- Structures in which the multiplier coefficients are precisely the coefficients of the transfer function are called direct form structures

3.1 Direct Form FIR Digital Filter Structures

- A direct form realization of an FIR filter can be readily developed from the convolution sum description as indicated below for $N=5$



3.1 Direct Form FIR Digital Filter Structures

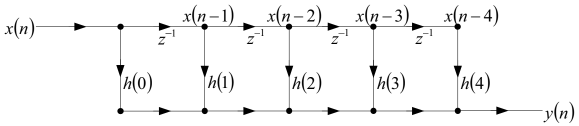
- An analysis of this structure yields

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3) + h(4)x(n-4)$$

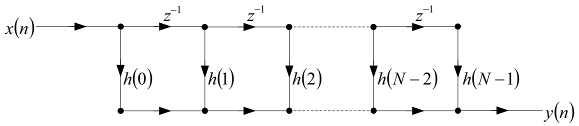
which is precisely of the form of the convolution sum description

- The direct form structure shown on the previous slide is also known as a tapped delay line or a transversal filter.

3.1 Direct Form FIR Digital Filter Structures



General Form



3.2 Cascade Form FIR Digital Filter Structures

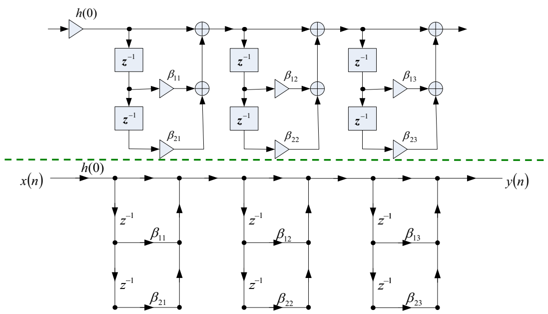
- A higher-order FIR transfer function can also be realized as a cascade of second order FIR sections and possibly a first-order section
- To this end we express $H(z)$ as

$$H(z) = h(0)\prod_{k=1}^{K}\left(1+\beta_{1k}z^{-1}+\beta_{2k}z^{-2}\right)$$

where $k = N/2$ if N is even, and $k = (N+1)/2$ if N is odd, with $\beta_{2k} = 0$

3.2 Cascade Form FIR Digital Filter Structures

- A cascade realization for $N = 6$ is shown below



3.3 Linear-Phase FIR Digital Filter Structures

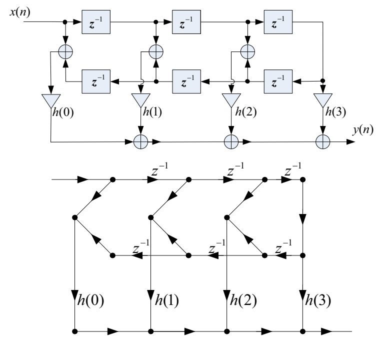
- Linear-phase FIR filter of length N is characterized by the symmetric impulse response
$$h(n) = h(N-1-n)$$
- An antisymmetric impulse response condition
$$h(n) = -h(N-1-n)$$
results in a constant group delay and "linear-phase" property
- Symmetry of the impulse response coefficients can be used to reduce the number of multiplications

3.3 Linear-Phase FIR Digital Filter Structures

- Length N is odd (N=7)

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$
$$+h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$
$$= h(0)(1+z^{-6}) + h(1)(z^{-1}+z^{-5})$$
$$+h(2)(z^{-2}+z^{-4}) + h(3)z^{-3}$$

3.3 Linear-Phase FIR Digital Filter Structures

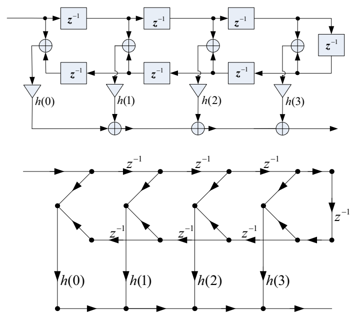


3.3 Linear-Phase FIR Digital Filter Structures

- Length N is even (N=8)

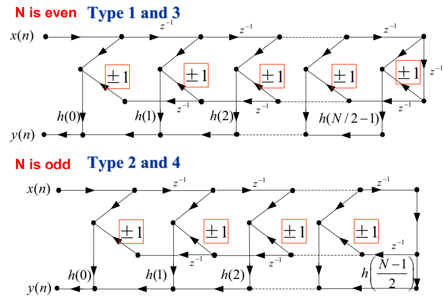
$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$
$$+h(3)z^{-4} + h(2)z^{-5} + h(1)z^{-6} + h(0)z^{-7}$$
$$= h(0)(1+z^{-7}) + h(1)(z^{-1}+z^{-6})$$
$$+h(2)(z^{-2}+z^{-5}) + h(3)(z^{-3}+z^{-4})$$

3.3 Linear-Phase FIR Digital Filter Structures



3.3 Linear-Phase FIR Digital Filter Structures

General Form



***N*/2 multipliers**

Direct Form needs *N* multipliers

(*N*+1) /2 multipliers

4. IIR Digital Filter Structures

- **Direct Form**
- **Cascade Form**
- **Parallel Form**

4.1 Direct Form IIR Digital Filter Structures

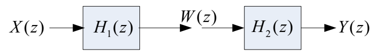
- The causal IIR digital filters we are concerned with in this course are characterized by a real rational transfer function of z^{-1} or, equivalently by a constant coefficient difference equation.
- From the difference equation representation, it can be seen that the realization of the causal IIR digital filters requires some form of feedback.

4.1 Direct Form IIR Digital Filter Structures

- Direct forms -- Coefficients are directly the transfer function coefficients
- Consider for simplicity a 3rd-order IIR filter with a transfer function (assuming $d_0 = 1$)

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

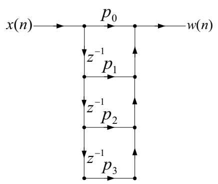
- We can implement $H(z)$ as a cascade of two filter sections as shown below



4.1 Direct Form IIR Digital Filter Structures

- where $H_1(z) = P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}$

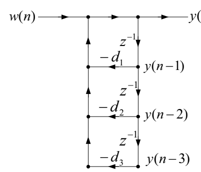
  $H_2(z) = 1/D(z)$
- The filter section $H_1(z)$ can be seen to be an FIR filter and can be realized as shown below



4.1 Direct Form IIR Digital Filter Structures

- The time-domain representation of $H_2(z)$ is given by

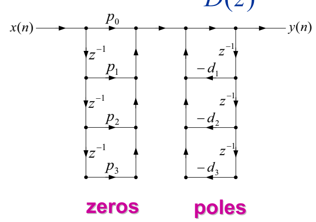
  $y(n) = w(n) - d_1 y(n-1) - d_2 y(n-2) - d_3 y(n-3)$
- Realization of $H_2(z)$ follows from the above equation and is shown below



4.1 Direct Form IIR Digital Filter Structures

- Considering the basic cascade realization results in *Direct form* I :

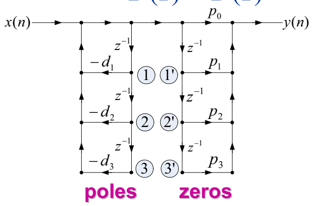
$$H(z) = P(z) \cdot \frac{1}{D(z)}$$



4.1 Direct Form IIR Digital Filter Structures

- Changing the order of blocks in cascade results in *Direct form* II :

$$H(z) = P(z) \cdot \frac{1}{D(z)} = \frac{1}{D(z)} \cdot P(z)$$

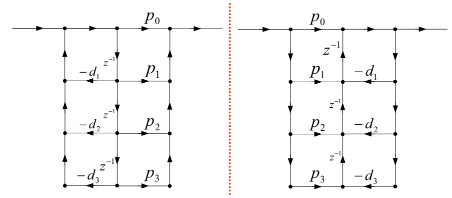


4.1 Direct Form IIR Digital Filter Structures

- Observe in the direct form structure shown below, the signal variable at nodes ① and ①' are the same, and hence the two top delays can be shared
- Likewise, the signal variables at nodes ② and ②' are the same, permitting the sharing of the middle two delays
- Following the same argument, the bottom two delays can be shared

4.1 Direct Form IIR Digital Filter Structures

- Sharing of all delays reduces the total number of delays to 3 resulting in a canonic realization shown below along with its transpose structure.



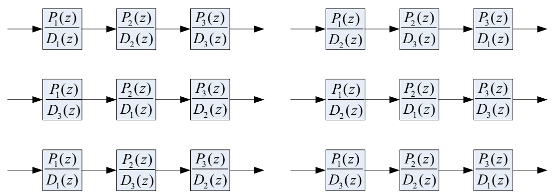
4.2 Cascade Realizations

- By expressing the numerator and the denominator polynomials of the transfer function as a product of polynomials of lower degree, a digital filter can be realized as a cascade of low-order filter sections (often sos)
- Consider, for example, $H(z)=P(z)/D(z)$ expressed as

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)P_2(z)P_3(z)}{D_1(z)D_2(z)D_3(z)}$$

4.2 Cascade Realizations

- Examples of cascade realizations obtained by different pole-zero pairings are shown below



4.2 Cascade Realizations

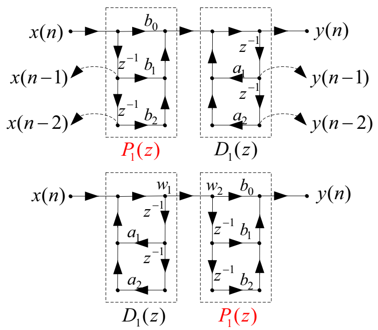
- There are altogether a total of 36 $(P_3^2 \cdot P_3^2)$ different cascade realizations of

$$H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)P_2(z)P_3(z)}{D_1(z)D_2(z)D_3(z)}$$

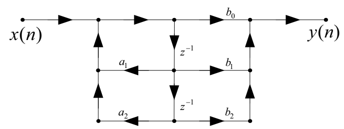
based on pole-zero-pairings and ordering

- Due to finite wordlength effects, each such cascade realization behaves differently from Others

4.2 Cascade Realizations



4.2 Cascade Realizations



- Usually, the polynomials are factored into a product of 1st-order and 2nd-order (sos) polynomials:

$$H(z) = p_0 \prod_k \left(\frac{1+\beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1+\alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

for a first-order factor $\alpha_{2k} = \beta_{2k} = 0$

4.2 Cascade Realizations

- Realizing complex conjugate poles and zeros with second order blocks results in real coefficients

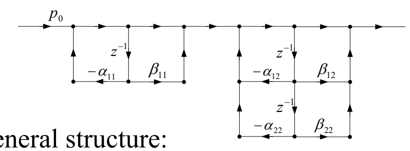
Example

- Third order transfer function

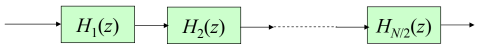
$$H(z) = \frac{P(z)}{D(z)} = p_0 \left(\frac{1+\beta_{11} z^{-1}}{1+\alpha_{11} z^{-1}} \right) \left(\frac{1+\beta_{12} z^{-1} + \beta_{22} z^{-2}}{1+\alpha_{12} z^{-1} + \alpha_{22} z^{-2}} \right)$$

4.2 Cascade Realizations

- One possible realization is shown below



- General structure:



4.3 Parallel Realizations

- Parallel realizations are obtained by making use of the partial fraction expansion of the transfer function

Parallel form I:

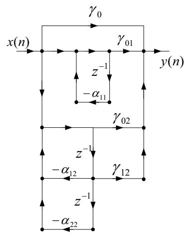
$$H(z) = \gamma_0 + \sum_k \left(\frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1+\alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

Parallel form II:

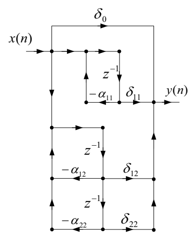
$$H(z) = \delta_0 + \sum_k \left(\frac{\delta_{1k} z^{-1} + \delta_{2k} z^{-2}}{1+\alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \right)$$

4.3 Parallel Realizations

- The two basic parallel realizations of a 3rd order IIR transfer function are shown below



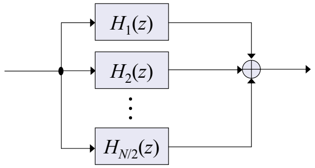
Parallel Form I



Parallel Form II

4.3 Parallel Realizations

- General structure:



- Easy to realize:
  No choices in section ordering and
  No choices in pole and zero pairing

4.3 Parallel Realizations

Example

- A partial-fraction expansion of

$$H(z)=\frac{0.44+0.362z^{-2}+0.002z^{-3}}{1+0.4z^{-1}+0.18z^{-2}-0.2z^{-3}}$$

in z^{-1} yields

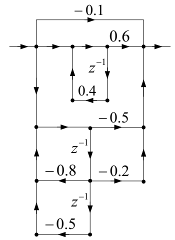
$$H(z)=-0.1+\frac{0.6}{1-0.4z^{-1}}+\frac{-0.5-0.2z^{-1}}{1+0.8z^{-1}+0.5z^{-2}}$$

- Likewise, a partial-fraction expansion of $H(z)$ in z yields

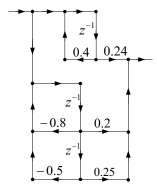
$$H(z)=\frac{0.24z^{-1}}{1-0.4z^{-1}}+\frac{0.2z^{-1}+0.25z^{-2}}{1+0.8z^{-1}+0.5z^{-2}}$$

4.3 Parallel Realizations

- Their realizations are shown below



Parallel Form I



Parallel Form II