Chapter 7B

LTI Discrete-Time Systems in the Transform-Domain

Part B

Simple Digital Filters

Simple Digital Filters

- Simple FIR Digital Filters
- Simple IIR Digital Filters
- Comb Filters

1. Simple FIR Digital Filters

- Later in the course we shall review various methods of designing frequency-selective filters satisfying prescribed specifications
- We now describe several low-order FIR and IIR digital filters with reasonable selective frequency responses that often are satisfactory in a number of applications

1. Simple FIR Digital Filters

- FIR digital filters considered here have integer-valued impulse response coefficients (quantified)
- These filters are employed in a number of practical applications, primarily because of their simplicity, which makes them amenable to inexpensive hardware implementations

1.1 Lowpass FIR Digital Filters

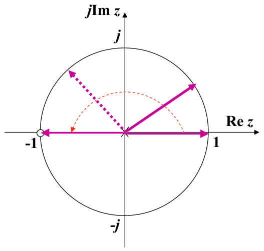
- The simplest lowpass FIR digital filter is the 2-point moving-average filter given by

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z}$$

- The above transfer function has a zero at $z=-1$ and a pole at $z = 0$
- Note that here the pole vector has a unity magnitude for all values of ω , thus

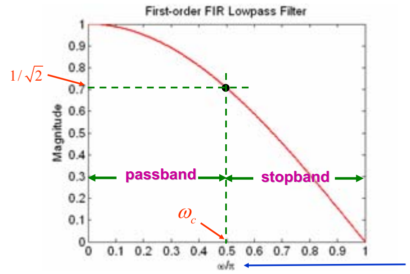
$$|H_0(e^{j\omega})| = 0.5|e^{j\omega} + 1|$$

1.1 Lowpass FIR Digital Filters



- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2, the diameter of the unit circle, to 0
- We can work out the frequency response $H_0(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2)$

1.1 Lowpass FIR Digital Filters

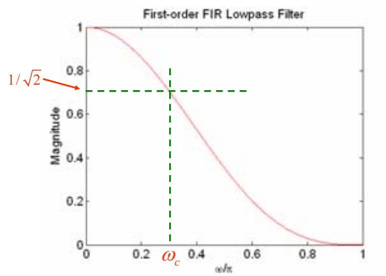


normalized digital angular frequency

ω_c $20\log(1/\sqrt{2}) = -3dB$ 3-dB cutoff frequency

1.1 Lowpass FIR Digital Filters

A cascade of 3 sections—an improved scheme

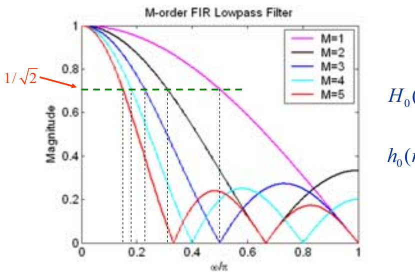


Notice:

The cascade of first-order sections yields a sharper magnitude response but at the expense of a decrease in the width of the passband

1.1 Lowpass FIR Digital Filters

M-order FIR Lowpass (M-order moving-average) Filter



$$H_0(z) = \frac{1}{M+1}\sum_{m=0}^{M} z^{-m}$$

$$h_0(n) = \frac{1}{M+1} R_{M+1}(n)$$

1.2 Highpass FIR Digital Filters

- The simplest highpass FIR filter is obtained from the simplest lowpass FIR filter by replacing z with $-z$
- This results in $H_1(z) = \frac{1}{2}(1 - z^{-1})$
- Corresponding frequency response is given by

$$H_1(e^{j\omega}) = je^{-j\omega/2}\sin(\omega/2)$$

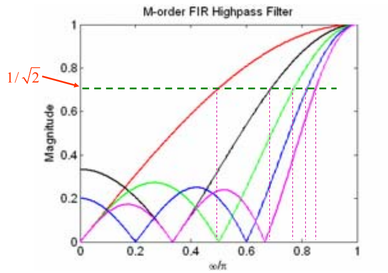
1.2 Highpass FIR Digital Filters

- Improved highpass magnitude response can again be obtained by **cascading** several sections of the first-order highpass filter
- Alternately, a **higher-order** highpass filter of the form

$$H_1(z) = \frac{1}{M+1}\sum_{n=0}^{M} (-1)^n z^{-n}$$

is obtained by replacing z with $-z$ in the transfer function of a moving average filter

1.2 Highpass FIR Digital Filters



2. Simple IIR Digital Filters

- **Lowpass IIR Digital Filters**
- **Highpass IIR Digital Filters**
- **Bandpass IIR Digital Filters**
- **Bandstop IIR Digital Filters**
- **Higher-order IIR Digital Filters**

2.1 Lowpass IIR Digital Filters

- A first-order causal lowpass IIR digital filter has a transfer function given by

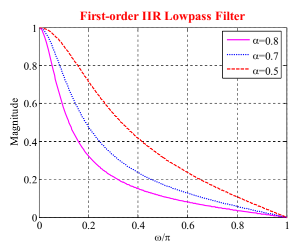
$$H_{LP}(z) = \frac{1-\alpha}{2}\,\frac{1+z^{-1}}{1-\alpha z^{-1}}$$

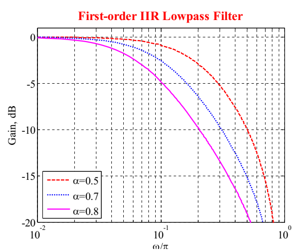
  where $|\alpha| < 1$ for stability
- The above transfer function has a zero at $z=-1$ i.e., at $\omega = \pi$ which is in the stopband
- $H_{LP}(z)$ has a real pole at $z = \alpha$

2.1 Lowpass IIR Digital Filters

- As ω increases from 0 to π , the magnitude of the zero vector decreases from a value of 2 to 0, whereas, for a positive value of α , the magnitude of the pole vector increases from value of $1-\alpha$ to $1+\alpha$
- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$

2.1 Lowpass IIR Digital Filters





2.2 Highpass IIR Digital Filters

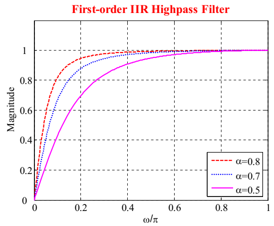
- A first-order causal highpass IIR digital filter has a transfer function given by

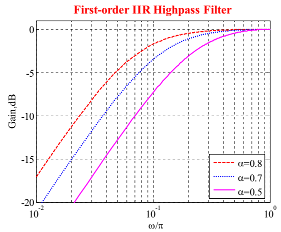
$$H_{HP}(z) = \frac{1+\alpha}{2}\,\frac{1-z^{-1}}{1-\alpha z^{-1}}$$

  where $|\alpha| < 1$ for stability
- The above transfer function has a zero at $z=1$ i.e., at $\omega = 0$ which is in the stopband

2.2 Highpass IIR Digital Filters

- Magnitude and gain responses of $H_{LP}(z)$ are shown below





2.3 Bandpass IIR Digital Filters

- A 2nd-order bandpass digital transfer function is given by

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

- Its squared magnitude function is

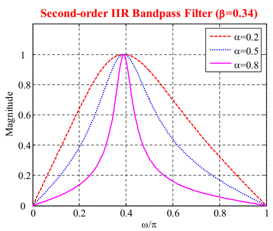
$$\left|H_{BP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2 (1-\cos 2\omega)}{2[1+\beta^2(1+\alpha)^2 + \alpha^2 - 2\beta(1+\alpha)^2 \cos\omega + 2\alpha\cos 2\omega]}$$

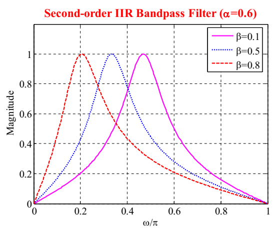
2.3 Bandpass IIR Digital Filters

- $\left|H_{BP}(e^{j\omega})\right|^2$ goes to zero at $\omega = 0$ and $\omega = \pi$
- It assumes a maximum value of 1 at $\omega = \omega_0$, called the center frequency of the bandpass filter, where
- The frequencies ω_{c1} and ω_{c2} where the squared magnitude becomes 1/2 are called the 3-dB cutoff frequencies
- The difference between the two cutoff frequencies, is called the 3-dB bandwidth

2.3 Bandpass IIR Digital Filters

- The transfer function is a BR function if $|\alpha| < 1$ and $|\beta| < 1$





2.4 Bandstop IIR Digital Filters

- A 2nd-order bandstop digital filter has a transfer function given by

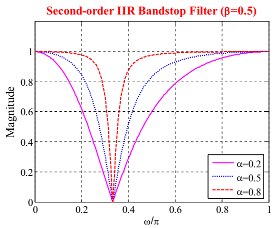
$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

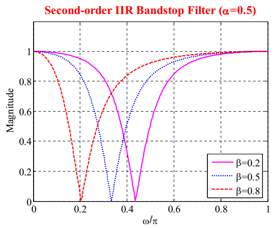
- The transfer function is a BR function if $|\alpha| < 1$ and $|\beta| < 1$
- Its magnitude response is plotted in the next slide

2.4 Bandstop IIR Digital Filters

- Here, the magnitude function takes the maximum value of 1 at $\omega = 0$ and $\omega = \pi$
- It goes to 0 at $\omega = \omega_0$, where ω_0, called the notch frequency, is given by $\omega_0 = \cos^{-1}\beta$
- The digital transfer function is more commonly called a notch filter
- The difference between the two cutoff frequencies is called the 3-dB notch bandwidth

2.4 Bandstop IIR Digital Filters





2.5 Higher-Order IIR Digital Filters

- By cascading the simple digital filters discussed so far, we can implement digital filters with sharper magnitude responses
- Consider a cascade of K first-order lowpass sections characterized by the transfer function

$$G_{LP}(z) = \left(\frac{1-\alpha}{2} \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)^K$$

2.5 Higher-Order IIR Digital Filters

- The corresponding squared-magnitude function is given by

$$\left| G_{LP}(e^{j\omega}) \right|^2 = \left[\frac{(1-\alpha)^2 (1+\cos\omega)}{2(1+\alpha^2 - 2\alpha\cos\omega)} \right]^K$$

- To determine the relation between its 3-dB cutoff frequency ω_c and the parameter α ,we set

$$\left[\frac{(1-\alpha)^2 (1+\cos\omega_c)}{2(1+\alpha^2 - 2\alpha\cos\omega_c)} \right]^K = \frac{1}{2}$$

2.5 Higher-Order IIR Digital Filters

which when solved for α, yields for a stable $G_{LP}(z)$:

$$\alpha = \frac{1 + (1-C)\cos\omega_c - \sin\omega_c \sqrt{2C - C^2}}{1 - C + \cos\omega_c}$$

where $C = 2^{(K-1)/K}$

- It should be noted that the expression for given earlier reduces to

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c} \qquad \text{for } K=1$$

3. Comb Filters

- The simple filters discussed so far are characterized either by a single passband and/or a single stopband
- There are applications where filters with multiple passbands and stopbands are required
- The comb filter is an example of such filters

3. Comb Filters

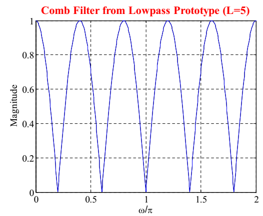
- In its most general form, a comb filter has a frequency response that is a periodic function of ω with a period $2\pi / L$, where L is a positive integer
- If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $G(z)=H(z^L)$

3. Comb Filters

- If $|H(e^{j\omega})|$ exhibits a peak at ω_p , then $|G(e^{j\omega})|$ will exhibit L peaks at $\omega_p k / L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega \le 2\pi$
- Likewise, if $|H(e^{j\omega})|$ has a notch at ω_0 ,then $|H(e^{j\omega})|$ will have L notches at $\omega_0 k / L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega \le 2\pi$
- A comb filter can be generated from either an FIR or an IIR prototype filter

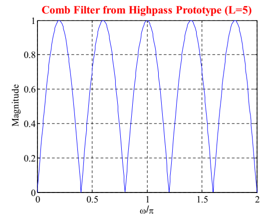
3. Comb Filters

- For example, the comb filter generated from the prototype lowpass FIR filter $H_0(z) = (1/2)(1+z^{-1})$ has a transfer function $G_0(z) = (1/2)(1+z^{-L})$
- $|G_0(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi / L$ and L peaks at $\omega = 2k\pi / L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega \le 2\pi$



3. Comb Filters

- For example, the comb filter generated from the prototype highpass FIR filter $H_1(z) = (1/2)(1-z^{-1})$ has a transfer function $G_1(z) = (1/2)(1-z^{-L})$
- $|G_1(e^{j\omega})|$ has L peaks at $\omega = (2k+1)\pi / L$ and L notches at $\omega = 2k\pi / L$ $0 \le k \le L-1$, in the frequency range $0 \le \omega \le 2\pi$



3. Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter
- For example, the M-point moving average filter

$$H(z) = \frac{1-z^{-M}}{M(1-z^{-1})}$$

has been used as a prototype

3. Comb Filters

- This filter has a peak magnitude at $\omega = 0$, and $M-1$ notches at $\omega = 2\pi l / M$, $1 \le l \le M-1$
- The corresponding comb filter has a transfer function

$$G(z) = \frac{1-z^{-LM}}{M(1-z^{-L})}$$

whose magnitude has L peaks at $\omega = 2k\pi / L$, $0 \le k \le L-1$ and $L(M-1)$ notches at $\omega = 2k\pi / LM$ $1 \le k \le L(M-1)$

3. Comb Filters