Chapter 7A

LTI Discrete-Time Systems
in the Transform Domain

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Types of Transfer Functions

- The time-domain classification of an LTI digital transfer function sequence is based on the length of its impulse response:

  -- Finite impulse response (FIR) transfer function

  -- Infinite impulse response (IIR) transfer function

Types of Transfer Functions

- In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications

  (1) Classification based on the shape of the magnitude function $|H(e^{j\omega})|$

  (2) Classification based on the form of the phase function $\theta(\omega)$

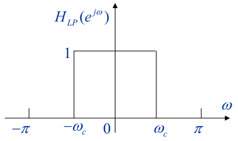
1.1 Ideal Filters

- Based on the shape of the magnitude function， four types of ideal filters are usually defined：*lowpass* , *highpass,* *bandpass* and *bandstop*
- A digital filter designed to pass signal components of certain frequencies without distortion should have a frequency response equal to one at these frequencies, and should have a frequency response equal to zero at all other frequencies

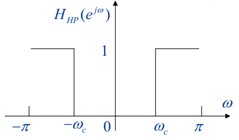
1.1 Ideal Filters

- The range of frequencies where the frequency response takes the value of one is called the passband
- The range of frequencies where the frequency response takes the value of zero is called the stopband
- Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown in the next slide

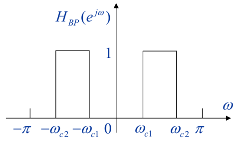
1.1 Ideal Filters



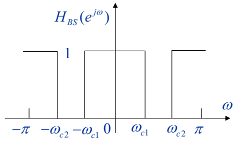
Lowpass



Highpass



Bandpass



Bandstop

1.1 Ideal Filters

- Earlier in the course we derived the inverse DTFT of the frequency response of the ideal lowpass filter:

$$h_{LP}(n) = \frac{\sin \omega_c n}{n\pi}, \quad -\infty \le n \le \infty$$

- We have also shown that the above impulse response is *not absolutely summable*, and hence, the corresponding transfer function is *not BIBO stable*

1.1 Ideal Filters

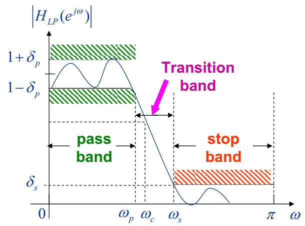
- Also, is *not causal* and is of doubly infinite length
- The remaining three ideal filters are also characterized by *doubly infinite, noncausal* impulse responses and are *not absolutely summable*
- Thus, the ideal filters with the ideal “brick wall” frequency responses cannot be realized with finite dimensional LTI filters

1.1 Ideal Filters

- To develop stable and realizable transfer functions, the ideal frequency response specifications are relaxed by including a *transition band* between the passband and the stopband.
- This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband.

1.1 Ideal Filters

- Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband.
- Typical magnitude response specifications of a lowpass filter are shown in the figure.



1.2 Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ is defined as a bounded real (BR) transfer function if
$$\left|H(e^{j\omega})\right| \le 1 \quad \text{for all values of } \omega$$
- Let $x(n)$ and $y(n)$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs

1.2 Bounded Real Transfer Functions

- Then the condition $\left|H(e^{j\omega})\right| \le 1$ implies that
$$\left|Y(e^{j\omega})\right|^2 \le \left|X(e^{j\omega})\right|^2$$
- Integrating the above from $-\pi$ to π, and applying Parseval's relation we get
$$\sum_{n=-\infty}^{\infty} \left|y(n)\right|^2 \le \sum_{n=-\infty}^{\infty} \left|x(n)\right|^2$$

1.2 Bounded Real Transfer Functions

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a passive structure
- If $\left|H(e^{j\omega})\right| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a lossless system

1.2 Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ with $\left|H(e^{j\omega})\right| = 1$ is thus called a lossless bounded real (LBR) transfer function
- The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity

1.3 Allpass Transfer Function

Definition

- An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,
$$\left|A(e^{j\omega})\right|^2 = 1, \quad \text{for all } \omega$$
is called an allpass transfer function
- An M-th order causal real-coefficient allpass transfer function is of the form
$$A_M(z) = \pm\frac{\boxed{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}} \rightarrow z^{-M}D_M(z^{-1})}{\boxed{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}} \rightarrow D_M(z)}$$

1.3 Allpass Transfer Function

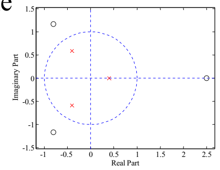
- Hence, $A_M(z)$ can be written as
$$A_M(z) = \pm\frac{z^{-M}D_M(z^{-1})}{D_M(z)}$$
- Note from the above that if $z=z_0$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z=1/z_0$
- The numerator of a real-coefficient allpass transfer function is said to be the mirror-image polynomial of the denominator, and vice versa

1.3 Allpass Transfer Function

- The expression

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real coefficient allpass function exhibit mirror-image symmetry in the z-plane



An example

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

1.3 Allpass Transfer Function

- To show that $\left|A(e^{j\omega})\right|^2 = 1$, we observe that

$$A_M(z^{-1}) = \pm \frac{z^{M} D_M(z)}{D_M(z^{-1})}$$

- Therefore,

$$A_M(z^{-1}) A_M(z) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^{M} D_M(z)}{D_M(z^{-1})} = 1$$

- Hence,

$$\left|A(e^{j\omega})\right|^2 = A_M(z^{-1}) A_M(z)\Big|_{z=e^{j\omega}} = 1$$

1.3 Allpass Transfer Function

- Now, the poles of a causal stable transfer function must lie inside the unit circle in the z-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle
- Figure in the next slide shows the principal value of the phase of the former example

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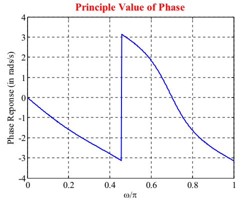
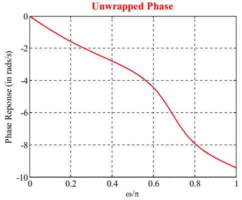
1.3 Allpass Transfer Function





- Note the discontinuity by the amount of 2π in the phase $\theta(\omega)$
- The unwrapped phase function is a continuous function of ω

1.3 Allpass Transfer Function

Properties

- A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure
- The magnitude function of a stable allpass function $A(z)$ satisfies:

$$\left|A(z)\right| \begin{cases} < 1 & for \ |z| > 1 \\ = 1 & for \ |z| = 1 \\ > 1 & for \ |z| < 1 \end{cases}$$

1.3 Allpass Transfer Function

- Let $\tau(\omega)$ denote the group delay function of an allpass filter $A(z)$, i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}\left[\theta_c(e^{j\omega})\right]$$

The unwrapped phase function of a stable allpass function is a monotonically decreasing function of ω so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$. An M-th order stable real-coefficient allpass transfer function satisfies:

$$\int_0^{\pi} \tau(\omega) d\omega = M\pi$$

1.3 Allpass Transfer Function

A Simple Application

- A simple but often used application of an allpass filter is as a delay equalizer
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

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1.3 Allpass Transfer Function

- Since $\left|A(e^{j\omega})\right|^2 = 1$, we have

$$\left|G(e^{j\omega})A(e^{j\omega})\right| = \left|G(e^{j\omega})\right|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $A(z)$

Part A Types of Transfer Functions

1. Based on Magnitude Characteristics
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2. Based on Phase Characteristics
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   - 2.2 Linear-Phase Transfer Functions
   - 2.3 Minimum-Phase and Maximum-Phase Transfer Functions

2.1 Zero-Phase Transfer Functions

- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband
- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase** characteristic

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2.1 Zero-Phase Transfer Functions

- However, it is not possible to design a causal digital filter with a zero phase (pp. 287-288)
- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement

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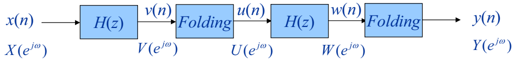
2.1 Zero-Phase Transfer Functions

- One zero-phase filtering scheme is sketched below



- From the figure, we can arrive at

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega})U^*(e^{j\omega}) = H^*(e^{j\omega})V(e^{j\omega})$$
$$= H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega}) = \underbrace{\left|H(e^{j\omega})\right|^2}_{\text{Real and Zero-Phase}} X(e^{j\omega})$$

2.2 Linear-Phase Transfer Functions

- Linear-Phase $H(e^{j\omega}) = e^{-j\omega D}$

$$\left|H(e^{j\omega})\right| = 1 \qquad \tau(\omega) = D$$

- The output $y(n)$ of this filter to an input $x(n) = Ae^{j\omega n}$ is then given by

$$y(n) = Ae^{-j\omega D}e^{j\omega n} = Ae^{j\omega(n-D)}$$

- If $x_a(t)$ and $y_a(t)$ represent the continuous time signals whose sampled versions, sampled at $t = nT$, are $x(n)$ and $y(n)$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount D

2.2 Linear-Phase Transfer Functions

- If D is an integer, then $y(n)$ is identical to $x(n)$, but delayed by D samples
- If D is not an integer, $y(n)$, being delayed by a fractional part, is not identical to $x(n)$
- In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed D units of time

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2.2 Linear-Phase Transfer Functions

- If it is desired to pass input signal components in a certain frequency range **undistorted in both magnitude and phase**, then the transfer function should exhibit a **unity magnitude** response and a **linear-phase** response in the band of interest
- Figure in the next slide shows the frequency response if a lowpass filter with a linear-phase characteristic in the passband

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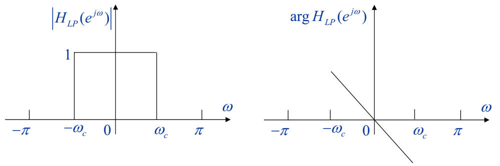
2.2 Linear-Phase Transfer Functions



- Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape

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2.2 Linear-Phase Transfer Functions

- It is nearly impossible to design a linear-phase IIR transfer function
- It is always possible to design an FIR transfer function with an exact linear-phase response
- Consider a causal FIR transfer function $H(z)$ of length N, i.e., of order $N-1$

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

2.2 Linear-Phase Transfer Functions

- The above transfer function has a linear-phase, if its impulse response $h(n)$ is either symmetric, i.e.,

$$h(n) = h(N-1-n), \quad 0 \le n \le N-1$$

or is antisymmetric, i.e.,

$$h(n) = -h(N-1-n), \quad 0 \le n \le N-1$$

- There are two types linear phase

$$\theta(\omega) = \theta_0 - \tau\omega = \begin{cases} -\tau\omega, & \theta_0 = 0 \\ -\dfrac{\pi}{2} - \tau\omega, & \theta_0 = -\dfrac{\pi}{2} \end{cases} \Rightarrow \frac{d\theta(\omega)}{d\omega} = -\tau$$

2.2 Linear-Phase Transfer Functions

Proof

$$H(e^{j\omega})=\sum_{n=0}^{N-1}h(n)e^{-j\omega n}=\sum_{n=0}^{N-1}h(n)\cos\omega n-j\cdot\sum_{n=0}^{N-1}h(n)\sin\omega n$$

- If $h(n)$ is a real sequence, we have

$$\theta(\omega)=\arg\tan\left\{-\frac{\sum_{n=0}^{N-1}h(n)\sin\omega n}{\sum_{n=0}^{N-1}h(n)\cos\omega n}\right\}$$

2.2 Linear-Phase Transfer Functions

- If this transfer function has a linear phase, such as $\theta(\omega)=-\tau\omega$
- We obtain the following relationship

$$\theta(\omega)=\arg\tan\left\{-\frac{\sum_{n=0}^{N-1}h(n)\sin\omega n}{\sum_{n=0}^{N-1}h(n)\cos\omega n}\right\}=-\tau\omega$$

- Taking tan(.) on both sides of the above equation

2.2 Linear-Phase Transfer Functions

$$\tan\tau\omega=\frac{\sin\tau\omega}{\cos\tau\omega}=\frac{\sum_{n=0}^{N-1}h(n)\sin\omega n}{\sum_{n=0}^{N-1}h(n)\cos\omega n}$$

i.e.,

$$\sum_{n=0}^{N-1}h(n)\cos\omega n\sin\tau\omega-\sum_{n=0}^{N-1}h(n)\sin\omega n\cos\tau\omega=0$$

or

$$\sum_{n=0}^{N-1}h(n)\sin\left[(\tau-n)\omega\right]=0 \qquad \bigstar$$

2.2 Linear-Phase Transfer Functions

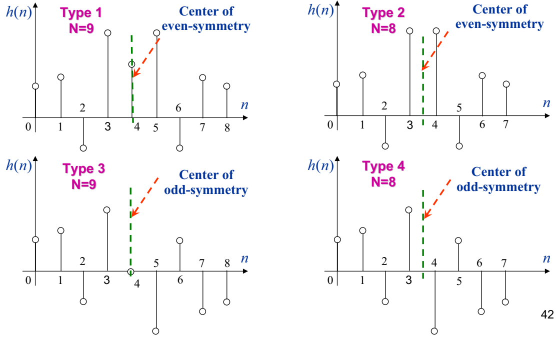
- $\sin\left[(\tau-n)\omega\right]$ is odd-symmetry on $\tau=n$
- Let $\tau=(N-1)/2$, thus equation $\bigstar$ holds if $h(n)$ is even-symmetry on $n=(N-1)/2$
- In other words, $h(n)=h(N-1-n),\quad 0\le n\le N-1$
- Similarly, if $\theta(\omega)=-\pi/2-\tau\omega$,we can arrive at

$$h(n)=-h(N-1-n),\quad 0\le n\le N-1$$

2.2 Linear-Phase Transfer Functions

- Since the length of the impulse response can be either even or odd, we can define four types of linear-phase FIR transfer functions
- For an antisymmetric FIR filter of odd length, i.e., N odd : $h\{(N-1)/2)\}=0$
- We examine next the each of the 4 cases

2.2 Linear-Phase Transfer Functions



2.2 Linear-Phase Transfer Functions

- Consider first an FIR filter with a symmetric impulse response: $h(n) = h(N-1-n)$
- Its transfer function can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{N-1} h(N-1-n) z^{-n}$$

- By making a change of variable $m=N-1-n$,we can write

$$H(z) = \sum_{m=0}^{N-1} h(m) z^{-(N-1-m)} = z^{-(N-1)} \sum_{m=0}^{N-1} h(m) z^{m}$$
$$= z^{-(N-1)} H(z^{-1})$$

2.2 Linear-Phase Transfer Functions

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a mirror-image polynomial (MIP)
- In the case of anti-symmetric impulse response, the corresponding expression is

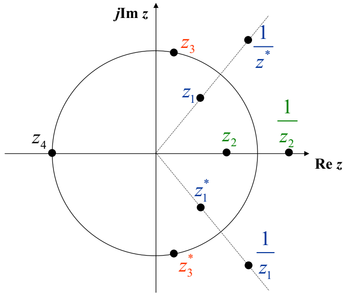
$$H(z) = -z^{-(N-1)} H(z^{-1})$$

which is called an antimirror-image polynomial (AIP)

2.2 Linear-Phase Transfer Functions

- It follows the relation $H(z) = \pm z^{-(N-1)} H(z^{-1})$ that if $z=z_i$ is a zero of $H(z)$, so is $z=1/z_i$
- Moreover, for an FIR filter with a real impulse response, the zeros of $H(z)$ occur in complex conjugate pairs
- Hence, a zero at $z=z_i$ is associated with a zero at $z=z_i^*$

2.2 Linear-Phase Transfer Functions



2.2 Linear-Phase Transfer Functions

- Since a zero at $z=\pm 1$ is its own reciprocal, it can appear only singly
- Now a Type 2 FIR filter satisfies

$$H(z) = z^{-(N-1)} H(z^{-1})$$

with degree $N-1$ odd

- Hence, $H(-1) = (-1)^{-(N-1)} H(-1) = -H(-1)$ implying $H(-1)=0$,i.e., $H(z)$ must have a zero at $z=-1$

2.2 Linear-Phase Transfer Functions

- Likewise, for a Type 3 or 4 filter,

$$H(1) = -H(1)$$

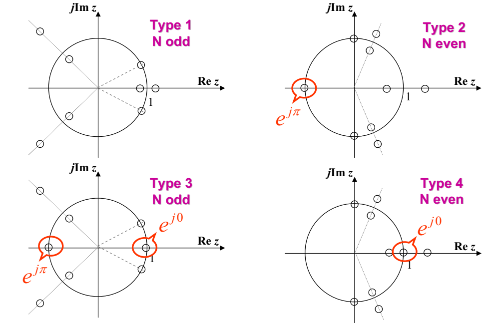
implying $H(z)$ must have a zero at $z=1$

- On the other hand, only the Type 3 FIR filter is restricted to have a zero at $z=-1$ since here the degree $N-1$ is even and hence,

$$H(-1) = -(-1)^{-(N-1)} H(-1) = -H(-1)$$

2.2 Linear-Phase Transfer Functions

Typical zero locations shown below



2.2 Linear-Phase Transfer Functions

Summary

- A **Type 2** FIR filter cannot be used to design a highpass filter since it always has a zero $z=-1$
- A **Type 3** FIR filter has zeros at both $z = 1$ and $z=-1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter
- A **Type 4** FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at $z = 1$
- **Type 1** FIR filter has no such restrictions and can be used to design almost any type of filter

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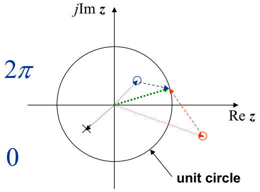
2.3 Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle

$$\underbrace{\Delta\arg}_{\omega}\left\{\begin{matrix}\text{zero or ploe vector}\\ \text{inside the unit circle}\end{matrix}\right\}\Bigg|_0^{2\pi}=2\pi$$

$$\underbrace{\Delta\arg}_{\omega}\left\{\begin{matrix}\text{zero or ploe vector}\\ \text{outside the unit circle}\end{matrix}\right\}\Bigg|_0^{2\pi}=0$$



2.3 Minimum-Phase and Maximum-Phase Transfer Functions

- It assumed that a causal stable transfer function has M zeros (with m_i inside the UC and m_0 outside the UC) and N poles (with n_i inside the UC and n_0 outside the UC)

$$\underbrace{\Delta\arg}_{\omega}\left[H(e^{j\omega})\right]\Big|_0^{2\pi}=2\pi m_i-2\pi n_i+2\pi(N-M)$$

$$=2\pi\left(n_0-m_0\right)$$

- Since $N= n_i$ and i.e., $n_0=0$, we have

$$\underbrace{\Delta\arg}_{\omega}\left[H(e^{j\omega})\right]\Big|_0^{2\pi}=-2\pi m_0$$

2.3 Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros inside the unit circle (m_o=0) is called a minimum-phase transfer function
- A causal stable transfer function with all zeros outside the unit circle (m_o=M) is called a maximum-phase transfer function
- A transfer function with zeros inside and outside the unit circle is called a mixed-phase transfer function

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2.3 Minimum-Phase and Maximum-Phase Transfer Functions

- Questions
  - An LTI system is said to be minimum-phase if the system and its inverse are causal and stable. (A(z)B(z)=1)
  - Is a causal stable allpass filter minimum or maximum phase?
  - What is case for a linear phase FIR filter?