# Chapter 6C

z-Transform

# Part C

The Transfer Function

## Part C: The Transfer Function

- Definition
- Transfer Function Expression
- Frequency Response from Transfer Function
- Geometric Interpretation of Frequency Response Computation
- Stability Condition in Terms of Poles Locations



## 1. Definition

- The concept of transfer function is a generalization of the frequency response $H(e^{j\omega})$
- The z-transform $H(z)$ of the impulse response $h(n)$ of the filter is called the *transfer function* or the *system function*
- The transfer function is derived through the method similar to that of the frequency response



## 1. Definition

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right)z^{-n} \\
&= \sum_{k=-\infty}^{\infty} h(k)\left(\sum_{n=-\infty}^{\infty} x(n-k)z^{-n}\right) \\
&\overset{l=n-k}{=} \sum_{k=-\infty}^{\infty} h(k)\left(\sum_{l=-\infty}^{\infty} x(l)z^{-(l+k)}\right) \\
&= \sum_{k=-\infty}^{\infty} h(k)\left(\sum_{l=-\infty}^{\infty} x(l)z^{-l}\right)z^{-k} \\
&= H(z)X(z)
\end{aligned}$$



## 1. Definition

**Example 1**

- Consider the $M$-point **moving average** FIR filter with an impulse response

$$h(n) = \begin{cases} 1/M, & 0 < n < M-1 \\ 0, & \text{otherwise} \end{cases}$$

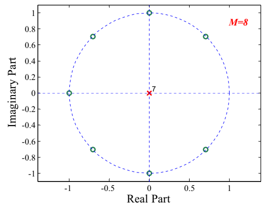
- Its transfer function is then given by

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^{M}-1}{M \cdot z^{M-1}(z-1)}$$

## 1. Definition

- The transfer function has $M$ zeros on the unit circle at $z = e^{j2\pi k/M}$, $K = 0, 1, 2, \ldots, M-1$
- There are poles at $z = 0$ and a single pole at $z = 1$

The pole at $z = 1$ exactly cancels the zero at $z = 1$. The ROC is the entire z-plane except $z = 0$




## 1. Definition

**Example 2**

- A **causal** LTI IIR digital filter is described by a constant coefficient difference equation, given by

$$y(n) = x(n-1) - 1.2x(n-2) + x(n-3) + 1.3y(n-1) - 1.04y(n-2) + 0.222y(n-3)$$

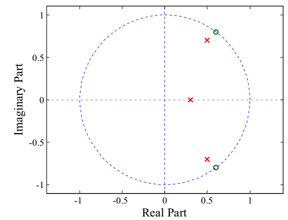
- Its transfer function is therefore given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$



## 1. Definition

**Note:** Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$ ROC: $|z| > \sqrt{0.74}$






## 2. Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- For a **real coefficient** transfer function $H(z)$, it can be shown that

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$



## 2. Frequency Response from Transfer Function

- For a stable rational transfer function in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z - \xi_k)}{\prod_{k=1}^{N}(z - \lambda_k)}$$

- The factored form of the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$



## 2. Frequency Response from Transfer Function

- It is convenient to visualize the contributions of the zero factor $(z - \xi_k)$ and the pole factor $(z - \lambda_k)$ from the factored form of the frequency response. The magnitude function is given by

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \left|e^{j\omega(N-M)}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - \lambda_k\right|} = \left|\frac{p_0}{d_0}\right| \cdot \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - \lambda_k\right|}$$

## 2. Frequency Response from Transfer Function

- The phase response for a rational transfer function is of the form

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M} \arg(e^{j\omega}-\xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega}-\lambda_k)$$



## 3. Geometric Interpretation of Frequency Response Computation

$$|H(e^{j\omega})| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M} |e^{j\omega}-\xi_k|}{\prod_{k=1}^{N} |e^{j\omega}-\lambda_k|}$$

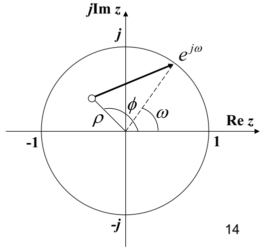
The product of magnitudes of *M* vectors

The product of magnitudes of *N* vectors

constant term

zero or pole vector $(e^{j\omega}-\rho e^{j\phi})$

represents a vector starting at the point $z=\rho e^{j\phi}$ and ending on the unit circle at $z=e^{j\omega}$






## 3. Geometric Interpretation of Frequency Response Computation

- Thus, an *approximate plot* of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by *examining the pole and zero locations*
- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$



## 3. Geometric Interpretation of Frequency Response Computation

- To highly *attenuate* signal components in a specified frequency range, we need to *place zeros very close to or on the unit circle* in this range.
- Likewise, to highly *emphasize* signal components in a specified frequency range, we need to *place poles very close to or on the unit circle* in this range.



## 4. Stability Condition in Terms of Poles Locations

- A causal LTI digital filter is BIBO stable if and only if its impulse response $h(n)$ is absolutely summable, i.e.,

$$S = \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$



## 4. Stability Condition in Terms of Poles Locations

- The ROC of the $z$-transform $H(z)$ of the impulse response sequence $h(n)$ is defined by values of $|z|=r$ for which $h(n)r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z|=1$, then the digital filter is stable, and vice versa
- For LTI system causality we require that $h(n)=0$,for $n<0$. This implies that the ROC of $H(z)$ must be outside some circle of radius $R_{x-}$

## 4. Stability Condition in Terms of Poles Locations

**Theorem**

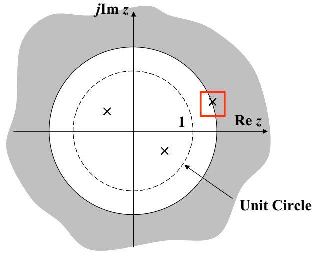
A causality LTI system is stable if and only if the system function $H(z)$ has all its poles inside the unit circle. It is a easy way to judge the causality and stability of an LTI system. So, the ROC will include the unit circle and entire $z$-plane including the point $z=\infty$

- An FIR digital filter with bounded impulse response is always stable



## 4. Stability Condition in Terms of Poles Locations

**Proof:** (reduction to absurdity)






## 4. Stability Condition in Terms of Poles Locations

**Example**

**Under what conditions, the following system is stable or causal ?**

$$H(z)=\frac{-3z^{-1}}{2-5z^{-1}+2z^{-2}}$$

- **Solution:**

**Step 1----Determine the zeros and poles of H(z)**

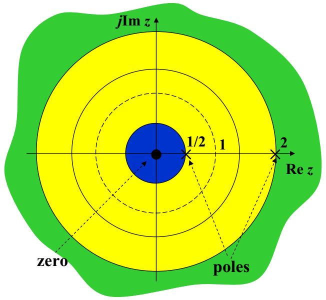
$$H(z)=\frac{-3z^{-1}}{2-5z^{-1}+2z^{-2}}=\frac{-3z}{(2z-1)(z-2)}$$

$z_{zero}=0$, $z_{pole}=0.5, 2$



## 4. Stability Condition in Terms of Poles Locations

**Step 2----plot the zeros and poles on the z-plane**




**Step 3----Determine all possible ROCs according to the distribution of the zeros and poles**

ROC1={|z|<0.5}
—Blue Area

ROC2={0.5<|z|<2}
—yellow Area

ROC3={|z|>2}
—Green Area



## 4. Stability Condition in Terms of Poles Locations

**Step4----Discuss the system's stability and causality**

**Case 1: ROC1={|z|<0.5} Because the unit circle does not lie in this area and the ROC is inside of the circle with radius 0.5, the system is anti-causal and unstable.**

**Case 2: ROC2={0.5<|z|<2} Because the unit circle lies in this area and the ROC is an annulus bounded by 0.5 and 2 , the system is anti-causal and stable.**

**Case 3: ROC3={|z|>2} Because the unit does not lie in this area and the ROC is outside of the circle with radius 2, the system is causal and unstable.**