Chapter 6B

z-Transform

Part B

Inverse z-Transform
and ZT Properties

Inverse z-Transform

- Inverse z-Transform
- z-Transform Properties

1. Inverse z-Transform

1.1 General Expression

- Recall that, for $z = re^{j\omega}$, the z-transform $G(z)$ given by

$$G(z) = G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) r^{-n} e^{-j\omega n}$$

is merely the DTFT of the modified sequence $g(n)r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$g(n)r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})e^{j\omega n}d\omega$$

1.1 General Expression

- By making a change of variable $z = re^{j\omega}$, the previous equation can be converted into a contour integral given by

$$g(n) = \mathcal{Z}^{-1}[G(z)] = \frac{1}{2\pi j}\oint_{c} G(z) z^{n-1} dz$$

where c is a counterclockwise contour of integration defined by $|z| = r$

- But the integral remains unchanged when c is replaced with any contour c' encircling the point $z=0$ in the ROC of $G(z)$

1.1 General Expression

- The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g(n)=\sum\left[\text{residues of } G(z)z^{n-1} \text{ at the poles inside } c\right]$$

$$=-\sum\left[\begin{array}{l}\text{residues of } G(z)z^{n-1} \text{ at the poles outside } c\\ \text{only if there are any higher-order poles inside } c\end{array}\right]$$

- The above equation needs to be evaluated at all values of n and is not pursued here

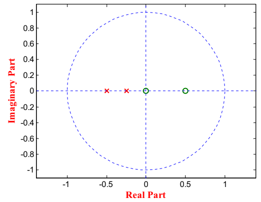
1.1 General Expression

Example:

$$G(z)=\frac{1-0.5z^{-1}}{1+0.75z^{-1}+0.125z^{-2}}$$

Zeros: $z=0 \quad z=0.5$

Poles: $z=-0.5 \quad z=-0.25$



Three ROCs:

$$|z|<0.25$$

$$0.25<|z|<0.5$$

$$|z|>0.5$$

1.1 General Expression

$$G(z)z^{n-1}=\frac{(z-0.5)z^{n}}{z^{2}+0.75z+0.125}$$

Case 1: $|z|<0.25$

If $n\geqslant 0$, there is no poles inside c. Thus, $g(n)=0$ when $n\geqslant 0$

If $n<0$, there is an $|n|$-order pole at $z=0$ which is inside c. In this case, we can compute the summation of the residues outside c instead of that inside

$$g(n)=-\left\{\text{Res}\{z=-0.5\}+\text{Res}\{z=-0.25\}\right\}$$

1.1 General Expression

$$g(n)=-(z+0.5)\frac{(z-0.5)z^{n}}{z^{2}+0.75z+0.125}\bigg|_{z=-0.5}$$

$$-(z+0.25)\frac{(z-0.5)z^{n}}{z^{2}+0.75z+0.125}\bigg|_{z=-0.25}$$

$$=-4(-0.5)^{n}+3(-0.25)^{n} \qquad n\leq -1$$

1.1 General Expression

Case 2: $0.25<|z|<0.5$

If $n\geqslant 0$, there is only one pole at $z=-0.25$ inside c

$$g(n)=(z+0.25)\frac{(z-0.5)z^{n}}{z^{2}+0.75z+0.125}\bigg|_{z=-0.25}$$

$$=-3(-0.25)^{n} \qquad n\geq 0$$

1.1 General Expression

If $n<0$, there are one first-order pole and one $|n|$th-order pole at $z=-0.25$ and $z=0$ inside c, respectively. Thus, we can compute the summation of the residues outside c instead of that inside

$$g(n)=-(z+0.5)\frac{(z-0.5)z^{n}}{z^{2}+0.75z+0.125}\bigg|_{z=-0.5}$$

$$=-4(-0.5)^{n} \qquad n\leq -1$$

Hence, we can rewrite $g(n)$ as follows

$$g(n)=-3(-0.25)^{n}u(n)-4(-0.5)^{n}u(-n-1)$$

1.1 General Expression

Case 3: $|z| > 0.5$

If $n \geqslant 0$, there are two first-order poles at $z=-0.25$ and $z=-0.5$ inside c

$$g(n) = (z+0.5)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.5} + (z+0.25)\frac{(z-0.5)z^n}{z^2+0.75z+0.125}\bigg|_{z=-0.25} = 4(-0.5)^n - 3(-0.25)^n \qquad n \geq 0$$

1.1 General Expression

If $n<0$, there are two first-order poles and one $|n|$th-order pole at $z=-0.25$, $z=-0.25$ and $z=0$ inside c, respectively. Thus, we can compute the summation of the residues outside c instead of that inside. Because there is no poles outside c. Thus, $g(n)=0$ in this case

Summary:

$$g(n) = \begin{cases} -4(-0.5)^n u(-n-1) + 3(-0.25)^n u(-n-1), & |z| < 0.25 \\ -3(-0.25)^n u(n) - 4(-0.5)^n u(-n-1), & 0.25 < |z| < 0.5 \\ 4(-0.5)^n u(n) - 3(-0.25)^n u(n), & |z| > 0.5 \end{cases}$$

1.2 Inverse z-Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g(n)$ has an ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g(n)$ by summing the inverse transform of the individual simpler terms in the expansion

1.2 Inverse z-Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \sum_{i=0}^{M} p_i z^{-i} \bigg/ \sum_{i=0}^{N} d_i z^{-i}$$

- If then $G(z)$ can be re-expressed as

$$G(z) = \sum_{l=0}^{M-N} \eta_l z^{-l} + \frac{P_1(z)}{D(z)}$$

Proper Fraction (真分数)

where the degree of $P_1(z)$ is less than N

1.2 Inverse z-Transform by Partial-Fraction Expansion

Solutions:

Step 1-- Converting $G(z)$ into the form of proper fractions by long division

Step 2-- Summing the inverse transform of the individual simpler terms in the expansion

Assume that $g(n)$ is causal

1.2 Inverse z-Transform by Partial-Fraction Expansion

Example:

$$G(z) = \frac{2+0.8z^{-1}+0.5z^{-2}+0.3z^{-3}}{1+0.8z^{-1}+0.2z^{-2}}$$

By long division we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5+2.1z^{-1}}{1+0.8z^{-1}+0.2z^{-2}}$$

$-3.5 \leftrightarrow -3.5\delta(n)$

$1.5z^{-1} \leftrightarrow 1.5\delta(n-1)$

1.2 Inverse z-Transform by Partial-Fraction Expansion

Let

$$H(z)=\frac{5.5+2.1z^{-1}}{1+0.8z^{-1}+0.2z^{-2}}$$

$$=\frac{2.75+0.25i}{1-(-0.4+0.2i)z^{-1}}+\frac{2.75-0.25i}{1-(-0.4-0.2i)z^{-1}}$$

$$(2.75+0.25i)(-0.4+0.2i)^n u(n)$$

$$(2.75-0.25i)(-0.4-0.2i)^n u(n)$$

1.3 Partial-Fraction Expansion Using MATLAB

- **[r,p,c]= `residuez`(num,den)** develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors **num** and **den**
- Vector **r** contains the residues
- Vector **p** contains the poles
- Vector **c** contains the constants η_l

1.3 Partial-Fraction Expansion Using MATLAB

- **[num,den]=`residuez`(r,p,c)** converts a z-transform expressed in a partial-fraction expansion form to its rational form

1.4 Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a **causal** sequence $\{g(n)\}$ can be expanded in a **power series** in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g(n)$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division.

1.4 Inverse z-Transform via Long Division

Example

- Consider $X(z)=\dfrac{1+2z^{-1}}{1+0.4z^{-1}-0.12z^{-2}}$

  Long division of the numerator by the denominator yields

$$X(z)=1+1.6z^{-1}-0.52z^{-2}+0.4z^{-3}-0.224z^{-4}+\cdots$$

- Hence

$$\{x(n)\}=\{\underset{\uparrow\, n=0}{1},\ 1.6,\ -0.52,\ 0.4,\ -0.2224,\ldots\}\quad n\geq 0$$

1.5 Inverse z-Transform Using MATLAB

- The function `impz` can be used to find the inverse of a rational z-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

2. z-Transform Properties

- Some useful properties of z-Transform are listed in Table 6.2
- This section is devoted to the computation of z-Transform by means of these properties

Example 1

Consider the two-sided sequences

$$v(n) = \alpha^n u(n) - \beta^n u(-n-1)$$

2. z-Transform Properties

- Let $x(n) = \alpha^n u(n) \quad y(n) = \beta^n u(-n-1)$

  with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms
- Now

$$X(z) = \frac{1}{1-\alpha z^{-1}} \quad |z| > |\alpha|$$

$$Y(z) = \frac{1}{1-\beta z^{-1}} \quad |z| < |\beta|$$

- Using the linearity property we arrive at

2. z-Transform Properties

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

- The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$
- If $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$
- If $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist

2. z-Transform Properties

Example 2

$$y(n) = (n+1)\alpha^n u(n)$$

$y(n)$ can be rewritten as $y(n) = nx(n) + x(n)$

where $x(n) = \alpha^n u(n)$

- The z-transform of $x(n)$ is given by

$$X(z) = \frac{1}{1-\alpha z^{-1}} \quad |z| > |\alpha|$$

2. z-Transform Properties

- Using the differentiation property, we arrive at the z-transform of $nx(n)$ as

$$-z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \quad |z| > |\alpha|$$

- Using the linearity property we finally obtain

$$Y(z) = \frac{1}{\left(1-\alpha z^{-1}\right)^2} \quad |z| > |\alpha|$$