Chapter 6

z-Transform

Part A

z-Transform

Contents

Part A: z-Transform

- **z-Transform**
- **Region of Convergence (ROC) of a Rational z-Transform**

1. z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems.
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist.
- As a result, it is not possible to make use of such frequency-domain characterization in these cases.

1. z-Transform

- In general, **ZT** can be thought of as a generalization of the **DTFT**. **ZT** is more complex than DTFT (both literally and figuratively), but provides a great deal of insight into system design and behavior. For discrete-time systems, **ZT** plays the similar role of **Laplace-transform** does in continuous-time systems. **ZT** characterizes signals or LTI systems in **complex frequency domain**.

1. 1 Definition of z-Transform

- Recall that the definition of DTFT of a sequence $g(n)$ can be expressed by

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n}$$

where $G(e^{j\omega})$ can be viewed as a Fourier series and $g(n)$ is the coefficients of this series. The basic building block in DTFT is $e^{j\omega}$.

1. 1 Definition of z-Transform

- $e^{j\omega}$ is a one dimensional (single-variable) function which can be expressed in one-dimensional plane. In order to extend the DTFT to ZT, it is possible to replace the basic building block $e^{j\omega}$ by a two dimensional (two-variable) function. Hence, the new basic building block can described in a two-dimensional plane
- Define a new two dimensional variable $z = re^{j\omega}$, we obtain the expression of z-transform

1. 1 Definition of z-Transform

- A generalization of the DTFT defined by leads

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n}$$

to the z-transform

- z-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of z-transform techniques permits simple algebraic manipulations

1. 1 Definition of z-Transform

- Consequently, z-transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g(n)$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g(n) z^{-n}$$

where $z = \mathrm{Re}(z) + j\mathrm{Im}(z)$ is a complex variable.

1. 1 Definition of z-Transform

- If we let $z = re^{j\omega}$, then the z-transform reduces to

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) r^{-n} e^{-j\omega n}$$

- The above can be interpreted as the DTFT of the modified sequence $\{g(n) r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), z-transform reduces to its DTFT, provided the latter exists.

1. 1 Definition of z-Transform

- The contour $|z| = 1$ is a circle in the z-plane of unity radius and is called the *unit circle*.
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g(n) z^{-n}$$

- For a given sequence, the set $\mathcal{R}$ of values of z for which its z-transform converges is called the *region of convergence* (ROC).

1. 1 Definition of z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) r^{-n} e^{-j\omega n}$$

converges if $g(n)z^{-n}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left| g(n) r^{-n} \right| < \infty$$

1. 1 Definition of z-Transform

- In general, the ROC $\mathcal{R}$ of a *z*-transform of a sequence $g(n)$ is an annular region of the *z*-plane:

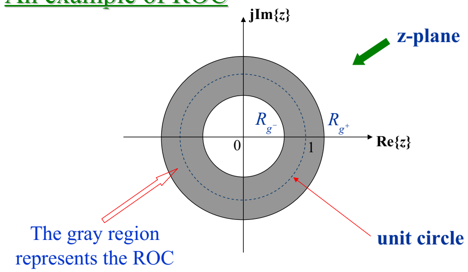
$$R_{g^-} < |z| < R_{g^+}$$

where $0 \le R_{g^-} < R_{g^+} \le \infty$

- Note: The *z*-transform is a form of a Laurent series and is an analytic function at every point in the ROC.

1. 1 Definition of z-Transform

An example of ROC



1. 1 Definition of z-Transform

Comments

① The complex variable z is called the complex frequency given by $z = re^{j\omega}$, where r is the attenuation and ω is the real frequency.

② Since the ROC is defined in terms of the magnitude r, the shape of the ROC is an annulus. Note that R_{g^-} may be equal to 0 and/or R_{g^+} could possibly be infinity.

1. 1 Definition of z-Transform

③ If $R_{g^+} < R_{g^-}$, then the ROC is a null space and the ZT does not exist.

④ The function $r=1$ (or $z = e^{j\omega}$) is a circle of unit radius in the *z*-plane and is called the unit circle. If the ROC contains the unit circle, then we can evaluate $G(z)$ on the unit circle.

$$G(z)|_{z=e^{j\omega}} = G(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} g(n) e^{-j\omega n}$$

1. 1 Definition of z-Transform

Therefore the discrete-time Fourier transform $G(e^{j\omega})$ may be viewed as a special case of the *z*-transform $G(z)$.

⑤ If $g(n)=h(n)$ is the impulse response of some system, its *z*-transform $G(z)=H(z)$ is called as System Function or Transfer Function of this system.

1. 1 Definition of z-Transform

Example 1:

Calculate the ZT of $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$
$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Note that the above equation holds only for $|az^{-1}| < 1$, i.e. $|z| > |a|$

Region of convergence

1. 1 Definition of z-Transform

Example 2:

Calculate the ZT of $x(n) = -a^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=-\infty}^{-1} \left(az^{-1}\right)^n$$
$$= \sum_{n=1}^{\infty} \left(az^{-1}\right)^{-n} = \sum_{n=1}^{\infty} \left(a^{-1}z\right)^n = \frac{z}{z-a}$$

Note that the above equation holds only for $|a^{-1}z| < 1$, i.e. $|z| < |a|$

Region of convergence

1. 1 Definition of z-Transform

From the above two examples, we find that

- Very different time functions can have the same z-transform. Because ROC plays an important role in computing the z-transform or inverse z-transform.
- So we must specify not only the z-transform corresponding to a time function, but its ROC as well.

1. 2 Rational z-Transform

- In the case of LTI discrete-time systems we are concerned with in this course, all involved z-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots d_{N-1} z^{-(N-1)} + d_{N-1} z^{-N}}$$

1. 2 Rational z-Transform

- The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots d_{N-1} z + d_{N-1}}$$

1. 2 Rational z-Transform

- A rational z-transform can be alternatively written in factored form as

$$G(z) = \frac{p_0}{d_0} \frac{\prod_{l=1}^{M}\left(1-\xi_l z^{-1}\right)}{\prod_{l=1}^{N}\left(1-\lambda_l z^{-1}\right)}$$
$$= z^{(N-M)} \frac{p_0}{d_0} \frac{\prod_{l=1}^{M}\left(z-\xi_l\right)}{\prod_{l=1}^{N}\left(z-\lambda_l\right)}$$

1. 2 Rational z-Transform

- At a root $z = \xi_l$ of the numerator polynomial, $G(\xi_l) = 0$, and as a result, these values of z are known as the zeros of $G(z)$
- At a root $z = \lambda_l$ of the denominator polynomial, $G(z_l) = 0$, and as a result, these values of z are known as the poles of $G(z)$

1. 2 Rational z-Transform

Consider: $G(z) = z^{(N-M)} \dfrac{p_0 \prod_{l=1}^{M}(z-\xi_l)}{d_0 \prod_{l=1}^{N}(z-\lambda_l)}$

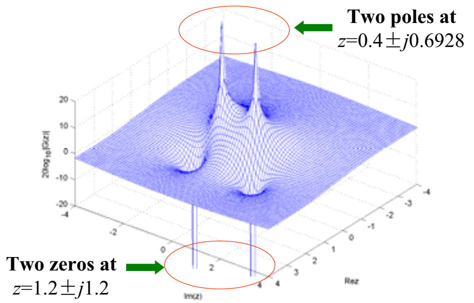
- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

1. 2 Rational z-Transform

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

1. 2 Rational z-Transform



2. Region of Convergence of a Rational z-Transform

- ROC of a z-transform is an important concept.
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its z-transform.
- Hence, the z-transform must always be specified with its ROC.

2. Region of Convergence of a Rational z-Transform

- Moreover, if the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle.
- There is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability.

2. Region of Convergence of a Rational z-Transform

- The ROC of a rational *z*-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a *z*-transform

2.1 General Form of ROC

- In general, there are four types of ROCs for *z*-transforms, and they depend on the type of the corresponding time functions.
  - – Finite-length sequence
  - – Right-sided sequence
  - – Left-sided sequence
  - – Two-sided (infinite duration) sequence

2.1 General Form of ROC

– Finite-length Sequence

A **finite-length sequence** $g(n)$ is defined for $-M \leqslant n \leqslant N$ with M and N positive, and $|g(n)|<\infty$.

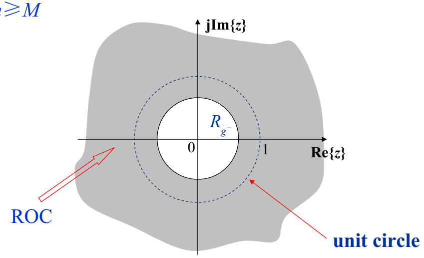
In general, its ROC includes the entire *z*-plane except possible $z=0$ or/and $z=\infty$

For finite duration sequences, the condition of convergence is that every term in the ZT is convergent. Except the $z=0$ and $z=\infty$, the ZT of a finite sequence is convergent in the entire *z*-plane.

2.1 General Form of ROC

– Right-sided Sequence

A **right-sided sequence** $u(\mathrm{n})$ with nonzero sample values only for $n \geqslant M$



2.1 General Form of ROC

If $M \geqslant 0$, $R_{g^-} < |z| \leq \infty$ $\quad R_{g^+} = \infty$

If $M < 0$, $R_{g^-} < |z| < \infty$ $\quad R_{g^+} < \infty$

If $M=0$, $u(n)$ is called a **causal sequence**

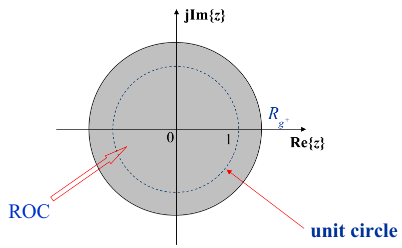
Comment

All causal sequences (or the impulse responses of LTI systems) are right-sided, while not all right-sided sequences correspond to causal systems.

2.1 General Form of ROC

– Left-sided Sequence

A **left-sided sequence** $v(\mathrm{n})$ with nonzero sample values only for $n \leqslant N$



2.1 General Form of ROC

If $N>0$, $0<|z|<R_{g+}$ $R_{g-}>0$

If $N\leq 0$, $0\leq|z|<R_{g+}$ $R_{g-}=0$

If N=0, $v(n)$ is called a **anticausal sequence**

2.1 General Form of ROC

– Two-sided Sequence

The z-Transform of a **two-sided sequence** $w(n)$ can be expressed as

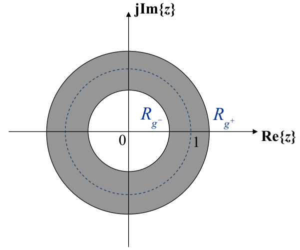
$$W(z)=\sum_{n=-\infty}^{\infty}w(n)z^{-n}=\sum_{n=0}^{\infty}w(n)z^{-n}+\sum_{n=-\infty}^{-1}w(n)z^{-n}$$

A right-sided sequence + A left-sided sequence

$|z|>R_{g-}$ $|z|<R_{g+}$

2.1 General Form of ROC

Obviously, the ROC of $W(z)$ is the intersection of $|z|>R_{g-}$ and $|z|<R_{g+}$. If $R_{g+}>R_{g-}$, its ROC has the following form



But, if $R_{g+}<R_{g-}$, its ROC is a null space, i.e., the transform does not exist

2.1 General Form of ROC

An example

Consider the two-sided sequence $x(n)=a^n$, where a can be either complex or real. Its z-Transform is given by

$$X(z)=\sum_{n=0}^{\infty}a^n z^{-n}+\sum_{n=-\infty}^{-1}a^n z^{-n}$$

$|z|>|a|$ $|z|<|a|$

There is no overlap between these two regions. Hence, its z-transform does not exist

2.1 General Form of ROC

Summary

- In general, if the rational z-transform has N poles with R distinct magnitudes, then it has R+1 ROCs
- Thus, there are R+1 distinct sequences with the same z-transform
- Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform.

2.2 Determine the ROC by MATLAB

- The ROC of a rational z-transform can be easily determined using MATLAB

  [z,p,k] = `tf2zp`(num,den)

  determines the *zeros, poles,* and the *gain constant* of a rational z-transform with the numerator coefficients specified by the vector **num** and the denominator coefficients specified by the vector **den**.

2.2 Determine the ROC by MATLAB

- [**num**,**den**] = **`zp2tf`**(**z**,**p**,**k**) implements the reverse process
- The factored form of the *z*-transform can be obtained using **sos** = **`zp2sos`**(**z**,**p**,**k**) where **sos** stands for second-order section
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix **sos**

2.2 Determine the ROC by MATLAB

$$\mathbf{sos} = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

2.2 Determine the ROC by MATLAB

- The pole-zero plot is determined using the function **`zplane`**
- The *z*-transform can be either described in terms of its zeros and poles:

  `zplane`(**zeros**,**poles**)

  or, it can be described in terms of its numerator and denominator coefficients:

  `zplane`(**num**,**den**)