Chapter 5

Finite Length
Discrete Transforms

Part A

The Discrete Fourier Transform (DFT)

Discrete Fourier Transform

- **Definition**
- **Matrix Relations**
- **DFT Computation Using MATLAB**
- **Relation between DTFT and DFT and their inverses**

1. Definition

Definition

- The simplest relation between a length-N sequence $x(n)$, defined for $0 \le n \le N-1$ and its DTFT $X(e^{j\omega})$ is obtained by uniformly sampling $X(e^{j\omega})$ on the ω-axis between $0 \le \omega \le 2\pi$ at $\omega_k = 2\pi\, k/N$, $0 \le k \le N-1$
- *DFT is obtained by sampling the DTFT over one principal value interval in the frequency domain*

1. Definition

- From the definition of the DTFT we thus have

$$X(k) = X(e^{j\omega})\big|_{\omega = 2\pi k/N}$$
$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad 0 \le k \le N-1$$

- $X(k)$ is also a length-N sequence in the frequency domain
- The sequence $X(k)$ is called the discrete Fourier transform (DFT) of the sequence $x(n)$

1. Definition

- Using the notation $W_N = e^{-j2\pi/N}$, the DFT is usually expressed as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1$$

- The inverse discrete Fourier transform (IDFT) is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad 0 \le n \le N-1$$

1. Definition

- $W_N = e^{-j2\pi/N}$ which is usually called twiddle factor has many useful features
  - It is the first root of the N N-th roots of unity
  - The modulus is 1 (on the unit circle)
  - $W_N^k = W_N^{k+N}$ $\quad W_N^{k+N/2} = W_N^{-k}$ $\quad \sum_{k=1}^{N-1} W_N^k = 0$

    $W_N^0 = 1$ $\quad W_N^{N/2} = -1$

1. Definition

- To verify the validity of IFDT, we multiply both sides of the expression by W_N^{ln} and sum the result from $n = 0$ to $n=N-1$, resulting in

$$\sum_{n=0}^{N-1} x(n)W_N^{ln} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}\right)W_N^{ln}$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X(k)W_N^{-(k-l)n}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sum_{n=0}^{N-1} W_N^{-(k-l)n}$$

1. Definition

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-l)n} = \begin{cases} N, & \text{for } k-l = rN, r \text{ is an integer} \\ 0, & \text{otherwise} \end{cases}$$

- Hence

$$\sum_{n=0}^{N-1} x(n)W_N^{ln} = X(l)$$

1. Definition

- Mapping Relations between time-domain and frequency-domain transforms

(Time-domain)		(Frequency-domain)
Continuous	⟺	Aperiodical
Discrete	⟺	Periodical
Periodical	⟺	Discrete
Aperiodical	⟺	Continuous

1. Definition

- ***Type* 1:** Continuous-Time Fourier Transform (CTFT)

Continuous Aperiodical $\quad x_a(t) \Longleftrightarrow X_a(j\Omega)\quad$ Aperiodical Continuous

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt$$

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}d\Omega$$

1. Definition

- ***Type* 2:** Continuous-Time Fourier Series (CTFS)

Continuous Periodical $\quad x_a(t) \Longleftrightarrow X_a(jk\Omega_0)\quad$ Aperiodical Discrete

$$X_a(jk\Omega_0) = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x_a(t)e^{-jk\Omega_0 t}dt$$

$$x_a(t) = \sum_{k=-\infty}^{\infty} X_a(jk\Omega_0)e^{jk\Omega_0 t}$$

1. Definition

- ***Type* 3:** Discrete-Time Fourier Transform (DTFT)

Discrete Aperiodical $x(n) \Longleftrightarrow X(e^{j\omega})$ Periodical Continuous

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

1. Definition

- ***Type* 4:** Discrete Fourier Transform (DFT)

Discrete Periodical $x(n) \Longleftrightarrow X(k)$ Periodical Discrete

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \quad 0 \le n \le N-1$$

1. Definition

Examples

- Rectangular Pulse $R_N(n)$, width N
- Its N-point DFT is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} = \sum_{n=0}^{N-1} W_N^{kn} = \frac{1-W_N^{kN}}{1-W_N^{k}}$$
$$= \frac{W_N^{kN/2}}{W_N^{k/2}}\,\frac{W_N^{-kN/2}-W_N^{kN/2}}{W_N^{-k/2}-W_N^{k/2}}$$
$$= e^{-j\frac{N-1}{N}k\pi}\,\frac{\sin(k\pi)}{\sin(k\pi/N)}$$

Geometric Progression

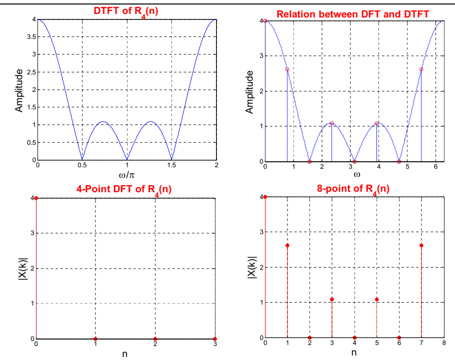
1. Definition

- Its $2N$-point DFT is given by

$$X(k) = \sum_{n=0}^{2N-1} x(n)W_{2N}^{kn} = \sum_{n=0}^{N-1} W_{2N}^{kn}$$
$$= \frac{1-W_{2N}^{kN}}{1-W_{2N}^{k}} = e^{-j\frac{N-1}{2N}k\pi}\,\frac{\sin(k\pi/2)}{\sin(k\pi/2N)}$$

- So the length of DFT plays a very important role in DFT

1. Definition



2. Matrix Relations

- This part is introduced for the purpose of computation using MATLAB
- Since MATLAB stands for *MAtrix LABoratory*, we represent DFT definition in terms of matrix form

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad 0 \le k \le N-1$$

$$\mathbf{X} = \mathbf{D}_N\mathbf{x}$$

2. Matrix Relations

- where $\mathbf{X}=\left[X(0) \quad X(1) \quad \cdots \quad X(N-1)\right]^T$

  $\mathbf{x}=\left[x(0) \quad x(1) \quad \cdots \quad x(N-1)\right]^T$

$$\mathbf{D}_N=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N}$$

2. Matrix Relations

- Likewise, the IDFT relations can be expressed in $\mathbf{x}=\mathbf{D}_N^{-1}\mathbf{X}$

$$\mathbf{D}_N^{-1}=\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}_{N\times N}$$

2. Matrix Relations

- Obviously, the relation between the two coefficient matrices can be expressed as follows

$$\mathbf{D}_N^{-1}=\frac{1}{N}\mathbf{D}_N^*$$

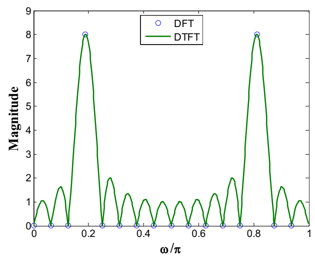
- Therefore, the algorithms designed for DFT are applicable to IDFT

3. DFT Computation Using MATLAB

- The functions to compute the DFT and the IDFT are `fft` and `ifft`
- These functions make use of FFT algorithms which are computationally highly efficient compared with the direct computation
- Figure in the next slide shows the DFT and DTFT of the sequence

$$\cos(6\pi n/16) \quad 0\le n\le 15$$

3. DFT Computation Using MATLAB



4. Relations between DTFT and DFT and their inverses

- **DTFT from DFT by *interpolation***
- ***Sampling* the DTFT**
- **Numerical computation of the DTFT using DFT**

$$X(k) \underset{\text{sampling}}{\overset{\text{interpolation}}{\rightleftarrows}} X(e^{j\omega})$$

4.1 DTFT from DFT by interpolation

- The N-point DFT $X(k)$ of a length-N sequence $x(n)$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points

$$\omega_k = 2\pi k/N, \quad 0 \le k \le N-1$$

- Given the N-point DFT $X(k)$ of a length-N sequence $x(n)$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X(k)$

4.1 DTFT from DFT by interpolation

- We use the IFDT relation and the definition of DTFT to study the relation between DFT and DTFT

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} = \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1} X(k)W_N^{-kn}\right]e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sum_{n=0}^{N-1} e^{-j[\omega-(2\pi k/N)]n}$$

IDFT $x(n)$

Exchange of the order of summations

4.1 DTFT from DFT by interpolation

- Let $S = \sum_{n=0}^{N-1} e^{-j[\omega-(2\pi k/N)]n}$ and $r = e^{-j[\omega-(2\pi k/N)]}$
- Thus

$$S = \sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r} = \frac{1-e^{-j(\omega N-2\pi k)}}{1-e^{-j[\omega-(2\pi k/N)]}}$$

$$= \frac{\sin\left(\dfrac{\omega N-2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} \cdot e^{-j[\omega-(2\pi k/N)][(N-1)/2]}$$

4.1 DTFT from DFT by interpolation

- There, DTFT can be determined by the following *interpolation formula*

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\frac{\sin\left(\dfrac{\omega N-2\pi k}{2}\right)}{\sin\left(\dfrac{\omega N-2\pi k}{2N}\right)} \cdot e^{-j[\omega-(2\pi k/N)][(N-1)/2]}$$

4.2 Sampling the DTFT

- Consider a sequence $x(n)$ with a DTFT $X(e^{j\omega})$
- We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k=2\pi k/N, \quad 0 \le k \le N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
- These N frequency samples can be considered as an N-point DFT $Y(k)$ whose N-point IDFT is a length-N sequence $y(n)$

4.2 Sampling the DTFT

- Now $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$
- Thus

$$Y(k) = X(e^{j\omega_k}) = X(e^{j(2\pi k/N)})$$

$$= \sum_{l=-\infty}^{\infty} x(l)W_N^{kl}$$

- An IDFT of $Y(k)$ yields $\quad y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)W_N^{-kn}$

4.2 Sampling the DTFT

- i.e. $y(n)=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=-\infty}^{\infty}x(l)W_N^{kl}W_N^{-kn}$

$$=\sum_{l=-\infty}^{\infty}x(l)\left[\frac{1}{N}\sum_{k=0}^{N-1}W_N^{-k(n-l)}\right]$$

- Making use of the identity

$$\frac{1}{N}\sum_{k=0}^{N-1}W_N^{-k(n-l)}=\begin{cases}1, & \text{for } l=n+mN\\ 0, & \text{otherwise}\end{cases}$$

4.2 Sampling the DTFT

We arrive at the desired relation

$$y(n)=\sum_{m=-\infty}^{\infty}x(n+mN),\quad 0\le n\le N-1$$

- Thus $y(n)$ is obtained from $x(n)$ by adding an infinite number of shifted replicas of $x(n)$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0\le n\le N-1$

4.2 Sampling the DTFT

- To apply

$$y(n)=\sum_{m=-\infty}^{\infty}x(n+mN),\quad 0\le n\le N-1$$

to finite-length sequences, we assume that the samples outside the specified range are zeros

- Thus if $x(n)$ is a length-M sequence with $M\le N$, then $y(n)=x(n)$ for $0\le n\le N-1$

4.2 Sampling the DTFT

- If $M > N$, there is a time-domain aliasing of samples of $x(n)$ in generating $y(n)$, and $x(n)$ cannot be recovered from $y(n)$
- This is called *Sampling Theorem in Frequency-Domain* ($N\ge M$)
- Recall that the condition of *Sampling Theorem in Time-Domain is* $f_s\ge 2f_c$

Example Let $x(n)=\{0\ 1\ 2\ 3\ 4\ 5\}$

4.2 Sampling the DTFT

- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k=2\pi k/4$, $0\le k\le 3$, and then applying a 4-point IDFT to these samples, we arrive at the sequence $y(n)$ given by

$$y(n)=x(n)+x(n+4)+x(n-4)\ ,\ 0\le k\le 3$$

i.e. $y(n)=\{4\ 6\ 2\ 3\}$

$\Rightarrow$ $\{x(n)\}$ cannot be recovered from $\{y(n)\}$

4.3 Numerical Computation of the DTFT using DFT

- A practical approach to the numerical computation of the DTFT of a finite-length sequence
- Let $X(e^{j\omega})$ be the DTFT of a length-N sequence $x(n)$
- We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies , where $M >> N$:

$$\omega_k=\frac{2\pi k}{M},\quad 0\le k\le M-1$$

4.3 Numerical Computation of the DTFT using DFT

$$X\left(e^{j\omega_k}\right)=\sum_{n=0}^{N-1}x(n)e^{-j\omega_k n}=\sum_{n=0}^{N-1}x(n)e^{-\frac{j2\pi kn}{M}}$$

- Define a new sequence

$$x_e(n)=\begin{cases}x(n) & 0\le n\le N-1\\ 0 & N\le n\le M-1\end{cases}$$

- Then

$$X\left(e^{j\omega_k}\right)=\sum_{n=0}^{M-1}x_e(n)e^{-\frac{j2\pi kn}{M}}$$

4.3 Numerical Computation of the DTFT using DFT

- Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e(k)$ of the length-M sequence $x_e(n)$
- The DFT $X_e(k)$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2
- The function **`freqz`** employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in $e^{-j\omega}$