Chapter 4B

Digital Processing of Continuous-Time Signals

Part B

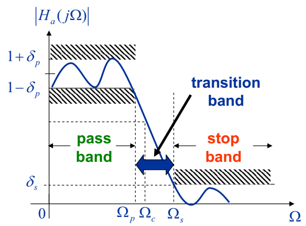
Design of Analog Filters

Design of Analog Filters

- **Analog Lowpass Filter Specifications**
- **Butterworth Approximation**
- **Design of Other Types of Analog Filters**

1. Analog Lowpass Filter Specifications

- Typical magnitude response of an analog lowpass filter may be given as indicated below



1. Analog Lowpass Filter Specifications

- In the **passband**, defined by $0 \le \Omega \le \Omega_p$, we require $1-\delta_p \le |H_a(j\Omega)| \le 1+\delta_p, \quad |\Omega| \le \Omega_p$ i.e., $|H_a(j\Omega)|$ approximates unity within an error of $\pm\delta_p$
- In the **stopband**, defined by $\Omega_s \le \Omega \le \infty$, we require $|H_a(j\Omega)| \le \delta_s, \quad \Omega_s \le \Omega \le \infty$ i.e., $|H_a(j\Omega)|$ approximates zero within an error of δ_s

1. Analog Lowpass Filter Specifications

Ω_p -- passband edge frequency
Ω_s -- stopband edge frequency
δ_p -- peak ripple value in the passband
δ_s -- peak ripple value in the stopband

Peak passband ripple

$$\alpha_p = -20\log_{10}(1-\delta_p)\ \text{dB}$$

Minimum stopband attenuation

$$\alpha_s = -20\log_{10}(\delta_s)\ \text{dB}$$

2. Butterworth Approximation

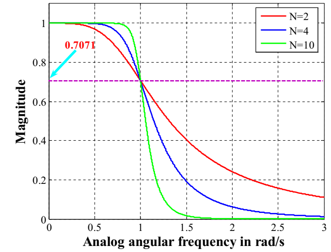
- The magnitude-square response of an N-th order analog lowpass Butterworth Filter is given by
$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$
- First $2N-1$ derivatives of $|H_a(j\Omega)|^2$ at $\Omega=0$ are equal to zero
- The Butterworth lowpass filter thus is said to have a maximally-flat magnitude at $\Omega=0$

2. Butterworth Approximation

- Gain in dB is $G(\Omega)=10\log_{10}|H_a(j\Omega)|^2$ dB
- As $G(0)$=0 dB and
$$G(\Omega_c)=10\log_{10}(0.5)=-3.0103\approx -3 \text{ dB}$$
Ω_c is called the 3-dB cutoff frequency

2. Butterworth Approximation

- Typical magnitude responses with $\Omega_c=1$



2. Butterworth Approximation

- Two parameters completely characterizing a Butterworth lowpass filter are Ω_c and N
- They are determined from the specified bandedges Ω_p and Ω_s, and minimum passband magnitude $|H_a(j\Omega_p)|$, and maximum stopband magnitude $|H_a(j\Omega_s)|$
- Ω_c and N are thus determined from

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \qquad |H_a(j\Omega_s)|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}}$$

2. Butterworth Approximation

- According to the definition of the loss function
$$\alpha = 10\log_{10}\frac{1}{|H(j\Omega)|^2}$$
- We know that
$$\frac{1}{|H(j\Omega)|^2} = 1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}$$
$$1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{\frac{\alpha_p}{10}} \qquad 1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = 10^{\frac{\alpha_s}{10}}$$

2. Butterworth Approximation

- Suppose
$$\lambda = \frac{\Omega_s}{\Omega_p} \qquad k = \sqrt{\frac{10^{\frac{\alpha_p}{10}}-1}{10^{\frac{\alpha_s}{10}}-1}}$$
- Then
$$N = -\frac{\lg k}{\lg \lambda}$$

2. Butterworth Approximation

- Since order N must be an integer, value obtained is rounded up to the next highest integer
- This value of N is used next to determine by satisfying either the stopband edge or the passband edge specification exactly
- If the stopband edge specification is satisfied, then the passband edge specification is exceeded providing a safety margin

2. Butterworth Approximation

- Since $H_a(s)H_a(-s)$ evaluated at $s = j\Omega$ is simply equal to $|H_a(j\Omega)|^2$, it follows that

$$H_a(s)H_a(-s) = \frac{1}{1+\left(-s^2/\Omega_c^2\right)^N}$$

- The poles of this expression occur on a circle of radius Ω_c at equally spaced points
- Because of the stability and causality, the transfer function itself will be specified by just the poles in the negative real half-plane of s

2. Butterworth Approximation

- Hence, the transfer function of an analog Butterworth lowpass filter has the form of

$$H_a(s) = \frac{C}{D_N(s)} = \frac{\Omega_c^N}{s^N + \sum_{l=0}^{N-1} d_l s^l} = \frac{\Omega_c^N}{\prod_{l=1}^{N}(s-p_l)}$$

Where

$$p_l = \Omega_c e^{j[\pi(N+2l-1)/2N]}, \quad l = 1,2,\ldots,N$$

- Denominator $D_N(s)$ is known as the Butterworth polynomial of order N

2. Butterworth Approximation

Example –

- Determine the lowest order of a Butterworth lowpass filter with a 1 dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz

$$10\log_{10}\left[\frac{1}{1+(2000\pi/\Omega_c)^{2N}}\right] = -1$$

$$10\log_{10}\left[\frac{1}{1+(10000\pi/\Omega_c)^{2N}}\right] = -40$$

$$N = \log_{10}\left\{\frac{10^{0.1}-1}{10^4-1}\right\} \Big/ \log_{10}\frac{1}{5} = 3.2811$$

3. Design of Other Types of Analog Filters

Highpass Filter

- *Step* 1 Develop of specifications of a prototype analog lowpass filter $H_{LP}(s)$ from specifications of desired analog filter $H_D(s)$ using a frequency (spectrum) transformation.
- *Step* 2 Design the prototype analog lowpass filter
- *Step* 3 Determine the transfer function $H_D(s)$ of desired analog filter by applying the inverse frequency transformation to $H_{LP}(s)$

3. Design of Other Types of Analog Filters

- Let s denote the Laplace transform variable of prototype analog lowpass filter $H_{LP}(s)$ and $\hat{s}$ denote the Laplace transform variable of desired analog filter $H_D(\hat{s})$
- The mapping from s -domain to $\hat{s}$ -domain is given by the invertible transformation $s = F(\hat{s})$

Then

$$H_D(\hat{s}) = H_{LP}(s)\big|_{s=F(\hat{s})}$$

$$H_{LP}(s) = H_D(\hat{s})\big|_{\hat{s}=F^{-1}(s)}$$

3. Design of Other Types of Analog Filters

- Spectral Transformation (Lowpass to Highpass)

$$s = \frac{\Omega_p \hat{\Omega}_P}{\hat{s}}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\hat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$

- On the imaginary axis the transformation is

$$\Omega = -\frac{\Omega_p \hat{\Omega}_P}{\hat{\Omega}}$$