Chapter 4

Digital Processing of Continuous-Time Signals

Part A

Sampling of Continuous-Time Signals

Part A: Sampling of Continuous-Time Signals

- **Introduction**
- **Sampling of Continuous-Time Signals**
  - **Effect of Sampling in the Frequency Domain**
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1. Introduction

- Digital processing of a continuous-time signal involves the following basic steps:

  (1) Conversion of the continuous-time signal into a discrete-time signal,

  (2) Processing of the discrete-time signal,

  (3) Conversion of the processed discrete-time signal back into a continuous-time signal

1. Introduction

- Conversion of a continuous-time signal into digital form is carried out by an analog-to-digital (A/D) converter
- The reverse operation of converting a digital signal into a continuous-time signal is performed by a digital-to-analog (D/A) converter

1. Introduction

- Since the A/D conversion takes a finite amount of time, a sample-and-hold (S/H) circuit is used to ensure that the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation
- To prevent **aliasing**, an analog anti-aliasing filter is employed before the S/H circuit.

1. Introduction

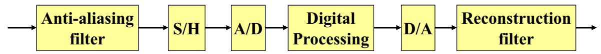
- S/H circuit often consists of a *capacitor* to store the analogue voltage, and an electronic *switch* or *gate* to alternately connect and disconnect the capacitor from the analogue input.

1. Introduction

- An anti-aliasing filter is a filter used before a signal sampler, to restrict the bandwidth of a signal to approximately satisfy the *sampling theorem*. Since the theorem states that *unambiguous interpretation* of the signal from its samples is possible only when the power of frequencies outside the *Nyquist bandwidth* is zero, the anti-aliasing filter would have to have perfect *stop-band rejection* to completely satisfy the theorem. Every realizable anti-aliasing filter will permit some aliasing to occur; the amount of aliasing that does occur depends on how good the filter is.

1. Introduction

- To smooth the output signal of the D/A converter, which has a staircase-like waveform, an analog reconstruction filter is used.
- The complete block-diagram is shown blew



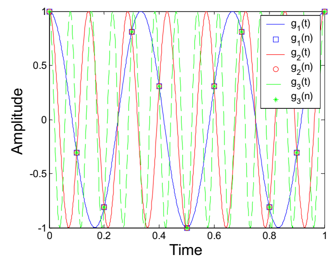
1. Introduction

- Since both the anti-aliasing filter and the reconstruction filter are **analog lowpass filters**, we will review the theory behind the design of such filters in this chapter
- Also, the most widely used IIR digital filter design method is based on the conversion of an **analog lowpass prototype**

2. Sampling of Continuous-Time Signals

- As indicated earlier, discrete-time signals in many applications are generated by sampling continuous-time signals
- It is obvious that identical discrete time signals may result from the sampling of more than one distinct continuous-time function
- In fact, there exists an infinite number of continuous-time signals, which when sampled lead to the same discrete-time signal

2. Sampling of Continuous-Time Signals



2. Sampling of Continuous-Time Signals

- However, under certain conditions, it is possible to relate a unique continuous-time signal to a given discrete-time signals
- If these conditions hold, then it is possible to recover the original continuous-time signal from its sampled values
- We next develop this correspondence and the associated conditions

2.1 Effect of Sampling in the Frequency Domain

- Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating the sequence $g(n)$ where $g(n)= g_a(nT)$ with T being the sampling period
- The reciprocal of T is called the sampling frequency F_T, i.e., $F_T = 1/T$
- Now, the frequency-domain representation of $g_a(t)$ is given by its continuous-time Fourier transform (CTFT):

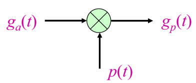
2.1 Effect of Sampling in the Frequency Domain

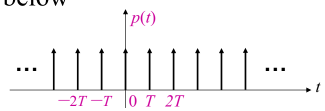
$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

- The frequency-domain representation of $g(n)$ is given by its discrete-time Fourier transform (DTFT): $G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n}$
- To establish the relation between $G_a(j\Omega)$ and $G(e^{j\omega})$, we treat the sampling operation mathematically as a multiplication of $g_a(t)$ by a periodic impulse train $p(t)$:

2.1 Effect of Sampling in the Frequency Domain

- $p(t)$ consists of a train of ideal impulses with a period T as shown below



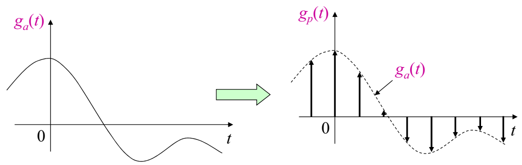


- The multiplication operation yields an impulse train:

$$g_p(t) = g_a(t)p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t-nT)$$

2.1 Effect of Sampling in the Frequency Domain

- $g_a(t)$ is a continuous-time signal consisting of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant



2.1 Effect of Sampling in the Frequency Domain

- There are two different forms of $G_p(j\Omega)$:
- One form is given by the weighted sum of the CTFTs of $\delta(t-nT)$:

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$

- To derive the second form, we note that $p(t)$ can be expressed as a Fourier series:

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\frac{2\pi}{T}kt} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\Omega_T kt}$$

where

$$\Omega_T = \frac{2\pi}{T}$$

2.1 Effect of Sampling in the Frequency Domain

- The impulse train $g_p(t)$ therefore can be expressed as

$$g_p(t) = \left(\frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{j\Omega_T kt} \right) g_a(t)$$

- From the frequency-shifting property of the CTFT, the CTFT of $e^{j\Omega_T kt} g_a(t)$ is given by

$$G_a(j(\Omega - k\Omega_T))$$

- Hence, an alternative form of the CTFT of is given by

$$G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

2.1 Effect of Sampling in the Frequency Domain

- Therefore, $G_p(j\Omega)$ is a periodic function of Ω consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by $1/T$
- The term on the RHS of the previous equation for $k = 0$ is the baseband portion of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions of $G_p(j\Omega)$

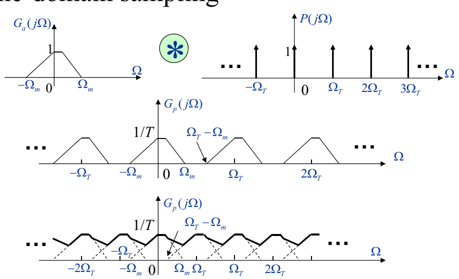
2.1 Effect of Sampling in the Frequency Domain

- The frequency range $-\Omega_T/2 \le \Omega \le \Omega_T/2$ is called the baseband or Nyquist band
- The above result is more commonly known as the sampling theorem:
- Let $g_a(t)$ be a band-limited signal with $G_a(j\Omega) = 0$ for $|\Omega| > \Omega_m$, then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty \le n \le \infty$,if

$$\Omega_T \ge 2\Omega_m \text{ where } \Omega_T = \frac{2\pi}{T}$$

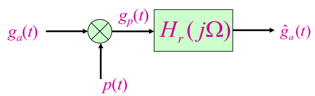
2.1 Effect of Sampling in the Frequency Domain

- Illustration of the frequency-domain effects of time-domain sampling



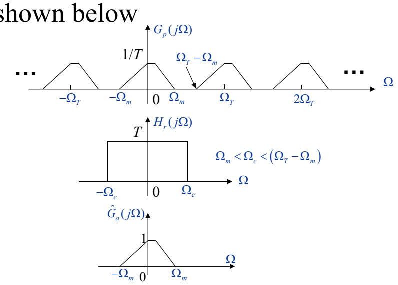
2.1 Effect of Sampling in the Frequency Domain

- ➡ If $\Omega_T \ge 2\Omega_m$, $g_a(t)$ can be recovered exactly from $g_p(t)$ by passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c greater than Ω_m and less than $\Omega_T - \Omega_m$ as shown below



2.1 Effect of Sampling in the Frequency Domain

- The spectra of the filter and pertinent signals are shown below



2.1 Effect of Sampling in the Frequency Domain

- On the other hand, if $\Omega_T < 2\Omega_m$, due to the overlap of the shifted replicas of $G_a(j\Omega)$, the spectrum $G_a(j\Omega)$ cannot be separated by filtering to recover because of the distortion caused by a part of the replicas $G_a(j\Omega)$ immediately outside the baseband folded back or aliased into the baseband.
- Several terms:

2.1 Effect of Sampling in the Frequency Domain

	$\Omega_T \geq 2\Omega_m$	**Nyquist condition**
	$\Omega_T / 2$	**Folding frequency**
	Ω_m	**Nyquist frequency**
	$2\Omega_m$	**Nyquist rate**
When	$\Omega_T > 2\Omega_m$	**Oversampling**
When	$\Omega_T < 2\Omega_m$	**Undersampling**
When	$\Omega_T = 2\Omega_m$	**Critical sampling**

Note: A pure sinusoid may not be recoverable from its critically sampled version

2.1 Effect of Sampling in the Frequency Domain

Example 1

- In high-quality analog music signal processing, a bandwidth of 20 kHz has been determined to preserve the fidelity (保真度)
- Hence, in compact disc (CD) music systems, a sampling rate of 44.1 kHz, which is slightly higher than twice the signal bandwidth, is used

2.1 Effect of Sampling in the Frequency Domain

Example 2

- Consider the three continuous time sinusoidal signals:

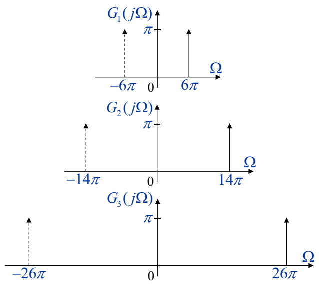
$$g_1(t) = \cos(6\pi t),\ g_2(t) = \cos(14\pi t),\ g_3(t) = \cos(26\pi t)$$

- Their corresponding CTFTs are:

$$G_1(j\Omega) = \pi\left[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)\right]$$
$$G_2(j\Omega) = \pi\left[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)\right]$$
$$G_3(j\Omega) = \pi\left[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)\right]$$

2.1 Effect of Sampling in the Frequency Domain

- These three transforms are plotted below



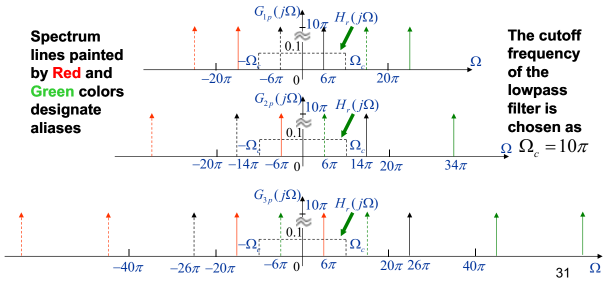
2.1 Effect of Sampling in the Frequency Domain

- These continuous-time signals sampled at a rate of $T = 0.1$ sec, i.e., with a sampling frequency $\Omega_T = 20\pi$ rad/sec
- The sampling process generates the continuous-time impulse trains, $g_{1p}(t)$, $g_{2p}(t)$, and $g_{3p}(t)$
- Their corresponding CTFTs are given by

$$G_{lp}(j\Omega) = 10 \sum_{k=-\infty}^{\infty} G_l(j(\Omega - k\Omega_T)), \quad 1 \leq l \leq 3$$

2.1 Effect of Sampling in the Frequency Domain

- Plots of the 3 CTFTs are shown below



2.1 Effect of Sampling in the Frequency Domain

- We now derive the relation between the DTFT of $g(n)$ and the CTFT of $g_p(t)$
- To this end we compare

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n}$$

with

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$

and make use of

$$g(n) = g_a(nT) \quad -\infty < n < \infty$$

2.1 Effect of Sampling in the Frequency Domain

- Observation: We have

$$G(e^{j\omega}) = G_p(j\Omega)\big|_{\Omega=\omega/T}$$

or, equivalently,

$$G_p(j\Omega) = G(e^{j\omega})\big|_{\omega=\Omega T}$$

- From the above observation and

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

2.1 Effect of Sampling in the Frequency Domain

- We arrive at the desired result given by

$$G(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - jk\Omega_T\right)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} - jk\Omega_T\right)$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

2.1 Effect of Sampling in the Frequency Domain

- The relation derived on the previous slide can be alternately expressed as

$$G(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - jk\Omega_T\right)$$

form $G(e^{j\omega}) = G_p(j\Omega)\big|_{\Omega=\omega/T}$

or from $G_p(j\Omega) = G(e^{j\omega})\big|_{\omega=\Omega T}$

2.1 Effect of Sampling in the Frequency Domain

- It follows that $G(e^{j\omega})$ is obtained from $G_p(j\Omega)$ by applying the mapping $\Omega = \omega / T$
- Now, the CTFT $G_p(j\Omega)$ is a periodic function of Ω with a period $\Omega_T = 2\pi / T$
- Because of the mapping, the DTFT $G(e^{j\omega})$ is a periodic function of ω with a period 2π

2.2 Recovery of the Analog Signal

- We now derive the expression for the output $\hat{g}_a(t)$ of the ideal lowpass reconstruction filter $H_r(j\Omega)$ as a function of the samples $g(n)$
- The impulse response $h_r(t)$ of the lowpass reconstruction filter is obtained by taking the inverse DTFT of :

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

2.2 Recovery of the Analog Signal

- Thus, the impulse response is given by

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\Omega_c t/2}, \quad -\infty \le t \le \infty$$

- The input to the lowpass filter is the impulse train $g_p(t)$

$$g_p(t) = \sum_{n=-\infty}^{\infty} g(n)\delta(t-nT)$$

2.2 Recovery of the Analog Signal

- Therefore, the output $\hat{g}_a(t)$ of the ideal lowpass filter is given by:

$$\hat{g}_a(t) = g_p(t) * h_r(t) = \sum_{n=-\infty}^{\infty} g(n)h_r(t-nT)$$

- Substituting $h_r(t) = \sin(\Omega_c t)/\Omega_c t/2$ in the above and assuming $\Omega_c = \Omega_T/2 = \pi/T$ for simplicity, we get

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g(n)\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

2.2 Recovery of the Analog Signal

- It can be shown that when $\Omega_c = \Omega_T/2$ in

$$h_r(t) = \frac{\sin \Omega_c t}{\Omega_T t/2}$$

$h_r(0)=1$ and $h_r(nT)=0$ for $n \neq 0$

- As a result, from

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g(n)\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

we observe $\hat{g}_a(rT) = g(r) = g_a(rT)$

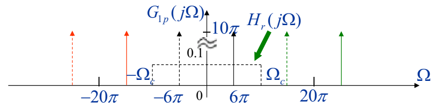
for all integer values of r in the range $-\infty < r < \infty$

2.2 Recovery of the Analog Signal

- The relation $\hat{g}_a(rT) = g(r) = g_a(rT)$ holds whether or not the condition of the sampling theorem is satisfied
- However, $\hat{g}_a(t) = g_a(t)$ for all values of t only if the sampling frequency Ω_T satisfies the condition of the sampling theorem

2.3 Implication of the Sampling Process

- Consider again the three continuous-time signals: $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$
- The plot of the CTFT $G_{1p}(j\Omega)$ of the sampled version of $g_1(t)$ is shown below



2.3 Implication of the Sampling Process

- From the plot, it is apparent that we can recover any of its frequency-translated versions $\cos\left[\left(20k \pm 6\right)\pi t\right]$ outside the baseband by passing through an ideal analog bandpass filter $g_{1p}(t)$ with a passband centered at $\Omega = \left(20k \pm 6\right)\pi$
- For example, to recover the signal $\cos(34\pi t)$, it will be necessary to employ a bandpass filter with a frequency response

$$H_r(j\Omega) = \begin{cases} 0.1, & (34-\Delta)\pi \le |\Omega| \le (34+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

A small number

2.3 Implication of the Sampling Process

- Likewise, we can recover the aliased baseband component $\cos(6\pi t)$ from the sampled version of either $g_{2p}(t)$ or $g_{3p}(t)$ by passing it through an ideal lowpass filter with a frequency response:

$$H_r(j\Omega) = \begin{cases} 0.1, & 0 \le |\Omega| \le (6+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

2.3 Implication of the Sampling Process

- There is no aliasing distortion unless the original continuous-time signal also contains the component $\cos(6\pi t)$
- Similarly, from either $g_{2p}(t)$ or $g_{3p}(t)$ we can recover any one of the frequency-translated versions, including the parent continuous-time signal $\cos(14\pi t)$ or $\cos(26\pi t)$ as the case may be, by employing suitable filters

3. Sampling of Bandpass Signals

- The conditions developed earlier for the unique representation of a continuous-time signal by the discrete-time signal obtained by uniform sampling assumed that the continuous-time signal is bandlimited in the frequency range from dc to some frequency
- Such a continuous-time signal is commonly referred to as a lowpass signal

3. Sampling of Bandpass Signals

- There are applications where the continuous-time signal is bandlimited to a higher frequency range $\Omega_L \le |\Omega| \le \Omega_H$ with $\Omega_L > 0$
- Such a signal is usually referred to as the bandpass signal
- To prevent aliasing a bandpass signal can of course be sampled at a rate greater than twice the highest frequency, i.e. by ensuring $\Omega_T \ge 2\Omega_H$

3. Sampling of Bandpass Signals

- However, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps
- Moreover, if Ω_H is very large, the sampling rate also has to be very large which may not be practical in some situations

3. Sampling of Bandpass Signals

- A more practical approach is to use under-sampling
- Let $\Delta\Omega = \Omega_H - \Omega_L$ define the bandwidth of the bandpass signal
- Assume first that the highest frequency Ω_H contained in the signal is an integer multiple of the bandwidth, i.e.,

$$\Omega_H = M(\Delta\Omega)$$

3. Sampling of Bandpass Signals

- We choose the sampling frequency Ω_T to satisfy the condition

$$\Omega_T = 2(\Delta\Omega) = \frac{2\Omega_H}{M}$$

which is smaller than $2\Omega_H$, the *Nyquist rate*

- Substitute the above expression for Ω_T in

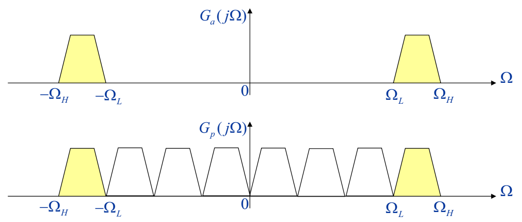
$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - jk\Omega_T\right)$$

3. Sampling of Bandpass Signals

- This leads to $G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - j2k(\Delta\Omega)\right)$
- As before, $G_p(j\Omega)$ consists of a sum of $G_a(j\Omega)$ and replicas of $G_a(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta\Omega$ and scaled by $1/T$
- The amount of shift for each value of k ensures that there will be no overlap between all shifted replicas ➡ *no aliasing*

3. Sampling of Bandpass Signals

- Figure below illustrates the idea behind



3. Sampling of Bandpass Signals

- As can be seen, $g_a(t)$ can be recovered from $g_p(t)$ by passing it through an ideal bandpass filter with a passband given by $\Omega_L \le |\Omega| \le \Omega_H$ and a gain of T
- Note: Any of the replicas in the lower frequency bands can be retained by passing $g_p(t)$ through bandpass filters with passbands $\Omega_L - k(\Delta\Omega) \le |\Omega| \le \Omega_H - k(\Delta\Omega)$, providing a translation to lower frequency ranges $1 \le k \le M-1$