Chapter 2: Analytic Functions

Li, Yongzhao

State Key Laboratory of Integrated Services Networks, Xidian University

September 28, 2010

Introduction

- Ch. 1 is focused on the algebraic operation of a complex number z
- From this chapter, we shall study function $f(z)$ defined on these complex variables
- Our objective is to mimic the concepts, theorems, and mathematical structure of calculus; such as **differentiating** and **integrating** the function $f(z)$
- The notation of a derivative is far more subtle in the complex case because of the intrinsically two-dimensional nature of the complex variables

Review of Functions of a Real Variable

- If f assigns the value y to the element x in A, we write
$$y = f(x)$$
and call y the image of x under f
- The set A is the **domain of definition** of f, and the set of all images $f(x)$ is the **range** of f
- Sometimes we refer to f as a **mapping** of A into B

Functions of a Complex Variable

- Now we consider the complex-valued functions of a complex variable
- The **domains of definition** and the **ranges** are subset of the complex numbers
- See an example:

$$f(z) = \frac{z^2 - 1}{z^2 + 1}$$

We take the domain of f to be the set of all z for which the formula is well defined (hence, $\pm i$ are excluded)

Functions of a Complex Variable (cont'd)

- Denote w as the value of the function $f(z)$ at point z. Then we write $w = f(z)$
- Just as z decomposes into real and imaginary parts as $z = x + iy$, the real and imaginary parts of w are each (real) functions of z or, equivalently, of x and y, and so we customarily write

$$w = u(x, y) + iv(x, y)$$

where u and v denoting the real and imaginary parts, respectively, of w
- Thus a complex valued function of a complex variable is a pair functions of two real variables (**Example** 1 on page 54)

Functions of a Complex Variable (cont'd)

- Unfortunately, it is generally impossible to draw the graph of a complex function; to display two real functions of two real variables graphically would require four dimensions
- Instead, we can visualize some of the properties of a complex function $w = f(z)$ by sketching of domain of definition in the z-plane and its range in the w-plane (**Examples** 2 and 3)
- The function $f(z) = 1/z$ is called the inversion mapping. It is an example of a one-to-one function because it maps distinct points to distinct points, i.e., if $z_1 \neq z_2$, then $f(z_z) \neq f(z_2)$

Limit of a Sequence of Complex Numbers

- The definition of absolute value can be used to designate the distance between two complex numbers
- Having a concept of distance, we can proceed to introduce the notions of limit and continuity
- When we have an infinite sequences $z_1, z_2, z_3, \cdots$ of complex numbers, we say that the number z_0 is the limit of the sequence if the z_n eventually (i.e., for large enough n) stay arbitrarily close to z_0

Definition of Limit of a Sequence of Complex Numbers

Definition

A sequence of complex numbers $\{z_n\}_1^\infty$ is said to have the limit z_0 or to converge to z_0 and we write

$$\lim_{n\to\infty} z_n = z_0$$

or equivalently,

$$z_n \to z_0 \quad \text{as} \quad n \to \infty$$

if for any $\varepsilon > 0$, there exists an integer N such that $|z_n - z_0| < \varepsilon$ for all $n > N$ (see Fig. 2.3 on page 59)

Definition of Limit of a Complex-Valued Function

Definition

Let f be a function defined in some neighborhood of z_0, with the possible exception of the point z_0 itself. We say that the limit of $f(z)$ as z approaches z_0 is the number w_0 and write

$$\lim_{z\to\ z_0} f(z) = w_0$$

or equivalently,

$$f(z) \to w_0 \quad \text{as} \quad z \to\ z_0$$

if for any $\varepsilon > 0$, there exists a positive number δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

Relation between the limit of a function and the limit of a sequence

- If $\lim_{z\to z_0} f(z) = w_0$, then for every sequence $\{z_n\}_1^\infty$ converging to $z_0(z_n \neq z_0)$ the sequence $\{f(z_n)\}_1^\infty$ converges to w_0, and vice versa
- The definitions of this section are direct analogous of concepts introduced in elementary calculus. Hence, many of the familiar theorems on real sequences, limits, and continuity remain valid in the complex case

Condition of Continuity

Definition

Let f be a function defined in a neighborhood of z_0. Then f is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

- In other words, for f to be continuous at z_0, it must have a limiting value at z_0, and this limiting value must be $f(z_0)$
- A function f is said to be continuous on a set S if it is continuous at each point of S

Some Comments

- One can show that $f(z)$ approaches a limit precisely when its real and imaginary parts approach limits
- Theorems 1 and 2 on page 61 are also derived from the familiar theorems on real sequences
- It is easy to show that the constant function $f(z) = z$ is continuous on the whole plane C. Then, we can deduce that:
  - The polynomial function $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ is continuous on the whole plane
  - The rational function $\dfrac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + a_2 z^2 + \ldots + b_m z^m}$ is continuous at each point where the denominator does not vanish

Limits Involving Infinity

- We say "$z_n \to \infty$" if, for each positive number M (no matter how large), there is an integer N such that $|z_n| > M$ whenever $n > M$
- Similarly,"$\lim_{z_n \to z_0} f(z) = \infty$" means that for each positive number M (no matter how large), there is a $\delta > 0$ such that $|f(z)| > M$ whenever $0 < |z - z_0| < \delta$
- Essentially, we are saying that complex numbers approach infinity when their magnitudes approach infinity

Distinction Between the Concepts of Real and Complex Cases

- There is an important distinction between the concepts of limit in the (one-dimensional) real and (two-dimensional) complex case
- For latter situation, a sequence $\{z_n\}_1^\infty$ may approach a limit z_0 from any direction in the plane, or even along a spiral
- Thus, the manner in which a sequence of numbers approaches its limit can be much more complicated in the complex case

Introduction

- The theory of analytic functions is the main topic of this course. Before we discuss this topic, we will give an informal preview of what it is we want to achieve.
- In the real calculus, we don't deal with the function that looks like $3 + 4\sqrt{2}$.
- This is because we we treat this number as an indivisible module and we don't ever perform separate algebraic operations on integer part and a different operation on the $\sqrt{\cdot}$ part
- We seek to classify the complex functions that behave this same way with regard to their complex argument

Introduction (Cont'd)

- We treat the complex functions in the same way, i.e., we treat the complex variable z as a single quantity and don't perform different algebraic operations on the x and a different one on the y.
- Actually, we did in the same way in finding the limit of a complex sequence or function
- For a complex variable $z = x + iy$, the complex function $f(z)$ also has its real and imaginary parts u and v:

$$f(z) = u + iv = u(x, y) + iv(x, y)$$

Introduction (Cont'd)

- We want to admit the functions such as: z, z^2, z^3, and $1/z$, and there basic arithmetic combinations (sums, products, quotients, powers, and roots)
- But we want to ban such functions as $\Re z = x$, $\Im z = y$, and $x^2 - y^2 + i3xy$
- $\overline{z}$ is also banned because if we admit it we will open the gate to $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$
- Similarly, admitting $|z|$ would be a mistake as well, since $\overline{z} = |z|^2/z$
- The function e^z is more vexing. We suspect it to be admissible but postpone the official verification until the next section

Derivative of a Complex Function

- In the following chapters, we will see that the criterion of analyticity we are seeking can be expressed simply in terms of differentiability

Definition

Let f be a complex-valued function defined in a neighborhood of z_0. Then the derivative of f at z_0 is given by

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0))}{\triangle z}$$

provided this limit exists. (Such as f is said to be differentiable at z_0)

Derivative of a Complex Function (Cont'd)

- The catch here is that $\triangle z$ is a complex number, so it can approach zero in many different ways (from the right, from below, along a spiral, etc.); but the difference quotient must tend to a unique limit $f'(z_0)$ independent of the manner in which $\triangle z \to 0$
- Example 2 on page 67-68 shows why analyticity disqualifies $\overline{z}$
- Theorem 3 on page 69 is corresponding to the rules of elementary calculus
- It should be also noted that differentiability implies continuity, as in the real case
- We see then that for purpose of differentiation, polynomial and rational functions in z can be treated as if z were a real variable

Definition of Analyticity

Definition

A complex-valued function $f(z)$ is said to be analyticity on an open set G if it has a derivative at every point of G

- Here we emphasize that analyticity is a property defined over open sets, while differentiability could conceivably hold at one point only
- A point where f is not analytic but which is the limit of points where f is analytic is known as a singular point or singularity
- If $f(z)$ is analytic on the whole plane, then it is said to be entire

Comments to Analyticity

- As we will see in the next few chapters, analyticity is the criterion that we have been seeking, for functions to respect the complex structure of the variable z
- We will demonstrate later that all analytic fucntions can be written in terms of z alone (not x, y, or $\overline{z}$)
- When a function is given in terms of real and imaginary parts as $u(x,y)+iv(x,y)$, it may be very tedious to apply the definition to determine if f is analytic
- The next section will provide a test that is easier to use

The Relationship Between $u(x,y)$ and $v(x,y)$

- The property of analyticity for a function indicates some type of connection between its real and imaginary parts
- If $f(z)=u(x,y)+iv(x,y)$ is differentiable at $z_0=x_0+iy_0$, then the limit
$$f'(z_0)=\lim_{\triangle z\to 0}\frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}$$
can be computed by allowing $\triangle z=\triangle x+i\triangle y$ to approach zero from any convenient direction in the complex plane

The Relationship Between $u(x,y)$ and $v(x,y)$ (Cont'd)

- If it approaches horizontally, then $\triangle z=\triangle x$, we obtain

$$\begin{aligned} f'(z_0) &= \lim_{\triangle x\to 0}\frac{u(x_0+\triangle x,y_0)+iv(x_0+\triangle x,y_0)-u(x_0,y_0)-iv(x_0,y_0))}{\triangle x} \\ &= \lim_{\triangle x\to 0}\left[\frac{u(x_0+\triangle x,y_0)-u(x_0,y_0)}{\triangle x}\right]+i\lim_{\triangle x\to 0}\left[\frac{v(x_0+\triangle x,y_0)-v(x_0,y_0)}{\triangle x}\right]\end{aligned}$$

- Since the limits of the bracketed expressions are just the first partial derivatives of u and v with respect to x, we deduce that
$$f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0) \qquad (1)$$

The Relationship Between $u(x, y)$ and $v(x, y)$ (Cont'd)

- If $\triangle z$ approaches zero vertically, then $\triangle z = i\triangle y$ and we obtain the following relation similarly

$$f'(z_0) = \lim_{\triangle y \to 0} \left[\frac{u(x_0, y_0 + \triangle y) - u(x_0, y_0)}{i\triangle y}\right] + i \lim_{\triangle y \to 0} \left[\frac{v(x_0 +, y_0 + \triangle y) - v(x_0, y_0)}{i\triangle y}\right]$$

- Hence

$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \qquad (2)$$

Cauchy-Riemann Equations

- By equating real and imaginary part in (1) and (2), we get the famous Cauchy-Riemann Equations as follows
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
- A necessary condition for a function $f(z) = u(x, y) + iv(x, y)$ to be differentiable at point z_0 is that the Cauchy-Riemann equation hold at z_0
- Consequently, if f is analytic in an open set G, then then Cauchy-Riemann equations must hold at every point of G

Cauchy-Riemann Equations (Cont'd)

- An easy way to recall the Cauchy-Riemann equations: Horizontal derivative must equal the vertical derivative, i.e.,
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial iy} \quad \text{or} \quad \frac{\partial (u + iv)}{\partial x} = \frac{\partial (u + iv)}{\partial iy}$$
- By equating the real and imaginary parts, we get
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Comments to Cauchy-Riemann Equations

- The Cauchy-Riemann equations alone are not sufficient to ensure differentiability. One needs the additional hypothesis of continuity of the first partial derivatives of u and v
- Theorem 5: Let $f(z) = u(x, y) + iv(x, y)$ be defined in some open set G containing the point z_0. If the first derivatives of u and v exist in G, are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0, then f if differentiable at z_0
- Theorem 6: If $f(z)$ is analytic in a domain D and if $f'(z) = 0$ everywhere in D, then $f(z)$ is constant in D

Comments to Cauchy-Riemann Equations (Cont'd)

- One easy consequence of Theorem 6 is the fact that if f and g are two functions analytic in a domain D whose derivatives are identical in D, then $f = g + constant$ in D
- Using Theorem 6 and Cauchy-Riemann equations, you can further show that an analytic function $f(z)$ must be constant when any one of the following conditions hold in a domain D:

  $\Re f(z)$ is constant

  $\Im f(z)$ is constant

  $|f(z)|$ is constant
- We can also use Cauchy-Riemann Equations and continuity of the partial derivatives to verify the analyticity of e^z