Chapter 1: Complex Numbers

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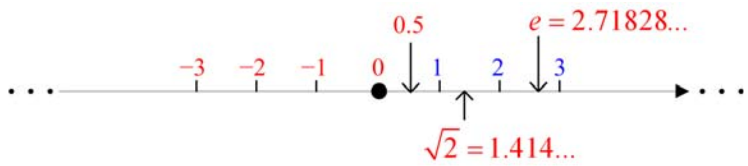
Review of Real Numbers

- Initially, we learned the **positive integers** $1, 2, 3, \ldots$
- **Zero** 0 is an interesting number
- Sometimes we need to calculate the equation $2 - 8$, so we introduced the solution -6 which is a **negative integer**
- An apple is cut into two pieces, each is half (0.5)
- Integers and **fractions** constitutes the **rational number system** (a/b)
- One solution to the equation $x^2 = 2$ is $\sqrt{2}$ which is an **irrational number**
- Rational and irrational numbers form the **real number system**

Review of Real Numbers (Cont.)



- We can compare the magnitudes of any two real numbers (larger, equal or smaller)
- One dimensional (represented by a straight line)
- Are real numbers enough?

Extend Real Numbers to Complex Numbers

- The problem of solving the equation $x^2 = -1$
  - One solution is $\sqrt{-1}$ (not a real number)
  - Use a symbol i (or j) to designate $\sqrt{-1}$
  - We get: $i^2 = -1$
- With the aid of i, we get the definition of a **Complex Number**

$$z := a + bi$$

where real numbers $a := \mathrm{Re}z = \Re z$ and $b := \mathrm{Im}z = \Im z$ are called the **Real Part** and **Imaginary Part** of z

Basic Operations of Complex Numbers

- Addition (or subtraction)

$$(a + bi) \pm (c + di) := (a \pm c) + (b \pm d)i$$

- Multiplication

$$(a + bi)(c + di) := (ac - bd) + (bc + ad)i$$

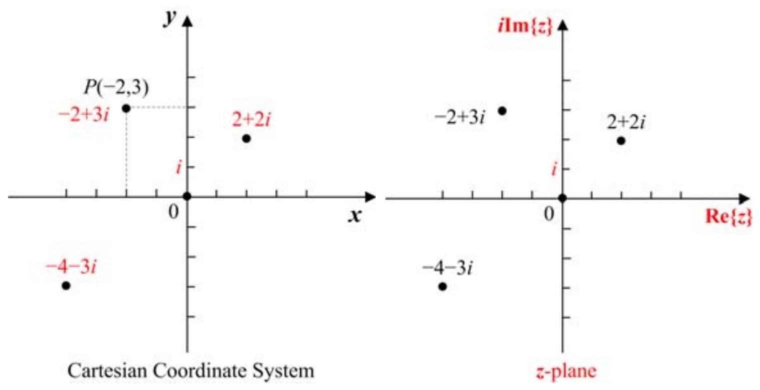
- Division

$$\frac{(a + bi)}{(c + di)} := \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Comments to Complex Numbers

- The set of all complex numbers is denoted as **C** (**R** for reals)
- No nature ordering for the elements of **C**
- The real part and imaginary part are independent of each other
- A complex number can be represented as a point in a two-dimensional plane
- Or it can be viewed as a vector with two entries $(a \quad b)$
- All reals are complex (a line in the two-dimensional plane)

Representing Complex Numbers in z-plane



Argand Diagram

Absolute Value of a Complex Number

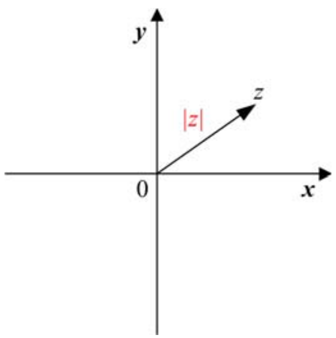
- The distance between two points $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ in the z-plane is $\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$
- When $z_2 = 0$ is the origin of the z-plane, we get the **absolute value** (or **modulus**) of z_1 which is denoted by $|z_1| := \sqrt{{a_1}^2 + {b_1}^2}$
- Hence, the distance between z_1 and z_2 can be written as $|z_1 - z_2|$
- Equation $|z - z_0| = r$ (where z_0 is a fixed complex number and r is a fixed non-negative real number) describes a circle of radius r centered at z_0

Complex Conjugate of a Complex Number

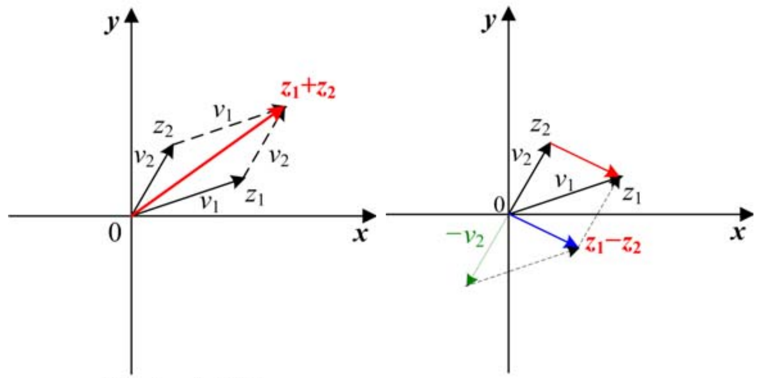
- Complex Conjugate ($\bar{z}$ or z^*)
  $$\bar{z} = \overline{a + bi} := a - bi$$
- The function of complex conjugate is to change the sign of the imaginary part of a complex number
- Features
  - $\mathrm{Re}z = a = (z + \bar{z})/2 \quad \mathrm{Im}z = b = (z - \bar{z})/2i \quad \overline{(\bar{z})} = z,$
  - $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$, $\overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}$
  - $|z| = |\bar{z}|, z\bar{z} = |z|^2$
  - $1/z = \bar{z}/|z|^2$

Vectors in the Complex-plane

- Each point z in the complex plane corresponds to a directed line segment from the origin to the point z
- The vector is determined by its length and direction
- The length equals to the modulus of z, namely $|z|$



Vector Addition and Subtraction



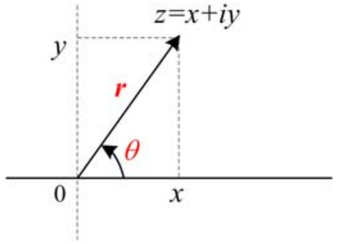
Vector Addition
The diagonal of the parallelogram

Vector Subtraction

Vector Addition and Subtraction (cont.)

- Parallelogram Law for addition of two vectors (or complex numbers)
- Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$
  The length of any side of a triangle is no greater than the sum of the lengths of the other two sides
- Corollary:$|z_2| \leq |z_1| + |z_2 - z_1| \Longrightarrow |z_2| - |z_1| \leq |z_2 - z_1|$
  The difference of the lengths of any two sides of a triangle is no greater than the length of the third side

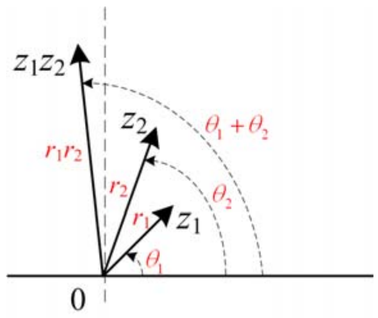
Polar Forms of Complex Numbers



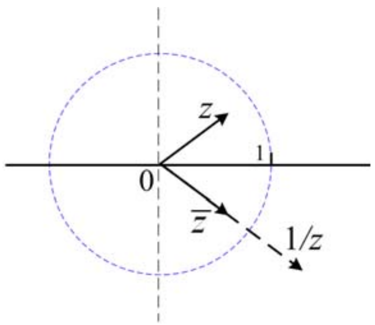
Polar Coordinates

- $(x, y) \Longrightarrow (r, \theta)$
- $x = r\cos\theta, \quad y = r\sin\theta$
- $z = r(\cos\theta + i\sin\theta) = r\text{cis}\theta$
- $r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \begin{cases} \arctan(y/x), & x > 0; \\ \arctan(y/x) \pm \pi, & x < 0 \end{cases}$
  I to IV quadrant
- $\arg z = \text{Arg}z + 2k\pi$
  $(k = 0, \pm 1, \pm 2, \ldots)$
  $\text{Arg}z$ is the principal value of $\arg z$
- *Q: How to represent a product of two complex numbers in a 2-D plane?*

Another Two Examples



Geometric interpretation of the product



Conjugate and the reciprocal

Euler's Equation

- The real exponential function $f(x) = e^x$ where x is a real number
- By replacing x with $z = x + iy$, we get the complex exponential function $f(z) = e^z$
- First, we postulate that the multiplication property should persist: $e^{x+iy} = e^x e^{iy}$, where e^x is still a real number and the second part e^{iy} needs to be defined
- According to Taylor' series expansion, we get the following equation

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \ldots \tag{1}$$

Euler's Equation (Cont'd)

- By replacing y with iy in Eq. (1), we get the Taylor's expansion of e^{iy} as follows

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \dots \quad (2)$$

- We know the identities

$$\begin{array}{l} i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1 \\ i^5 = i, \quad i^6 = -1, \quad i^7 = -i, \quad i^8 = 1, \dots \end{array} \quad (3)$$

- Hence, we deduce that $i^n = i^{n+4}$ is periodical function with period 4

Euler's Equation (Cont'd)

- By separating the real part and imaginary part of e^{iy}, Eq. (2) can be rewritten as

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) \quad (4)$$

- Note that the real part and the imaginary part of the above are just the Taylor's expansions of $\cos y$ and $\sin y$, respectively
- Hence, We arrive at the famous **Euler's equation** as follows

$$e^{iy} = \cos y + i \sin y$$

- By using the Euler's equation, we have the definition of a complex exponential function: $e^z := e^x(\cos y + i \sin y)$

Comments to Euler's Equation

- Since $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$, e^{iy} is a vector which locates on the circle of radius 1 about origin
- y is the angle of inclination of the vector e^{iy}, measured positively in a counterclockwise sense from the positive real axis
- Recall that any complex number z can be written as the polar form: $z = r(\cos\theta + i\sin\theta)$
- Euler's equation enables us to write it in another form: $z = re^{i\theta} = |z|e^{i \arg z}$

Application of Complex Exponential

- $\cos\theta = \Re e^{i\theta} = \dfrac{e^{i\theta} + e^{-i\theta}}{2}; \quad \sin\theta = \Im e^{i\theta} = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$
- Multiplication of two complex numbers:

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = (r_1 r_2) e^{i(\theta_1+\theta_2)}$$

- Division of two complex numbers:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

- Complex Conjugate: $\overline{z} = re^{-i\theta}$
- *De Moivre's Formula*: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$
  Q: Does this formula hold for arbitrary integers n (positive or negative)?

Powers of a Complex Number

- We can represent the complex number z in its polar form: $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
- The n-th power of z is calculated by two steps:
  - Step 1: The n-th power of the modulus: r^n
  - Step 2: The n-fold of the angle of inclination: $n\theta$
- Finally, we get the n-th power of z, namely, $z^n = r^n e^{in\theta}$
- The above rule is valid for both positive and negative integers
- The question arises whether the formula will work for $n = 1/m$

Roots of a Complex Number

- The computation of the roots is more complicated than powers
- Let $w = \rho(\cos\varphi + i\sin\varphi)$ be the m-th roots of $z = r(\cos\theta + i\sin\theta)$, so $w^m = z$ means

$$\rho^m(\cos m\varphi + i\sin m\varphi) = r(\cos\theta + i\sin\theta) \qquad (5)$$

- Eq.(5) means

$$\rho^m = r, \quad \cos m\varphi = \cos\theta, \quad \sin m\varphi = \sin\theta$$

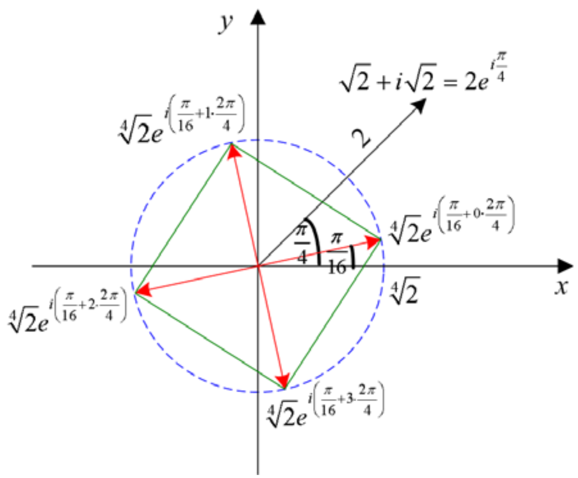
- Hence, $\rho = r^{1/m}, \quad m\varphi = \theta + 2k\pi \Longrightarrow \varphi = \dfrac{\theta + 2k\pi}{m}$

Roots of a Complex Number (Con't)

- When $k = 0, 1, 2, \ldots, m-1$, we get the m distinct roots for Eq.(5) as $\left\{ w_k = r^{1/m}\left(\cos\dfrac{\theta + 2k\pi}{m} + i\sin\dfrac{\theta + 2k\pi}{m}\right)\right\}$
- When $k = m, m+1, m+2, \ldots, 2m-1$, the same roots repeat again,...
- Hence, there are only m distinct roots for $z^{1/m}$

An Example of Finding the Roots

Find the Four fourth roots of $\sqrt{2} + i\sqrt{2}$



Planar Sets

- In the calculus of functions of a real variable, the main theorems are typically stated for functions defined on an interval, such as $(0, 1), \quad (0, 1], \quad [0, 1), \quad [0, 1]$
- The interval can be interpreted as a segment in the x-axis in z-plane
- A complex number is two-dimensional, hence for the functions of a complex variable, the basic results are formulated for functions defined on sets that are 2-dimensional "domains" or "closed regions"

Open Disk (Neighborhood) and Open Set

- The set of all points that satisfy the inequality
$$|z - z_0| < \rho$$
where ρ is a positive number, is called an **open disk** or **circular neighborhood** of z_0
- A point z_0 which lies in a set **S** is called an **interior point** of **S** if there is some circular neighborhood of z_0 that is completely contained in **S**
- If every point of a set **S** is an interior point of **S**, we say that **S** is an **open set**

Domain

- An open set **S** is said to be **connected** if every pair of points z_1, z_2 in **S** can be joined by a polygonal path that lies entirely in **S**. Roughly speaking, this means that **S** consists of a "Single Piece"
- An open connected set is called a **domain**
- For real variables, the derivative of the function equals zero implies that this function is identically constant on the defined interval

Domain (Cont'd)

- The extension result to functions of two real variables: Suppose $u(x, y)$ is a real-valued function defined in a domain D. If the first partial derivative of u satisfy
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$
at all points of D, then $u \equiv$constant in D
- If D is merely assumed to be an open set (not connected), the theorem is no longer true

Boundary

- A point z_0 is said to be a **boundary point** of a set S if every neighborhood of z_0 contains at least one point not in S
- The set of all boundary points of S is called the **boundary** or **frontier** of S
- Since each point of a domain D is an interior point of D, it follows that a domain cannot contain any of its boundary points
- A set S is said to be **closed** if it contains all of its boundary points. The set of points z that satisfy the inequality $|z - z_0| \leq \rho \quad (\rho > 0)$ is a **closed set**, for it contains its boundary $|z - z_0| = \rho$. We call this set a **closed disk**

Bounded and Region

- A set of points S is said to be **bounded** if there exists a positive real number R such that $|z| < R$ for every z in S
- A set is both closed and bounded is said to be **compact**
- A **region** is a domain together with some, none, or all of its boundary points. In particular, every domain is region