Chapter 5: Series Representations for Analytic Functions

Li, Yongzhao

State Key Laboratory of Integrated Services Networks, Xidian University

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Introduction

- In Ch. 2 we defined what is meant by convergence of a sequence of complex numbers; recall that the sequence $\{A_n\}_{n=1}^{\infty}$ has A as a limit if $|A - A_n|$ can be made arbitrarily small by taking n large enough
- For computational convenience it is often advantageous to use an element A_n of the sequence as an approximation to A
- The use of sequences, and in particular the kind of sequences associated with series, is an important tool in both the theory and applications of analytic functions
- This chapter is devoted to the development of this subject

Definition of a Series

Definition

A **series** is a formal expression of the form $c_0 + c_1 + c_2 + \cdots$, or equivalently $\sum_{j=0}^{\infty} c_j$, where the **terms** c_j are complex numbers. The n-th **partial sum** of the series, usually denoted S_n, is the sum of the first $n+1$ terms, that is, $S_n := \sum_{j=0}^{n} c_j$. If the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ has a limit S, the series is said to **converge**, or **sum** to S, and we write $S = \sum_{j=0}^{\infty} c_j$. A series that does not converge is said to **diverge**

- One way to demonstrate that a series converges to S is to show that the reminder after summing the first $n+1$ terms, $S - \sum_{j=0}^{n} c_j$, goes to zero as $n \to \infty$

Comparison and Ratio Tests

Theorem

(**Comparison Test**) *Suppose that the terms* c_j *satisfy the inequality*

$$|c_j| \leq M_j$$

for all integers j *larger that some number* J. *Then if the series* $\sum_{j=0}^{\infty} M_j$ *converge, so does* $\sum_{j=0}^{\infty} c_j$

Theorem

(**Ratio Test**) *Suppose that the terms of the series* $\sum_{j=0}^{\infty} c_j$ *have the property that the ratios* $|c_{j+1}/c_j|$ *approach a limit* L *as* $j \to \infty$. *Then the series converges if* $L < 1$ *and diverges if* $L > 1$

Uniform Convergence

- If we have a sequence of functions $F_1(z)$, $F_2(z)$, $F_3(z)$, $\ldots$, we must consider the possibility that for some values of z the sequence converges, while for others it diverges
- Similarly, a series of complex functions $\sum_{j=0}^{\infty} f_j(z)$ may **converge** for some values of z and **diverge** for others
- In applying this theory to analytic functions we need a somewhat stronger notion of convergence
- Figure 5.1 (page 238) shows an example of '**pointwise convergence**'

Uniform Convergence (Cont'd)

Definition

The sequence $\{F_n(z)\}_{n=1}^{\infty}$ is said to **converge uniformly to** $F(z)$ on the set T if for any $\varepsilon > 0$ there exists an integer N such that when $n > N$,

$$|F(z) - F_n(z)| < \varepsilon \qquad \text{for all } z \text{ in } T$$

Accordingly, the series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly to $f(z)$ on T if the sequence of its partial sums converges uniformly to $f(z)$ there

Uniform Convergence (Cont'd)

- The essential feature of uniform convergence is that for a given $\varepsilon > 0$, one must be able to find an integer N that is independent of z in T such that the error $|F(z) - F_n(z)|$ is less than ε for $n > N$
- In contrast, for pointwise convergence, N can depend upon z. Of course, uniform convergence on T implies pointwise convergence on T
- Example 3 and 4 show that the series $\sum_{j=0}^{\infty}(z/z_0)^j$ converges pointwise in the open disk $|z| < |z_0|$ and uniformly on any closed subdisk $|z| \leq r < |z_0|$

Introduction

- In Sec. 3.1, we learned the Taylor form of the polynomial $p_n(z)$, centered at z_0
- Suppose we want to find a polynomial $p_n(z)$ of degree at most n that approximates an analytic function $f(z)$ in a neighborhood of a point z_0
- Naturally there are differing criteria as to how well the polynomial approximates the function
- We shall construct a polynomial that "looks like" $f(z)$ at the point z_0 in the sense that its derivatives match those of f at z_0

Definition of Taylor Series

- The n-th-degree polynomial that matches f, f', f'', ..., $f^{(n)}$ at z_0 is

$$p_n(z) = f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\ldots+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Definition

If f is analytic at z_0, then the series

$$f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\ldots=\sum_{j=0}^{\infty}\frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$$

is called the **Taylor series** for f around z_0. When $z_0 = 0$, it is known as the **Maclaurin series** of f

Convergence of Taylor Series

Theorem

If f is analytic in the disk $|z-z_0| < R$, then the Taylor series converges to $f(z)$ for all z in this disk. Furthermore, the convergence of the series is uniform in any closed subdisk $|z-z_0| \leq R' < R$

- The theorem implies that the Taylor series will converge to $f(z)$ everywhere inside the largest open disk, centered at z_0, over which f is analytic

Derivatives of Taylor Series

Theorem

If f is analytic at z_0, the Taylor series for f' around z_0 can be obtained by termwise differentiation of the Taylor series for f around z_0 and converges in the same disk as the series for f

Linearity of Taylor Series

Theorem

If f and g be analytic functions with Taylor series $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$ and $g(z) = \sum_{j=0}^{\infty} b_j(z-z_0)^j$ around the point z_0 [that is $a_j = f^{(j)}(z_0)/j!$ and $b_j = g^{(j)}(z_0)/j!$]. Then
(i) the Taylor series for $cf(z)$, c a constant, is $\sum_{j=0}^{\infty} ca_j(z-z_0)^j$
(ii) the Taylor series for $f(z) \pm g(z)$ is $\sum_{j=0}^{\infty}(a_j \pm b_j)(z-z_0)^j$

Product of Two Taylor Series

Definition

The **Cauchy product** of two Taylor series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ and $\sum_{j=0}^{\infty} b_j(z-z_0)^j$ is defined to be the (formal) series $\sum_{j=0}^{\infty} c_j(z-z_0)^j$, where c_j is given by

$$a_j b_0 + a_{j-1}b_1 + a_{j-2}b_2 + \ldots + a_1 b_{j-1} + a_0 b_j = \sum_{l=0}^{j} a_{j-l}b_l$$

Theorem

Let f and g be analytic functions with Taylor series $f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j$ and $g(z) = \sum_{j=0}^{\infty} b_j(z-z_0)^j$ around the point z_0. Then the Taylor series for the product **fg** *around z_0 is given by the Cauchy product of these two series*

Comments

- The proof of the validity of the Taylor expansion substantiates the claim, made in Sec. 2.3, that **any analytic function can be displayed with a formula involving z alone**, and not $\overline{z}$, x, or y

Definition of Power Series

- Actually, a Taylor series for an analytic function appears to be a special instance of a certain general type of series of the form $\sum_{j=0}^{\infty} a_j(z-z_0)^j$. Such series have a name of **Power Series**

Definition

A series of the form $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ is called a **power series**. The constants a_j are the **coefficients** of the power series

The Goal of This Section

- Consider an arbitrary power series, such as

$$\sum_{j=0}^{\infty} \frac{z^j}{(j+1)^2} = 1 + \frac{z}{4} + \frac{z^2}{9} + \frac{z^3}{16} + \cdots$$

- We will answering the following questions
  - For what values of z does the series converge?
  - Is the sum an analytic function?
  - Is the power series representation of a function unique?
  - Is every power series a Taylor series?

Convergence of the Power series

Theorem
For any power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ there is a real number R between 0 and ∞, inclusive, which depends only on the coefficients $\{a_j\}$, such that

(i) *the series converges for $|z-z_0| < R$*

(ii) *the series converges uniformly in any closed subdisk $|z-z_0| \le R' < R$*

(iii) *the series diverges for $|z-z_0| > R$.*

The number R is called the **radius of convergence** of the power series

Convergence of the Power series (Cont'd)

- In particular, when $R = 0$ the power series converges only at $z = z_0$, and when $R = \infty$ the series converges for all z
- For $0 < R < \infty$, the circle $|z-z_0| = R$ is called the **circle of convergence**, but no general convergence statement can be made for z lying on this circle

Lemma
If the power series $\sum_{j=0}^{\infty} a_j z^j$ converges at a point having modulus r, then it converges at every point in the disk $|z| < r$

Convergence of the Power series (Cont'd)

- To see the existence of the number R in Theorem for the power series $\sum_{j=0}^{\infty} a_j z^j$ we reason informally as follows:
  - Consider the set of all real numbers r such that the series converges at some point having modulus r
  - Let R be the "largest" of these numbers r
  - Then by Lemma, the series converges for $|z| < R$, and from the definition of R, the series diverges for all z with $|z| > R$
- If z is replaced by $(z-z_0)$, we deduce that the region of convergence of the general power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ must be a disk with center z_0
- The formula for the radius of convergence R will be given in Sec. 5.4 (will not be covered)

Uniform Convergence

- Uniform convergence is a powerful feature of a sequence, as the next three results show
- The first says that the uniform limit of continuous functions is itself continuous

Lemma
Let f_n be a sequence of functions continuous on a set $T \subset \mathbf{C}$ and converging uniformly to f on T. Then f is also continuous on T

Uniform Convergence (Cont'd)

- Knowing that the uniform limit of a sequence of continuous functions is continuous, we can integrate this limit. In fact the integral of the limit is the limit of integrals

Theorem
Let f_n be a sequence of functions continuous on a set $T \subset \mathbf{C}$ containing the contour Γ, and suppose that f_n converges uniformly to f on T. Then the sequence $\int_\Gamma f_n(z)dz$ converges to $\int_\Gamma f(z)dz$

- Combining these results with Morera's theorem (page 210), we can prove the following theorem in the next slide

Uniform Convergence (Cont'd)

Theorem
Let f_n be a sequence of functions analytic in a simple connected domain D and converging uniformly to f in D. Then f is analytic in D

- Since the partial sums of a power series are analytic functions (indeed, polynomials) and since they converge uniformly in any closed subdisk interior to the circle of convergence, we know that the limit function is analytic inside every such subdisk

Uniform Convergence (Cont'd)

- But any point within the circle of convergence lies inside every such a subdisk, so we can state the following

Theorem
A power series sums to a function that is analytic at every point inside its circle of convergence

Relationship Between Power Series and Taylor Series

Theorem

If $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ converges to $f(z)$ in some neighborhood of z_0 (that is, the radius of its circle of convergence is nonzero), then

$$a_j = \frac{f^{(j)}(z_0)}{j!} \qquad (j = 0, 1, 2, \ldots)$$

Consequently, $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ is the Taylor expansion of $f(z)$ around z_0

- If a power series converges inside some circle, it is the Taylor series of its (analytic) limit function and can be integrated and differentiated term by term inside this circle; moreover, this limit function must fail to be analytic somewhere on the circle of convergence

Introduction

- In this section, we wish to investigate the possibility of a series representation of a function f near a singularity
- After all, if the occurrence of a singularity is merely due to a vanishing denominator, might it not be possible to express the function as something like $A/(z-z_0)^p + g(z)$, where g is analytic and has a Taylor series around z_0?
- To be sure, not all singularities are of this type (recall $\text{Log} z$ at $z_0 = 0$)
- If the function is analytic in an annulus surrounding one or more of its singularities (note that $\text{Log} z$ does not have this property, due to its branch cut), we can display its "singular part" according to the following theorem

Definition of Laurent Series

Theorem

Let f be analytic in the annulus $r < |z-z_0| < R$. Then f can be expressed there as the sum of two series

$$f(z) = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$$

both series converging in the annulus, and converging uniformly in any closed subannulus $r < \rho_1 \leq |z-z_0| \leq \rho_2 < R$. The coefficients a_j are given by

$$a_j = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi-z_0)^{j+1}}d\xi \qquad (j = 0, \pm 1, \pm 2, \ldots)$$

where C is any positively oriented simple closed contour lying in the annulus and containing z_0 in its interior

Definition of Laurent Series (Cont'd)

- Such an expansion, containing negative as well as positive powers of $(z-z_0)$, is called the **Laurent series** for f in this annulus. It is usually abbreviated
$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$
- Note that if f is analytic throughout the disk $|z-z_0| < R$, the coefficients with negative subscripts are zero by Cauchy's theorem, and the others reproduce the Taylor series for f

Definition of Laurent Series (Cont'd)

- Replacing $(z - z_0)$ with $1/(z - z_0)$ in Theorem 7 (page 253), one easily sees that any formal series of the form $\sum_{j=1}^{\infty} c_{-j}(z - z_0)^{-j}$ will converge **outside** some "circle of convergence" $|z - z_0| = r$ whose radius depends on the coefficients, with uniform convergence holding in each region $|z - z_0| \geq r' > r$
- Thus termwise integration is justified by Theorem 8 (page 255), and proceeding in a manner analogous to that of Sec 5.3 we can prove the theorem in the next slide

Definition of Laurent Series (Cont'd)

Theorem

Let $\sum_{j=0}^{\infty} c_j(z - z_0)^j$ and $\sum_{j=1}^{\infty} c_{-j}(z - z_0)^{-j}$ be any two series with the following properties:

(i) $\sum_{j=0}^{\infty} c_j(z - z_0)^j$ *converges for* $|z - z_0| < R$

(ii) $\sum_{j=1}^{\infty} c_{-j}(z - z_0)^{-j}$ *converges for* $|z - z_0| > r$ *and*

(iii) $r < R$

Then there is a function $f(z)$, analytic for $r < |z - z_0| < R$, whose Laurent series in this annulus is give by $\sum_{j=-\infty}^{\infty} c_j(z - z_0)^j$

Introduction

- This section focuses on using the Laurent expansion to classify the behavior of an analytic function near its zeros and isolated singularities
- A **zero** of a function is a point z_0 where f is analytic and $f(z_0) = 0$
- An **isolated singularity** of f is a point z_0 such that f is analytic in some punctured disk $0 < |z - z_0| < R$ but **not analytic** at z_0 itself

Zeros of Complex-Valued Functions

Definition

A point z_0 is called a **zero of order** m for the function f if f is analytic at z_0 and f and its first $m - 1$ derivatives vanish at z_0, but $f^{(m)}(z_0) \neq 0$

- In other words, we have
$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \neq f^{(m)}(z_0)$$
- In this case the Taylor series for f around z_0 takes the form
$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + a_{m+2}(z-z_0)^{m+2} + \cdots$$
or
$$f(z) = (z-z_0)^m \left[a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \cdots\right]$$
where $a_m = f^{(m)}(z_0)/m! \neq 0$

Zeros of Complex-Valued Functions (Cont'd)

Theorem

Let f be analytic at z_0. Then f has a zero of order m at z_0 if and only if f can be written as

$$f(z) = (z - z_0)^m g(z)$$

where g is analytic at z_0 and $g(z_0) \neq 0$

Corollary

If f ia an analytic function such that $f(z_0) = 0$, then either f is identically zero in a neighborhood of z_0 or there is a punctured disk about z_0 in which f has no zeros

Zeros of Complex-Valued Functions (Cont'd)

- Notice that if f is nonconstant, analytic, and zero at z_0, the order of the zero must be a whole number
- The function $z^{1/2}$ could be said to have a zero of order $1/2$ at $z = 0$, but of course it is not analytic there

Singularities of Complex-Valued Functions

- We know that f has a Laurent expansion around any isolated singularity z_0;

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \qquad (1)$$

  for, say $0 < |z - z_0| < R$. (The r is zero for an isolated singularity)
- We can classify z_0 into one of the following three categories in the next slide

Singularities of Complex-Valued Functions (Cont'd)

Definition

Let f have an isolated singularity at z_0, and let (1) be the Laurent expansion of f in $0 < |z - z_0| < R$. Then

(i) If $a_j = 0$ for all $j < 0$, we say that z_0 is a **removable singularity** of f

(ii) If $a_{-m} \neq 0$ for some positive integer m but $a_j = 0$ for all $j < -m$, we say that z_0 is a **pole of order** m for f

(iii) If $a_{-m} \neq 0$ for an infinite number of negative values of j, we say that z_0 is an **essential singularity** of f

Removable Singularities

- When f has a removable singularity at z_0, its Laurent series takes the form
  $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R)$

Lemma
If f has a removable singularity at z_0, then
(i) *$f(z)$ is bounded in some punctured circular neighborhood of z_0*
(ii) *$f(z)$ has a (finite) limit as z approaches z_0, and*
(iii) *$f(z)$ can be redefined at z_0 so that the new function is analytic at z_0*

Removable Singularities (Cont'd)

- Conversely, if a function is bounded in some punctured neighborhood of an isolated singularity, that singularity is removable
- Clearly, removable singularities are not too important in the theory of analytic functions
- The concept is occasionally helpful in providing compact descriptions of the other kinds of singularities

A Pole of Order m

- The Laurent series for a function with a pole of order m looks like
  $$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (a_{-m} \neq 0)$$
  valid in some punctured neighborhood of z_0
- A pole of order 1 is called a simple pole

Lemma
If the function f has a pole of order m at z_0, then $|(z - z_0)^l f(z)| \to \infty$ as $z \to z_0$ for all integers $l < m$, while $(z - z_0)^m f(z)$ has a removable singularity at z_0. In particular, $|f(z)| \to \infty$ as z approaches a pole

A Pole of Order m (Cont'd)

Lemma
A function f has a pole of order m at z_0 if and only if in some punctured neighborhood of z_0

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where g is analytic at z_0 and $g(z_0) \neq 0$

(See Example 1 on page 281)

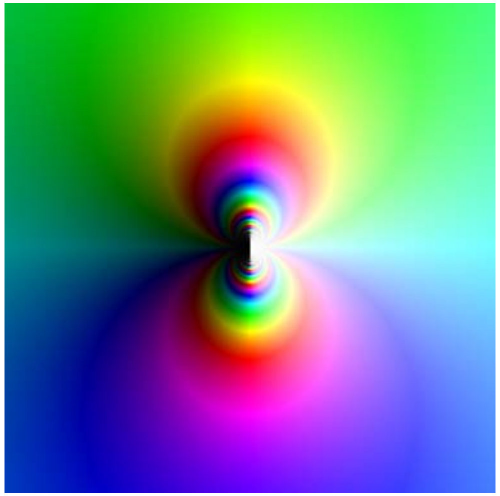
Essential Singularities

Theorem

(Picard's Theorem) *A function with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity*

- If neither $\lim_{z \to z_0} f$ nor $\lim_{z \to z_0} 1/f$ exist, then z_0 is an essential singularity of both f and $1/f$
- Another way to characterize an essential singularity is: The point z_0 is an essential singularity if and only if one (or both) of these of two conditions exist:
  (1) The function f has poles in every neighborhood of z_0, meaning that the singularity is not isolated
  (2) The Laurent series of f at the point z_0 has infinitely many negative degree terms

Essential Singularities (Cont'd)



Plot of the function $e^{1/z}$, centered on the essential singularity at $z_0 = 0$. The hue represents the complex argument, the luminance represents the absolute value. This plot shows how approaching the essential singularity from different directions yields different behaviors (as opposed to a pole, which would be uniformly white)

Comments to Three Types of Singularities

- Boundedness indicated a removable singularity, approaching ∞ indicated a pole, and anything else must indicate an essential singularity
- These characterizations are often useful in determining the nature of a singularity when it is inconvenient to find the Laurent expansion

Comments to Three Types of Singularities (Cont'd)

The following theorem summarize the various equivalent characterizations of the three types of isolated singularities

Theorem

If f has an isolated singularity at z_0, then the following equivalences hold:

(1) *z_0 is a removable singularity $\Leftrightarrow$ $|f|$ is bounded near z_0 $\Leftrightarrow$ $f(z)$ has a limit as $z \to z_0$ $\Leftrightarrow$ f can be redefined at z_0 so that f is analytic at z_0*

(2) *z_0 is a pole $\Leftrightarrow$ $|f(z)| \to \infty$ as $z \to z_0$ $\Leftrightarrow$ f can be written $f(z) = g(z)/(z - z_0)^m$ for some integer $m > 0$ and some function g analytic at z_0 with $g(z_0) \neq 0$*

(3) *z_0 is an essential singularity $\Leftrightarrow$ $|f(z)|$ neither is bounded near z_0 nor goes to infinity as $z \to z_0$ $\Leftrightarrow$ $f(z)$ assumes every complex number, with possible one exception, as a value in every neighborhood of z_0*

Some Observations

- The analytic property for a function f at a point z_0 places enormous restrictions on f; in particular, it must be infinitely differentiable, and expressed by its Taylor series in a neighborhood of z_0
- Now we find that if f is merely presumed to be defined, and analytic, in a punctured neighborhood of z_0 (like $0 < |z - z_0| < r$), then it is still strongly restricted
- One can characterized its behavior near z_0 by asking how many powers of $(z - z_0)$ would it take to "civilize" $f(z)$, in the sense that $(z - z_0)^m f(z)$ would have a finite, nonzero limiting values as $z \to z_0$

Some Observations (Cont'd)

- If the answer m is a positive integer, then f has a pole of order m at z_0 and it can be written as $g(z)/(z - z_0)^m$ with g analytic and nonzero at z_0
- If m is a negative integer, then f can be written as $g(z)(z - z_0)^{|m|}$ with g, again, analytic and nonzero at z_0. In this case, f exhibits a zero of order $|m|$ at z_0
- If m is zero, then f has a removable singularity at z_0
- The only other possibility is that no such m exists, that is, no power of $(z - z_0)$ can endow $(z - z_0)^m f(z)$ with a nonzero limit at z_0. Then unless f is identically zero, it has an essential singularity at z_0, taking all complex numbers as values in any neighborhood of z_0 (with possibly, one exception)

Introduction

- From last section we know that if a mapping is given by an analytic function possessing a pole, it carries points near that pole to indefinitely distant points
- It must have occurred to the reader that one might take the value of f at the pole to be ∞
- Before taking this plunge, however, we should be aware of all the ramifications
- Geometrically, we are speaking of the point of infinity, which can be reached by proceeding infinitely far **along any direction** in the complex plane

Concept of Point of Infinity

- A sequence of points z_n in **C** $(n = 1, 2, 3, \ldots)$ approaches ∞ if $|z_n|$ can be made arbitrarily large by taking n large
- Consequently, we shall write $f(z_0) = \infty$ when $|f(z)|$ increases without bound as $z \to z_0$ and shall write $f(\infty) = w_0$ when $f(z) \to w_0$ as $z \to \infty$
- We find it convenient to carry this notion still further and speak of functions that are **"analytic at ∞"**
- The analyticity properties of f at ∞ are classified by first performing the mapping $w = 1/z$, which maps the point to the origin, and then examining the behavior of the composite function $g(w) := f(1/w)$ at the origin $w = 0$

Concept of Point of Infinity (Cont'd)

Thus we say

1. $f(z)$ is analytic at ∞ if $f(1/w)$ is analytic (or has a removable singularity) at $w = 0$
2. $f(z)$ has a pole of order m at ∞ if $f(1/w)$ has a pole order at $w = 0$
3. $f(z)$ has an essential singularity at ∞ if $f(1/w)$ has an essential singularity at $w = 0$

We can interpret these conditions for a function analytic outside some disk as follows:

1'. $f(z)$ is analytic at ∞ if $|f(z)|$ is bounded for sufficiently large $|z|$

2'. $f(z)$ has a pole at ∞ if $f(z) \to \infty$ as $z \to \infty$

3'. $f(z)$ has an essential singularity at ∞ if $|f(z)|$ neither is bounded for large $|z|$ nor goes to infinity as $z \to \infty$