Chapter 4: Complex Integration

Li, Yongzhao

State Key Laboratory of Integrated Services Networks, Xidian University

October 10, 2010

Introduction

- The two-dimensional nature of the complex plane required us to generalize our notion of a derivative because of the freedom of the variable to approach its limit along any of an infinite number of directions.
- This two-dimensional aspect will have an effect on the theory of integration, necessitating the consideration of integrals along general curves in the plane not merely segments of the x-axis
- Fortunately, such well-known techniques as using antiderivatives to evaluate integrals carry over to the complex case

Introduction (Cont'd)

- When the function under consideration is analytic the theory of integration becomes an instrument of profound significance in studying its behavior
- The main result is the **theorem of Cauchy**, which roughly says that the integral of a function around a closed loop is zero if the function is analytic "inside and on" the loop
- Using this result, we shall derive the **Cauchy integral formula**, which explicitly displays many of the important properties of analytic function

Parametrization of a Curve

- To study the complex integration in a plane, the first problem is finding a mathematical explication of our intuitive concept of a curve in the xy-plane (or called z-plane)
- Although most of the applications described in this book involve only two simple types of curves – line segments and arc of circles – it will be necessary for proving theorems to nail down the definition of more general curves
- A curve γ can be constituted by the points $z(t) = x(t) + iy(t)$ over an interval of time $a \le t \le b$. Then the curve γ is the range of $z(t)$ as t varies between a and b
- In such a case, $z(t)$ is called the **parametrization** of γ

Smooth Curves (Cont'd)

Definition

A point set γ is called a **smooth closed curve** if it is the range of some continuous function $z = z(t)$, $a \le t \le b$, satisfying conditions i and ii and the following:

(iii') $z(t)$ is one-to-one on the half open $[a, b)$, but $z(b) = z(a)$ and $z'(b) = z'(a)$

Smooth Curves

Definition

A point set γ in the complex plane is said to be a **smooth arc** if it is the range of some continuous complex-valued function $z = z(t)$, $a \le t \le b$, that satisfies the following conditions:

(i) $z(t)$ has a continuous derivative on $[a, b]$

(ii) $z'(t)$ never vanishes on $[a, b]$
- $z'(t)$ must exist (no corners)
- $z'(t)$ is nonzero (no cusps)

(iii) $z(t)$ is one-to-one on $[a, b]$ (no self-intersections)

Smooth Curves (Cont'd)

- The phrase "γ is a smooth curve" means that γ is either a smooth arc or a smooth closed curve
- The conditions of the definition imply that smooth curve possesses a unique tangent at every point and the tangent direction varies continuous along the curve. Consequently a smooth curve has no corners or cusps
- To show that a set of points γ in the complex plane is a smooth curve, we have to exhibit a parametrization function $z(t)$ whose range is γ, and is "admissible" in the sense that it meets the criteria of the definition
- A given smooth curve γ will have many different admissible parameterizations, but we need produce only one admissible parametrization in order to show that a given curve is smooth

Directed Smooth Arcs

- A smooth arc, together with a specific ordering of its points, is called a **directed smooth arc**. The ordering can be indicated by an arrow
- The point $z(t_1)$ will precede $z(t_2)$ whenever $t_1 < t_2$. Since there are only two possible ordering, any admissible parametrization must fall into one the two categories, according to the particular ordering it respects
- If $z = z(t), a \leq t \leq b$, is an admissible parametrization consistent with one of the ordering, then $z = z(-t), -b \leq t \leq -a$, always corresponds to the opposite ordering

Concept of a Contour

- The general curves are formed by joining directed smooth curves together end-to-end; this allows self-intersection, cusps, and corners
- It will be convenient to include single isolated points as members of this class

Definition

A **contour** Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \ldots, n-1$. In this case one can write $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$

Directed Smooth Arcs (Cont'd)

- The points of a smooth closed curve have been ordered when (i) a designation of the initial point is made and (ii) one of the two "directions of transit" from this point is selected
- If this parametrization is given by $z = z(t), a \leq t \leq b$, then (i) the initial point must be $z(a)$ and (ii) the point $z(t_1)$ precedes the point $z(t_2)$ whenever $a < t_1 < t_2 < b$
- The phrase directed smooth curve will be used to mean either a directed smooth arc or a directed smooth closed curve
- Next, we are ready to specify the more general kinds of curves that will be used in the theory of integration

Concept of a Contour (Cont'd)

- The theory of contour is easier to express in terms of **contour parameterizations**
- One can say that $z = z(t), a \leq t \leq b$, is a parametrization of the contour $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ if there is a subdivision of $[a, b]$ into n subintervals $[\tau_0, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_{n-1}, \tau_n]$, where $a = \tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = b$, such that on each subinterval $[\tau_{k-1}, \tau_k]$ the function $z(t)$ is an admissible parametrization of the smooth curve γ_k, consistent with the direction on γ_k
- Since the endpoints of consecutive γ_k's are properly connected, $z(t)$ must be continuous on $[a, b]$. However $z'(t)$ may have jump discontinuities at the points γ_k

Parametrization of a Contour

- When we have admissible parameterizations of the components γ_k of a contour Γ. We can piece these together to get a contour parametrization for Γ by simply **rescaling** and **shifting** the parameter intervals for t (Example 2 on page 156)
- The (undirected) point set underlying a contour is known as a **piecewise smooth curve**
- We shall use the symbol Γ ambiguously to refer to both the contour and its underlying curve, allowing the context to provide the proper interpretation
- The opposite contour is denoted by $-\Gamma$

Jordan Curve Theorem

Theorem

Any simple closed contour separates the plane into two domains, each having the curves as its boundary. One of these domains, called the **interior**, *is* **bounded**; *the other, called the* **exterior**, *is* **unbounded**

- When the interior domain lies to the left, we say that Γ is positively oriented. Otherwise Γ is said to be oriented negatively.
- A positive orientation generalizes the concept of counterclockwise motion

Closed Contour

- Γ is said to be a **closed contour** or a loop if its initial and terminal points coincide
- A **simple closed contour** is closed contour with no multiple points other than its initial-terminal point; in other words, if $z = z(t), a \le t \le b$, is a parametrization of the closed contour, then $z(t)$ is one-to-one on the half-open interval $[a, b)$ (no self-intersections)
- There is an alternative way of specifying the direction along a curve if the curve happens to be a simple closed contour

The Length of a Contour

- If one admissible parametrization for curve γ is $z(t) = x(t) + iy(t), a \le t \le b$, let $s(t)$ be the length of the arc of γ traversed in going from the point $z(a)$ to the point $z(b)$. As shown in elementary calculus, we have

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left|\frac{dz}{dt}\right|$$

- Consequently, **the length of the smooth curve** is given by the important integral formula

$$l(\gamma) = \text{length of } \gamma = \int_a^b \frac{ds}{dt} dt = \int_a^b \left|\frac{dz}{dt}\right| dt \qquad (1)$$

The Length of a Contour (Cont'd)

- $l(\gamma)$ is a geometric quantity that depends only on the point set γ and is independent of the particular admissible parametrization used in the computation
- The **length of a contour** is simply defined to be the sum of the length of its component curves

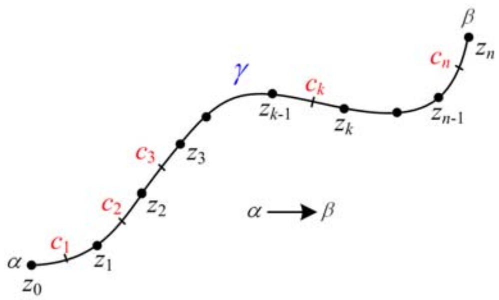
Introduction (Cont'd)

- We will accomplish this by first defining the integral along a single directed smooth curve and then defining integrals along a contour in terms of the integrals along its smooth components
- Finally, we once again obtain simple rules for evaluating integrals in terms of antiderivatives

Introduction

- In calculus, the definite integral of a real-valued function f over an interval $[a,\ b]$ is defined as the limit of certain sums $\sum_{k=1}^{n} f(c_k)\triangle x_k$ (called **Riemann sums**)
- However, the fundamental theorem of calculus lets us evaluate integrals more directly when an **antiderivative** is known
- The aim of this section is to use this notion of Riemann sums to define integral of a complex-valued function along a contour Γ in the z-plane

Riemann Sum



Partitioned Curve

Riemann Sum

- **Partition** $\mathcal{P}_n$ is a finite number of points $\{z_0, z_1, \ldots, z_n\}$ on γ such that $z_0 = \alpha$, $z_n = \beta$
- **Riemann sum** for the function f corresponding to the partition $\mathcal{P}_n$:
  $\mathbf{S}(\mathcal{P}_n) := f(c_1)(z_1 - z_0) + f(c_2)(z_2 - z_1) + \cdots + f(c_n)(z_n - z_{n-1})$
- On writing $z_k - z_{k-1} = \triangle z_k$, this becomes

$$\mathbf{S}(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\triangle z_k$$

- With the concept of Riemann Sum, we can generalize the definition of definite integral given in calculus

Integral of a Complex Function f along a Directed Smooth Curve γ (Cont'd)

Theorem

If f is continuous on the directed smooth curve γ, then f is integrable along γ

- This theorem is of great theatrical importance, but is gives us no information of how to compute the integral $\int_\gamma f(z)dz$
- Since we are ready skilled in evaluating the definite integral of calculus, it would certainly be advantageous if we could express the complex integral in terms of real integrals

Integral of a Complex Function f along a Directed Smooth Curve γ

Definition

Let f be a complex-valued function defined on the directed smooth curve γ. We say that f is integrable along γ if there exists a complex number L that is the limit of every sequence of Riemann sums $\mathbf{S}(\mathcal{P}_1)$, $\mathbf{S}(\mathcal{P}_2)$, …, $\mathbf{S}(\mathcal{P}_n)$, … corresponding to any sequence of partitions of γ satisfying $\lim_{n\to\infty} \mu(\mathcal{P}_n) = 0$; i.e.

$$\lim_{n\to\infty} \mathbf{S}(\mathcal{P}_n) = L \quad \text{whenever} \quad \lim_{n\to\infty} \mu(\mathcal{P}_n) = 0$$

The constant L is called the integral of f along γ, and we write

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k = \int_\gamma f(z)dz = \int_\gamma f$$

Contour Integrals Along a Directed Smooth Curve

- First consider the special case when γ is the real line segment $[a, b]$ directed from left to right
- Notice that if f happened to be a real-valued function defined on $[a, b]$, the definition of complex integral reduces to the integral $\int_a^b f(t)$ given in calculus
- When f is a complex-valued function continuous on $[a, b]$, we can write $f(t) = u(t) + iv(t)$, where u and v are each real-valued and continuous on $[a, b]$, then we have

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt \qquad (2)$$

this expresses the complex integral in terms of two real integrals

Contour Integrals Along a Directed Smooth Curve (Cont'd)

Theorem

If the complex-valued function f is continuous on $[a,b]$ and $F'(t)=f(t)$ for all t in $[a,b]$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

Theorem

Let f be a function continuous on the directed smooth curve γ. Then if $z=z(t), a\le t\le b$, is any admissible parametrization of γ consistent with its direction, we have

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Contour Integrals Along a Contour

Definition

Suppose that Γ is a contour consisting of the directed smooth curves $(\gamma_1,\gamma_2,\dots,\gamma_n)$, and let f be a function continuous on Γ. Then the contour integral of f along Γ is denoted by the symbol $\int_\Gamma f(z)dz$ and is defined by the equation

$$\int_\Gamma f(z)dz := \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \dots + \int_{\gamma_n} f(z)dz$$

If Γ consists of a single point, then for obvious reasons we set

$$\int_\Gamma f(z)dz := 0$$

Contour Integrals Along a Directed Smooth Curve (Cont'd)

- Since the integral of f along γ was defined independently of any parametrization, we immediately deduce the following corollary

Corollary

If f is continuous on the directed smooth curve γ and if $z=z_1(t), a\le t\le b$, and $z=z_2(t), c\le t\le d$, are any two admissible parameterizations of γ consistent with its direction, then

$$\int_a^b f(z_1(t))z_1'(t)dt = \int_c^d f(z_2(t))z_2'(t)dt$$

Contour Integrals Along a Contour (Cont'd)

- If we have a parametrization $z=z(t), a\le t\le b$, for the whole contour $\Gamma=(\gamma_1,\gamma_2,\dots,\gamma_n)$, we can get the following formula
$$\int_\Gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$
- Using this formula it is not difficult to prove that integration around simple closed contour is independent of the choice of the initial-terminal point
- In problems dealing with integrals along such contours, we need only specify the direction of transit, not the starting point

Upper Bound of the Magnitude of a Contour Integral

Theorem

If f is continuous on the contour Γ and if $|f(z)| \leq M$ for all z on Γ, then

$$\left| \int_\Gamma f(z) dz \right| \leq M l(\Gamma)$$

where $l(\Gamma)$ denotes the length of Γ. In particular, we have

$$\left| \int_\Gamma f(z) dz \right| \leq \max_{z \text{ on } \Gamma} |f(z)| \cdot l(\Gamma)$$

Introduction

- One of the important results in the theory of complex analysis is the extension of the Fundamental Theorem of Calculus to contour integrals
- It implies that in certain situations, the integral of a function is independent of the particular path joining the initial and terminal points, in fact, it completely characterize the conditions under which this property holds
- In this section, we will explore this phenomenon in detail. We will begin with the Fundamental Theorem, which enables us to evaluate integrals without introducing parameterizations, provided that an antiderivative of the integrand is known

Comments

- Although the real definite integral can be interpreted, among other things, as an area, no corresponding geometric visualization is available for contour integrals
- Nevertheless, the latter integrals are extremely useful in applied problems, as we shall see in subsequent chapters

Independence of Path

Theorem

Suppose that the function $f(z)$ is continuous in a domain D and has an antiderivative $F(z)$ throughout D; i.e., $dF(z)/dz = f(z)$ for each a in D. Then for any contour Γ lying in D, with initial point z_I and terminal point z_T, we have

$$\int_\Gamma f(z) dz = F(z_T) - F(z_I)$$

Note that the conditions of the theorem imply that $F(z)$ is analytic and hence continuous in D

Independence of Path (Cont'd)

- Since the endpoints of a loop, i.e., a closed contour, are equal, we have the following immediate consequence of the theorem

Corollary

If f is continuous in a domain D and has an antiderivative throughout D, then $\int_\Gamma f(z)dz = 0$ for all loops Γ lying in D

- Another important conclusion that can be drawn from the theorem is that when a function f has an antiderivative throughout a domain D, its integral along a contour in D depends only on the endpoints z_I and z_T; i.e., the integral is independent of the path Γ joining these two points

Cauchy's Integral Theorem

Theorem

If f is analytic in a simple connected domain D and Γ is any loop (closed contour) in D, then

$$\int_\Gamma f(z)dz = 0$$

Theorem

In a simple connected domain, an analytic function has an antiderivative, its contour integral are independent of path, and its loop integrals vanish

Independence of Path (Cont'd)

Theorem

let f be continuous in a domain D. Then the following are equivalent:

(i) *f has antiderivative in D*

(ii) *Every loop integral of f in D vanishes [i.e., if Γ is any loop in D, then $\int_\Gamma f(z)dz = 0$]*

(iii) *The contour integral of f are independent of path in D [i.e., if Γ_1 and Γ_2 are any two contours in D sharing the same initial and terminal points, then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$]*

Introduction

- From Cauchy's theorem we know that, if f is analytic inside and on the simple closed contour Γ, $\int_\Gamma f(z)dz = 0$
- Now the question is how about the integral $\int_\Gamma f(z)/(z - z_0)dz$, where z_0 is a point in the interior of Γ
- Obviously, there is no reason to expect that this integral is zero, because the integrand has a singularity inside the contour Γ
- In fact, as the primary result of this section, we shall show that for all z_0 inside Γ the value of the integral is proportional to $f(z_0)$

Cauchy's Integral Formula

Theorem

let Γ be a simple closed positively oriented contour. If f is analytic in some simple connected domain D containing Γ and z_0 is any point inside Γ, then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz$$

- One remarkable consequence of Cauchy's formula is that by merely knowing the values of the analytic function f on Γ we can compute the above integral and hence all the values of f inside Γ. In other words, the behavior of a function analytic in a region is completely determined by its behavior on the boundary

Cauchy's Integral Formula (Cont'd)

Theorem

If f is continuous in a domain D and if

$$\int_\Gamma f(z)dz = 0$$

for every closed contour Γ in D, then f is analytic in D

Theorem

If f is analytic inside and on the simple closed oriented contour Γ and if z is any point inside Γ, then

$$f^{(m-1)}(z_0) = \frac{(m-1)!}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^m}dz$$

Cauchy's Integral Formula (Cont'd)

Theorem

If f is analytic in a domain D, then all its derivatives f', f'', $\ldots$, $f^{(n)}$, $\ldots$ exist and are analytic in D

Theorem

If $f = u + iv$ is analytic in a domain D, then all partial derivatives of u and v exist and are analytic in D

Introduction

- Many interesting facts about analytic functions are uncovered when one considers upper bounds on their moduli
- We already have one result in this direction, namely, the integral estimate Theorem 5 of Sec 4.2
- When this is judiciously applied to the Cauchy integral formulas we obtain the Cauchy estimates for the derivatives of an analytic function

Cauchy Estimate for the Derivative of an Analytic Function

Theorem
Let f be analytic inside and on a circle C_R of radius R centered about z_0. If $|f(z)| \le M$ for all z on C_R, then the derivatives of f at z_0 satisfy

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n}$$

Theorem
Liouville's theorem: *The only bounded entire functions are the constant functions*

Mean-Value Property

- According to the Cauchy formula, for the function f, analytic inside and on the circle C_R around z_0, we have
$$f(z_0) = \frac{1}{2\pi i}\oint_{C_R} \frac{f(z)}{z - z_0} dz$$
- Parameterizing C_R by $z = z_0 + Re^{it}, 0 \le t \le 2\pi$, then we write the above formula as
$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it} dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it}) dt \end{aligned}$$

Fundamental Theorem of Algebra

- Clearly, nonconstant polynomials are unbounded (over the whole plane)
- We expect a polynomial of degree n to behave like z_n for large $|z|$, because the leading term will dominate the lower powers
- If $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, and we can see that
$$P(z)/z^n \to a_n \quad \text{as} \quad |z| \to \infty$$

Theorem
Every nonconstant polynomial with complex coefficients has at least one zero

Mean-Value Property (Cont'd)

- The mean-value formula helps us establish the following lemma

Lemma
Suppose that f is analytic in a disk centered at z_0 and that the maximum value of $|f(z)|$ over this disk is $|f(z_0)|$. Then $|f(z)|$ is constant in the disk

- The lemma says that the modulus of an analytic function cannot achieve its maximum at the center of the disk unless $|f|$ is constant

Maximum Modulus Principle

Theorem

If f ia analytic in a domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D, then f is constant in D

Theorem

A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary